CREDIT VALUE ADJUSTMENT FOR BILATERAL COUNTERPARTY RISK OF COLLATERALIZED CONTRACTS UNDER SYSTEMIC RISK

Cyril Durand
School of Mathematics and Statistics
University of New South Wales
NSW 2052, Australia

Marek Rutkowski∗
School of Mathematics and Statistics
University of Sydney
NSW 2006, Australia

August 15, 2010

Abstract

This paper departs from the usual methods for pricing contracts with counterparty risk found in the existing literature. In effect, typically, these models, first, do not account for either systemic effects or ‘at first default’ contagion, second, postulate that the contract value at default equals either the default-free value or the pre-default value, and third, do not take margin agreements into account. Instead, we propose a general framework which allows for the CVA computation under bilateral counterparty risk of a contract in the presence of systemic and right- or wrong-way risks, and under alternative settlement conventions and margin agreements.

∗The research of M. Rutkowski was supported under Australian Research Council’s Discovery Projects funding scheme (DP0881460). The authors thank J.P. Laurent for enlightening discussions and comments.
1 Introduction

The risk that a counterparty cannot meet its contractual obligations has become the hot subject of the moment in the realm of finance. Intertwined studies were recently conducted in different directions, especially the systemic risk (the risk of a domino effect following the bankruptcy of a major financial institution in Cont et al. (2009)), systemic impact of centralized clearing in Duffie and Zhu (2009), the effects of an asymmetric information in regard credit risk of securitized products and exposure of banks to these products in Gorton (2008), the liquidity effects (the risk of fire sales, impact of collateral policies of central banks in Liberti and Mian (2008)), and, last but not least, the counterparty risk (the computations of the Credit Value Adjustment (CVA) in Bielecki et al. (2010), Brigo and Capponi (2008), Crépey et al. (2010), Gregory (2009), Lipton and Sepp (2009), Pykhtin (2009), Elouerkahoui (2010), and Yi (2010a, 2010b), to mention a few).

While the effects of margin agreements and the intricacies of collateral markets (rehypothecation, especially) emerge as a crucial issue from all these different perspectives, the regulators are enforcing new constraints on liquidity, capital allocation and short term borrowing. They are also contemplating new ways of dealing with defaults (most notably, novation) and pushing for the development of additional compensation chambers for derivatives. Although some support for the introduction of central clearing-houses for some OTC derivatives (for instance, credit default swaps) was voiced, it is unclear whether the introduction of such clearing-houses would improve counterparty risk management considered as a whole (see Duffie and Zhu (2009)). In any case, the study of counterparty risk will still be necessary for OTC derivatives. It is also worthwhile to note that the bankruptcy of investment banks, such as Lehman Brothers, shed new light on legal issues related to the termination and settlement of derivative contracts in a context of multi-jurisdictions and diverse contractual terms, which calls for enhanced harmonization of legal proceedings. These issues are beyond the scope of this research, however.

In the aftermath of the recent financial crisis, it is now widely acknowledged that it is crucial to account for both counterparty risk and systemic risk in the pricing and hedging of financial instruments. In this paper, we depart from the usual methods for pricing contracts with counterparty risks developed in the existing literature. In effect, typically, these models, first, postulate that the contract’s value at default is equal either to the default-free value or the pre-default value. Second, they frequently do not take margin agreements into account. Finally, contagion effects are typically neglected. By contrast, we analyze here in some detail the computation of the CVA under several alternative settlement conventions. We also quantify the effects of various kinds of margin agreements, as typically specified in a Credit Support Annex (CSA). Finally, we deal with a particular Markovian contagion model in which the effects of correlation parameters such as the right- or wrong-way risk or the contagion effect at first default, as well as the systemic risk, are taken into account. In Section 2, we present our fairly general setting for the computation of CVA under bilateral counterparty risk. Note that we deal throughout with bilateral counterparty risk, so that the sign of the CVA can be either positive or negative, in principle. There is thus no need to introduce the terminological distinction between the concepts of CVA and DVA (Debt Value Adjustment), as is fairly common in those practical applications where the computations of CVA and DVA are conducted separately, assuming each time a unilateral counterparty risk. In Section 3, we focus our attention on the CVA for credit default swaps with bilateral counterparty risk. We apply there a general pricing procedure in the presence of counterparty and systemic risks and we briefly discuss the practical relevance of our approach.

2 Counterparty Risk and CVA

The hypothesis of a unilateral counterparty risk has been seen in the past as a practical, albeit somewhat rough, estimate for modeling contracts between a major financial institutions and their clients (see, e.g., Mashal and Naldi (2003)). But the realization that even the most prestigious investment banks could go bankrupt has shattered the foundations for resorting to unilateral models.
The clients of banks are nowadays prone to question such an assumption and are willing to ask for suitable adjustments of contractual terms in order to gain a better security on their financial instruments, as well as on their collaterals, in the event of the counterparty’s default. This situation explains why, for instance, the subject of re-hypothecation has given rise to much debate between investment banks and hedge funds.

2.1 Settlement Values without Margin Agreements

Let us first state a few basic definitions related to the counterparty risk in a general setting.

**Definition 2.1.** A default-free contract is an over-the-counter (OTC) contract entered into by two non-defaultable parties. A contract with unilateral counterparty risk is an OTC contract entered into by a non-defaultable investor and a defaultable counterparty. A contract with bilateral counterparty risk is an OTC contract entered into by a defaultable investor and a defaultable counterparty.

We denote by 1 and 2 the two parties which enter into a contract; they are referred to hereafter as the ‘investor’ and the ‘counterparty’. We write $c_1^t$ and $c_2^t$ to denote the level of creditworthiness of each party at time $t$.

**Definition 2.2.** By the creditworthiness $c_j^t$, $j = 1, 2$ of the $j$th party at time $t$, we mean some measure of the ability at time $t$ of this party to pay back its debts, given its financial and business situation.

As a proxy for the creditworthiness at time $t$ one may take, for instance, the current credit rating (or simply rating) of the counterparty, either based on an internal rating methodology or provided by a specialized rating agency. The counterparty risk is the risk that the party of an OTC contract may default and fail to meet contractual obligations. We assume that in the event of the counterparty’s default the contract is terminated and immediately settled. For a contract liquidly traded after the counterparty’s default, the settlement value could be based on the current market value of the contract. This is a rather unusual situation for an OTC contract to be actively traded after default of the counterparty, however. Therefore, there is a need to specify the settlement value in some conventional way. To this end, we first examine the concept of the mark-to-market value of a contract.

By the right-continuous with left-hand limits (i.e., càdlàg) process $M_i^t$ we denote hereafter the conventional mark-to-market value of a contract, as seen by the investor. The exact specification of the contractual mark-to-market value at the time of default in each of those situations depends on specific clauses in the contract’s agreement accepted by both parties. It is natural to assume that they agree on the convention $i$ in the event of default. The following alternative specifications of the mark-to-market value will be of our interest.

**Definition 2.3.** The mark-to-market value $M_i^t$ (MtM value, for short) at time $t$ under convention $i$ for $i = a, a', b, c, c'$ is the value of a contract with identical promised cash flows:

- $a$) entered by two risk-free companies,
- $a'$) entered by the non-defaulting party with a risk-free company,
- $b$) entered by the non-defaulting party with a party having the same rating at time $t$ as the initial rating of the defaulted party,
- $c$) entered by the non-defaulting party with a party having the same rating at time $t$ as the rating of the counterparty just before default,
- $c'$) entered by two companies with the same credit ratings as the investor’s and counterparty’s ratings just before default.

In principle, the actual value of the mark-to-market value at the time of default may depend on whether the counterparty or the investor defaulted. However, we assume here that they coincide. The settlement value is typically defined as a particular function of the contract’s MtM value at the time of default. If one of the counterparties defaults, it is common to settle the contract as follows.
If the MtM value $M^t_i$ of the contract at time $t$ of the counterparty’s default is negative for the investor (which means that the MtM value $-M^t_i$ is positive for the counterparty) then the counterparty receives the full value of the contract. Otherwise, that is, when the MtM value $M^t_i$ of the contract is positive, the investor claims the full value $M^t_i$, but, typically, she will receive only a portion of this value. An analogous rule applies in the event of the investor’s default. This prevailing market convention is known as the full two-way payment rule (or no-fault rule). In what follows, we always consider settlement values from the investor’s perspective assuming the full two-way payment rule. For any real number $a$, we denote $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$. Hence the equality $a = a^+ - a^-$ is satisfied for any real $a$.

**Definition 2.4.** The settlement value $S^{i,2}_t$ of the contract in the case of the counterparty’s default at time $t$ is given by the stochastic process $S^{i,2}_t = R^2_t(M^t_i)^+ - (M^t_i)^-$ where $R^2_t$ is the counterparty’s recovery rate. The settlement value $S^{i,1}_t$ of the contract in the case of the investor’s default at time $t$ is given by the stochastic process $S^{i,1}_t = (M^t_i)^+ - R^1_t(M^t_i)^-$ where $R^1_t$ is the investor’s recovery rate.

It is assumed that processes $R^1$ and $R^2$ take values in $[0, 1]$. Note that $S^{i,1}_t = M^t_i$ ($S^{i,2}_t = M^t_i$, resp.) whenever $R^1_t(1)$ (or $R^2_t = 1$, resp.), that is, when the claimed value is paid in full by a defaulted party. We will later adjust Definition 2.4 for the presence of a collateral posted or received by the investor (see Definition 2.11).

### 2.2 Replacement Costs at Default

We will now attempt to formalize the crucial concept of a replacement cost of a contract in the event of counterparty’s default. The following definition conveys the general idea of a replacement cost.

**Definition 2.5.** The replacement cost is the cost of entering at time of counterparty’s default an ‘equivalent’ contract enabling the non-defaulting party to keep a ‘similar position’ in the market.

The notion of a ‘similar position’ in the market is ambiguous, however, and thus several alternative specifications of the replacement cost can be postulated. Essentially, we have in mind a contract that is as close as possible to the original one, including the counterparty risk of one or both parties. We argue that it is crucial to make a distinction between the convention specification of the contract’s settlement value at the time of default of a counterparty and its actual replacement cost. For instance, the settlement clause with the original counterparty could specify that in the event of default of the counterparty, if the MtM value of the corresponding risk-free contract is positive (i.e., when $M^t_i > 0$), the investor receives 50% of this value (i.e., $R^1_t = .5$). This does not mean that the replacement cost is already uniquely defined, since the computation of a replacement cost should refer to an ‘equivalent’ contract (or a ‘similar position’) and in this example the corresponding risk-free contract is manifestly not ‘equivalent’ to the original contract with a default-risky counterparty, even if the investor’s default risk is neglected.

**Definition 2.6.** By an equivalent bilateral contract at time $t$, we refer to the same contract in terms of promised future cash flows, but entered into at time $t$ by:

- a) two risk-free companies,
- or, by the non defaulting party with either:
  - a') a risk-free company, or
  - b) a company with the same rating at time $t$ as the defaulting party’s rating $c_0$ at the inception of the contract, or
  - c) a company with the same rating at time $t$ as the defaulting party’s rating $c_0$ just before default, or
  - c') by two companies: a company with the same creditworthiness at time $t$ as the non defaulting party just before default and a company with the same rating at time $t$ as the defaulting party’s rating just before default.

Let $P^{i,j}_t$ stand for the value of an equivalent contract, as seen by the investor, in the case of the counterparty’s default before the maturity date and prior to the investor’s default, with the index $j$. 

Correspondingly, the replacement cost is now defined as the difference between the settlement value of the original contract and the settlement value of the equivalent contract:

$R^i_t = S^{i}_t - P^{i}_t$
referring to a particular convention \( a, a', b, c \) or \( c' \) chosen by the investor. Similarly, we denote by \( P_t^{i,k} \) the price of an equivalent contract in the case of the investor’s default prior to the maturity date and before the counterparty’s default, where the convention \( k \) is chosen by the counterparty.

### 2.3 Loss and Gain Given Default

Loss and gain given default are aimed to quantify the effects of default on both counterparties. Note that \( i \) stands for the common convention specified in the contract, whereas \( j \) (resp. \( k \) ) denotes a particular convention chosen by the investor (resp. the counterparty) for the MtM value and the value of an equivalent contract.

**Definition 2.7.** The **loss given default** \( L_t^{i,j,1} \) represents the losses incurred by the non-defaulting investor at time \( t \) if the counterparty defaults at this date. It is given as the difference between the replacement cost and the settlement value of the contract at the moment of default, that is, \( L_t^{i,j,1} = P_t^{i,j} - S_t^{i,2} \). The **gain given default** \( G_t^{i,j,1} \) represents the gains incurred by the investor at time \( t \) if his default occurs at this date and the counterparty has not yet defaulted. It is thus given as the difference between the settlement value of the contract and the MtM value at the moment of default, that is, \( G_t^{i,j,1} = S_t^{i,1} - M_t^{i,1} \). For the counterparty, we find it convenient to set \( L_t^{i,k,2} = S_t^{i,1} - P_t^{i,k} \) and \( G_t^{i,k,2} = M_t^{k,2} - S_t^{i,2} \) so that all losses and gains are seen from the perspective of the investor.

It is worth stressing that we do not postulate in Definition 2.7 that the value of the loss given default (resp. the gain given default) is necessarily positive. Also, the loss and gain given default processes for the investor and the counterparty do not mirror each other, in general. This in turn implies that that the contract’s value is asymmetric, in general. In financial terms, this asymmetry is a natural consequence of potentially different contingency plans in the event of default on each side of the contract.

Let \( \tau^1 \) and \( \tau^2 \) stand for the random times of default of the investor and the counterparty, respectively. They are defined on an underlying probability space \((\Omega, \mathcal{G}, \mathbb{Q})\), where \( \mathbb{Q} \) is the risk-neutral probability measure. We denote by \( \mathbb{E}_\mathbb{Q} \) the expectation under \( \mathbb{Q} \) and by \( \mathcal{G}_t \) the \( \sigma \)-field of all events observed by time \( t \). It is assumed throughout that \( \tau^1 \) and \( \tau^2 \) are stopping times with respect to the filtration \( \mathcal{G} = (\mathcal{G}_t)_{t \geq 0} \) and \( \mathbb{Q}(\tau^1 = \tau^2) = 0 \). In order to introduce the right- and wrong-way risks, we need first to formally define the loss processes associated with a contract.

**Definition 2.8.** The **loss process** \( \mathcal{L}^{i,j,1} \) represents the investor’s loss given default, specifically, \( \mathcal{L}^{i,j,1}_t = 1_{t \geq \tau^2} L_t^{i,j,1} \). The **loss process** \( \mathcal{L}^{i,k,2} \) represents the counterparty’s loss given default, specifically, \( \mathcal{L}^{i,k,2}_t = 1_{t \geq \tau^1} L_t^{i,k,2} \).

The next definition introduces the general concepts of the right- and wrong-way risks. We will later illustrate them by means of particular examples. For conciseness, we skip the indices \( i, j, k \).

**Definition 2.9.** The **investor right-way (resp. wrong-way) risk** is the positive (resp. negative) dependence between the counterparty’s rating \( c^2 \) and the **positive loss process** \( \mathcal{L}^{1,+} = [\mathcal{L}^1]^+ \). The **counterparty’s right-way (resp. wrong-way) risk** is the positive (resp. negative) dependence between the investor’s rating \( c^1 \) and the **positive loss process** \( \mathcal{L}^{2,+} = [\mathcal{L}^2]^+ \).

### 2.4 Settlement Values with Margin Agreements

The use of collateral has become an essential risk mitigation technique in wholesale financial markets. Financial institutions extensively employ collateral in cash lending/borrowing activities (e.g., in repo transactions), in securities lending and borrowing, and in derivatives markets. In addition, central banks typically require collateral in their credit operations (see Liberti and Mian (2008)). The recent financial crisis has shed new light on the importance of collateral management and its influence on
the overall financial system liquidity. Rehypothecation especially is considered as a major ‘driver of contagion’ during the financial crisis. For a comprehensive review on the liquidity crisis, we refer to Adrian and Shin (2009), Allen and Carletti (2008), Brunnermeier (2009), Danielsson and Shin (2002), Ewerhart and Tapkin (2008), Gorton and Metrick (2009), and Singh and Aitken (2009).

2.4.1 Collateralized Exposures

We begin by introducing basic notions associated with the Credit Support Annex (CSA), which is a legal document regulating credit support (collateral) for derivatives contracts. A CSA defines specific clauses under which collaterals are posted or transferred between counterparties to mitigate the exposure to the counterparty risk.

To alleviate our notation, we skip in Section 2.4 the superscript $i$, and thus we write $M_t$ instead of $M_t^i$. The uncollateralized exposure for the investor (resp. the counterparty) is thus simply equal to $M_t$ (resp. $-M_t$).

We denote by $C_t^1$ (resp. $C_t^2$) the absolute value of the basket of collaterals posted at time $t$ by the investor (resp. by the counterparty). It is assumed throughout that they exclude one another; more specifically, the equalities $\mathbb{I}_{M_t \geq 0} C_t^1 = 0$ and $\mathbb{I}_{M_t \leq 0} C_t^2 = 0$ hold for every $t$. Hence, according to our notational convention, the total value of the collateral process $C$, as seen by the investor, equals $C_t = C_t^2 - C_t^1 = \mathbb{I}_{M_t > 0} C_t^2 - \mathbb{I}_{M_t < 0} C_t^1 = \mathbb{I}_{M_t \geq 0} C_t^2 - \mathbb{I}_{M_t < 0} C_t^1$.

**Definition 2.10.** The collateralized exposure for the investor equals

$$E_t^i = M_t - C_t = \mathbb{I}_{M_t \geq 0} (M_t - C_t^2) + \mathbb{I}_{M_t < 0} (M_t + C_t^1).$$

The collateralized exposure for the counterparty equals $E_t^c = -E_t^i$, so that

$$E_t^c = C_t - M_t = \mathbb{I}_{M_t \geq 0} (C_t^2 - M_t) + \mathbb{I}_{M_t < 0} (-C_t^1 - M_t).$$

Note that the collateralized exposures may be either positive or negative. They underpin the settlement values of a collateralized contract in the sense that Definition 2.4 now applies to $E_t^1$ or $E_t^2$ and it is combined with the collateral process $C$. Formally, we amend Definition 2.4 of settlement values as follows:

**Definition 2.11.** In the case of investor’s default at time $t$, the settlement value $S_t^1$ of a collateralized contract equals $S_t^1 = C_t + (E_t^1)^+ - R_t^1(E_t^1)^-$ or, equivalently,

$$S_t^1 = \mathbb{I}_{M_t \geq 0} \left( C_t^2 + (M_t - C_t^2)^+ - R_t^1 (M_t - C_t^2)^- \right) - \mathbb{I}_{M_t < 0} \left( C_t^1 + R_t^1 (M_t + C_t^1)^- - (M_t + C_t^1)^+ \right). \quad (1)$$

In the case of counterparty’s default at time $t$, the settlement value $S_t^2$ of a collateralized contract equals $S_t^2 = C_t + R_t^2(E_t^1)^+ - (E_t^1)^-$ or, equivalently,

$$S_t^2 = \mathbb{I}_{M_t \geq 0} \left( C_t^2 + R_t^2 (M_t - C_t^2)^+ - (M_t - C_t^2)^- \right) - \mathbb{I}_{M_t < 0} \left( C_t^1 + (M_t + C_t^1)^- - R_t^2 (M_t + C_t^1)^+ \right). \quad (2)$$

In the case of a contract that is fully collateralized at time $t$, we have that $C_t = M_t$ (i.e., $C_t^2 = M_t^+$ and $C_t^1 = M_t^-$) and thus collateralized exposures $E_t^1$ and $E_t^2$ vanish, so that $S_t^1 = S_t^2 = M_t$. The so-called under-collateralization may occur if either the value of the collaterals decreases or the MtM value of the contract increases, as seen by the net creditor, during the margin period of risk.

**Definition 2.12.** The investor is under-collateralized on the event $\{E_t^1 > 0, M_t > 0\}$. The counterparty is under-collateralized on the event $\{E_t^1 < 0, M_t < 0\}$. 

Collaterals mitigate risk for the receiver, but creates risk for the giver. In effect, collaterals are generally ‘pledged’, which means that the posting of collaterals corresponds to a ‘transfer of title’: the ownership of collaterals is transferred from the net debtor to the net creditor. This implies that, on the event of the net creditor’s bankruptcy, the net debtor becomes an unsecured creditor with regard the collaterals in excess over the contract exposure. This feature of a collateralized contract is the source of over-collateralization.

**Definition 2.13.** The investor is over-collateralized on the event \( \{ E^1_t > 0, M_t < 0 \} \). The counter-party is over-collateralized on the event \( \{ E^1_t < 0, M_t > 0 \} \).

Contracts are commonly over-collateralized because of the haircut. In effect, the haircut, a risk mitigation tool against under-collateralization to the benefit of the net creditor, creates over-collateralization risk for the net debtor, unless the collaterals are managed in a segregated account.

### 2.4.2 Lock-Up Margins

We will now complement our previous analysis by considering another potential contract clause, namely, the lock-up margin.

**Definition 2.14.** The lock-up margin amount \( U_t \) at time \( t \) is a pledged amount that is excluded from the exposure calculation before the moment \( \tau^1 \wedge \tau^2 \) of the first default, but it is accounted for in the settlement procedure.

Let \( U^1_t \) (resp. \( U^2_t \)) be the lock-up margin posted at time \( t \) by the investor (resp. by the counter-party). We denote by \( V_t = U^2_t - U^1_t \) the net value of lock-up margins, as seen by the investor.

**Lemma 2.1.** In the case of the investor’s default at time \( t \), the settlement value \( S^1_t \) of a collateralized contract with the net lock-up margin \( U_t \) is equal to

\[
S^1_t = \mathbb{1}_{M_t \geq 0} \left( C^2_t - V_t + \left( M_t - C^2_t + V_t \right)^+ - R^1_t \left( M_t - C^2_t + V_t \right)^- \right)
- \mathbb{1}_{M_t < 0} \left( C^1_t + V_t - \left( M_t + C^1_t + V_t \right)^+ + R^1_t \left( M_t + C^1_t + V_t \right)^- \right).
\]

In the case of the counterparty’s default at time \( t \), the settlement value \( S^2_t \) of a collateralized contract with the net lock-up margin \( U_t \) is given by

\[
S^2_t = \mathbb{1}_{M_t \geq 0} \left( C^2_t - V_t + R^2_t \left( M_t - C^2_t + V_t \right)^+ - \left( M_t - C^2_t + V_t \right)^- \right)
- \mathbb{1}_{M_t < 0} \left( C^1_t + V_t - R^2_t \left( M_t + C^1_t + V_t \right)^+ + \left( M_t + C^1_t + V_t \right)^- \right).
\]

**Proof.** We now set \( E^1_t = M_t - (C_t - V_t) \) and thus, since \( C_t = \mathbb{1}_{M_t \geq 0} C^2_t - \mathbb{1}_{M_t < 0} C^1_t \), we obtain

\[
E^1_t = \mathbb{1}_{M_t \geq 0} \left( M_t - C^2_t + V_t \right) + \mathbb{1}_{M_t < 0} \left( M_t + C^1_t + V_t \right)
\]

and

\[
S^1_t = C_t - V_t + \left( E^1_t \right)^+ - R^1_t \left( E^1_t \right)^-.
\]

This yields the asserted equality (3). Note that for \( R^1_t = 1 \), we have that \( S^1_t = M_t \), as expected. To derive (4), it suffices to combine (5) with the equality \( S^2_t = C_t - V_t + R^2_t \left( E^1_t \right)^+ - \left( E^1_t \right)^- \). Observe that for \( R^2_t = 1 \), we get \( S^2_t = M_t \).

### 2.4.3 Segregation of Collaterals

Finally, we will examine the settlement values when collaterals are segregated.

**Definition 2.15.** The holding of collateral in a risk-free account strictly separated from the net creditor’s own accounts is termed the segregation of collateral.
The goal of segregation is to mitigate the over-collateralization risk for the net debtor party, forasmuch as the third party is assumed to be risk-free. Notice that this assumption could be also be relaxed, by accounting for the probability of default and recovery rate of the third party custodian.

**Lemma 2.2.** In the case of investor’s default at time \( t \), the settlement value \( S_1^1 \) of a collateralized contract with segregation is given by the formula

\[
S_1^1 = \mathbb{1}_{M_t \geq 0} M_t - \mathbb{1}_{M_t < 0} \left( C_t^1 + R_t^1 (M_t + C_t^1)^- - (M_t + C_t^1)^+ \right).
\]

In the case of counterparty’s default at time \( t \), the settlement value \( S_2^2 \) of a collateralized contract is given by

\[
S_2^2 = \mathbb{1}_{M_t \geq 0} \left( C_t^2 + R_t^2 (M_t - C_t^2)^+ - (M_t - C_t^2)^- \right) + \mathbb{1}_{M_t < 0} M_t.
\]

**Proof.** We first observe that segregation is only relevant for the first term in the right-hand side of equality (3) (resp. (4)). Assume that there is over-collateralization at time \( t \) equals 1, and thus equality (1) becomes

\[
S_1^1 = \mathbb{1}_{M_t \geq 0} M_t - \mathbb{1}_{M_t < 0} \left( C_t^1 + V_t - (M_t + C_t^1 + V_t)^+ + R_t^1 (M_t + C_t^1 + V_t)^- \right).
\]

This establishes equality (6). To prove (7), it suffices to set \( R_t^2 = 1 \) in the second term in the right-hand side of (2).

### 2.4.4 Lock-Up Margins and Segregation of Collaterals

Finally, we derive the settlement values under segregation of collaterals with lock-up margins.

**Lemma 2.3.** The settlement value \( S_1^1 \) of a collateralized contract with lock-up margins and segregation, is given by the following expression

\[
S_1^1 = \mathbb{1}_{M_t \geq 0} M_t - \mathbb{1}_{M_t < 0} \left( C_t^1 + V_t - (M_t + C_t^1 + V_t)^+ + R_t^1 (M_t + C_t^1 + V_t)^- \right).
\]

The settlement value \( S_2^2 \) of a collateralized contract with lock-up margins and segregation is given by the formula

\[
S_2^2 = \mathbb{1}_{M_t \geq 0} \left( C_t^2 - V_t + R_t^2 (M_t - C_t^2)^+ - (M_t - C_t^2)^- \right) + \mathbb{1}_{M_t < 0} M_t.
\]

**Proof.** Equality (8) (resp. (9)) is established by substituting \( R_t^1 \) (resp. \( R_t^2 \)) by 1 in the first term of the right-hand side of equality (3) (resp. (4)).

### 2.5 Loss and Gain Given Default: Special Case

To reduce the variety of cases under study and to simplify the presentation of simulation algorithms, we will sometimes postulate that the settlement value and the replacement costs for the investor and the counterparty refer to the same convention so that \( i = j = k \). We thus deal with the case where \( P^* = P^* = M \). It is worth stressing that these equalities are also tacit assumptions made in papers that omit the replacement cost, only refer to the MtM value upon default and consider a unique convention for the MtM value. Under the present assumptions, the equalities \( L^{i,1} = G^{i,1} \) and \( L^{i,2} = G^{i,2} \) hold and thus the contract’s valuation becomes symmetric. Once again, to alleviate notation we drop the superscript \( i \) and we first consider an uncollateralized contract. Our goal is to show that loss processes can be computed explicitly and they follow positive processes.
Lemma 2.4. On the event \( \{ \tau^2 = t \leq T, \tau^1 > \tau^2 \} \) of counterparty’s default, the loss given default for the investor equals \( L^1_t = (1 - R^2_t) M^+ \). On the event \( \{ \tau^1 = t \leq T, \tau^2 > \tau^1 \} \) of investor’s default, the gain given default for the investor equals \( G^1_t = (1 - R^1_t) M^- \).

Proof. We substitute the equivalent contract price to the MtM value in Definition 2.4 of the settlement value and we represent the settlement value in terms of the equivalent contract price, so that

\[
S^2_t = R^2_t M^+_t - M^- = M_t - (1 - R^2_t) M^+_t.
\]  

(10)

From Definition 2.7, we obtain \( L^1_t = M_t - S^2_t \) and thus the stated formula follows. Similarly,

\[
S^1_t = M^+_t - R^1_t M^- = M_t + (1 - R^1_t) M^-,
\]

and the asserted formula now follows since \( G^1_t = S^1_t - M_t \).

\[ \square \]

In view of convention adopted in Definition 2.7, for the counterparty we obtain

\[
L^2_t = (1 - R^1_t) M^-,
\]

\[
G^2_t = (1 - R^2_t) M^+_t.
\]

In what follows, we focus on \( L^1 \) and \( G^1 \); the formulae for \( L^2 \) and \( G^2 \) are analogous. In the presence of a margin agreement, Lemma 2.4 can be extended as follows.

Lemma 2.5. On the event \( \{ \tau^2 = t \leq T, \tau^1 > \tau^2 \} \) of counterparty’s default, the loss given default for the investor equals

\[
L^1_t = (1 - R^2_t) \left( \mathbf{1}_{M_t \geq 0} (M_t - C^2_t)^+ + \mathbf{1}_{M_t < 0} (M_t + C^1_t)^+ \right).
\]

On the event \( \{ \tau^1 = t \leq T, \tau^2 > \tau^1 \} \) of investor’s default, the gain given default for the investor equals

\[
G^1_t = (1 - R^1_t) \left( \mathbf{1}_{M_t \geq 0} (M_t - C^2_t)^- + \mathbf{1}_{M_t < 0} (M_t + C^1_t)^- \right).
\]

Proof. The lemma follows by a direct application of formulae (1) and (2) to Definition 2.7. \[ \square \]

Finally under segregation of collaterals and lock-up margins, we obtain the following result.

Lemma 2.6. On the event \( \{ \tau^2 = t \leq T, \tau^1 > \tau^2 \} \) of counterparty’s default, the loss given default for the investor equals

\[
L^1_t = \mathbf{1}_{M_t \geq 0} (1 - R^2_t) (M_t - C^2_t + V_t)^+.
\]

On the event \( \{ \tau^1 = t \leq T, \tau^2 > \tau^1 \} \) of investor’s default, the gain given default for the investor equals

\[
G^1_t = \mathbf{1}_{M_t < 0} (1 - R^1_t) (M_t + C^1_t + V_t)^-.
\]

(12)

Proof. Using (9), we obtain

\[
L^1_t = M_t - S^2_t
\]

\[
= M_t - \mathbf{1}_{M_t \geq 0} \left( C^2_t - V_t + R^2_t (M_t - C^2_t + V_t)^+ - (M_t - C^2_t + V_t)^- \right) - \mathbf{1}_{M_t < 0} M_t
\]

\[
= \mathbf{1}_{M_t \geq 0} M_t - \mathbf{1}_{M_t \geq 0} \left( C^2_t - V_t + R^2_t (M_t - C^2_t + V_t)^+ - (M_t - C^2_t + V_t)^- \right)
\]

\[
= \mathbf{1}_{M_t \geq 0} \left( M_t - C^2_t + V_t - R^2_t (M_t - C^2_t + V_t)^+ + (M_t - C^2_t + V_t)^- \right)
\]

\[
= \mathbf{1}_{M_t \geq 0} (1 - R^2_t) (M_t - C^2_t + V_t)^+.
\]

Similar computations based on (8) yield (12). \[ \square \]
2.6 CVA for Bilateral Counterparty Risk

Credit Value Adjustment (CVA) is the correction that should be made to the price of the contract with no counterparty risk in order to account for the potential losses of the parties at the moment of default, be the default triggered by the investor or the counterparty. We denote by $\Pi(t, T)$ the discounted cash flows of a default-free contract over the time period $[t, T]$, as seen by the investor. Note that the cash flows at times $t$ and $T$ are included in $\Pi(t, T)$. For any time $u$ after the current date $t$ and before the contract’s maturity $T$ (that is, for any $u$ such that $t < u < T$), by a slight abuse of notation, we denote by $\Pi(t, u)$ the cash flows that occur over the time period $[t, u)$, that is, with the date $u$ excluded. We denote by $P_t = \mathbb{E}_Q(\Pi(t, T) \mid \mathcal{G}_t)$ the risk-neutral price at time $t$ of a contract with no counterparty risk and by $\hat{P}_t$ the risk-neutral price of the corresponding contract with bilateral counterparty risk under convention $i$ for settlement values. Hence, on the event $\{t \leq \tau^1 \wedge \tau^2\}$,

$$\hat{P}_t^i = \mathbb{E}_Q\left[ \mathbf{1}_{\tau^1 \wedge \tau^2 > T} \Pi(t, T) + \mathbf{1}_{\tau^1 \leq T, \tau^2 > \tau^1} \left( \Pi(t, \tau^1) + \frac{B_t}{B_{\tau^1}} S_{\tau^1}^{i,1} \right) + \mathbf{1}_{\tau^2 \leq T, \tau^1 > \tau^2} \left( \Pi(t, \tau^2) + \frac{B_t}{B_{\tau^2}} S_{\tau^2}^{i,2} \right) \mid \mathcal{G}_t \right].$$

**Definition 2.16.** The *credit value adjustment* at time $t$ under convention $i$ of a contract with bilateral counterparty risk is given by the equality $\text{CVA}_t^i = P_t - \hat{P}_t^i$.

To obtain a general representation for the CVA for a collateralized contract, we find it convenient to introduce the following concept.

**Definition 2.17.** The *bilateral first-default-free contract* is the contract in which the investor (resp. the counterparty) is protected against a loss due to the counterparty’s default (resp. the investor’s default) but not against the potential losses of the equivalent contract under convention $j$ (under convention $k$). Therefore, the discounted cash flows of the contract on the event $\{t \leq \tau^1 \wedge \tau^2\}$ are

$$\tilde{\Pi}^{j,k}_t(t, T) = \mathbf{1}_{\tau^1 \wedge \tau^2 > T} \Pi(t, T) + \mathbf{1}_{\tau^1 \leq T, \tau^2 > \tau^1} \left( \Pi(t, \tau^1) + \frac{B_t}{B_{\tau^1}} P^{*,k}_t \right) + \mathbf{1}_{\tau^2 \leq T, \tau^1 > \tau^2} \left( \Pi(t, \tau^2) + \frac{B_t}{B_{\tau^2}} P^{*,j}_t \right)$$

where $P^{*,j}$ and $P^{*,k}$ are replacement cost processes chosen by the investor and the counterparty. The risk-neutral price at time $t$ of the bilateral first-default-free contract is denoted by $\tilde{P}_t^{j,k}$.

Note that for $j = k = a$ we obtain $\tilde{P}_t^{i,a} = P_t$, as expected. The next result furnishes a generic representation for the CVA for a contract with bilateral counterparty risk.

**Proposition 2.1.** The following equality holds, for every $t \in [0, T]$ on the event $\{t \leq \tau^1 \wedge \tau^2\}$,

$$\tilde{P}_t^i = \tilde{P}_t^{i,k} + P_t(\mathcal{L}_t^{i,k,2}) - P_t(\mathcal{L}_t^{i,j,1}).$$

Consequently, the credit value adjustment for a contract with bilateral counterparty risk is given by

$$\text{CVA}_t^i = P_t - \tilde{P}_t^{i,k} + P_t(\mathcal{L}_t^{i,j,1}) - P_t(\mathcal{L}_t^{i,k,2}).$$

**Proof.** Recall that $S_{\tau^1}^{i,2} = P^{*,j}_t - L^{i,j,1}_t$ and $S_{\tau^1}^{i,1} = P^{*,k}_t + L^{i,k,2}_t$ (see Definition 2.7). Hence

$$\tilde{P}_t^i = \mathbb{E}_Q\left[ \mathbf{1}_{\tau^1 \wedge \tau^2 > T} \Pi(t, T) + \mathbf{1}_{\tau^1 \leq T, \tau^2 > \tau^1} \left( \Pi(t, \tau^1) + \frac{B_t}{B_{\tau^1}} (P^{*,k}_t + L^{i,k,2}_t) \right) + \mathbf{1}_{\tau^2 \leq T, \tau^1 > \tau^2} \left( \Pi(t, \tau^2) + \frac{B_t}{B_{\tau^2}} (P^{*,j}_t - L^{i,j,1}_t) \right) \mid \mathcal{G}_t \right]$$

$$= \tilde{P}_t^{i,k} + P_t(\mathcal{L}_t^{i,k,2}) - P_t(\mathcal{L}_t^{i,j,1})$$
and thus (13) now follows from Definition 2.16.

Proposition 2.1 shows that the price $\tilde{P}_i^t$ of a contract with bilateral counterparty risk under convention i) is equal to the difference of the price $\tilde{P}_i^{i,k}$ of the first-default-free contract under convention i) and the price of the investor’s loss $L^{i,j:1}$ due to the counterparty’s default before the contract termination $T$, augmented by the price of the counterparty’s loss $L^{i,k:2}$ due to the investor’s default before the contract termination $T$.

Observe that we do not follow here the usual claim in the existing literature (see, for instance, Brigo and Capponi (2009)), that the CVA under the uncollateralized full two-way payment rule is given by the equality $CVA_i = P_i(L^{1,+}) - P_i(L^{2,+})$, where $P_i(L^{1,+})$ is the price of the protection leg of a contingent CDS on the positive investor’s loss and $P_i(L^{2,+})$ is the price of the protection leg of a contingent CDS on the positive counterparty’s loss. One can check, however, that formula (14) is indeed consistent with this equality, provided that it is postulated that $i = j = k = a$, that is, when the settlement value and losses given default are specified with respect to the default-free contract.

Let us assume, more generally, that $j = k = i$. Then $P_{i-j} = P_{i-k} = M^i$ and from Section 2.6 we know that the loss processes $L^{i,1}$ and $L^{i,2}$ are positive. Therefore, we may represent (14) as follows

$$CVA_i^j = P_i - \tilde{P}_i^j + P_i(L^{i,1,+}) - P_i(L^{i,2,+}).$$

Hence if one applies the formula $CVA_i^j = P_i(L^{i,1,+}) - P_i(L^{i,2,+})$ to compute the CVA, one implicitly assumes that the equality $P_i = \tilde{P}_i^j$ holds, which is inappropriate assumption under convention i), in general. In fact, except for convention a), the price of the contract with counterparty risk under convention i) is a function of the price of the equivalent contract under convention i), which itself is a function of an ‘equivalent’ contract under the same convention, and so on. Put another way, the loss at time of the counterparty’s default is a function of the potential loss of the subsequent equivalent contract, which is itself a function of the loss of the ‘equivalent’ contract for the first ‘equivalent’ contract, and so on. This convolution property of pricing under convention i) gives rise to a serious challenge, and it explains why it is by far more common to adopt the simplified convention $a)$. Indeed, under convention $a)$ the convolution property disappears and thus one can easily identify the CVA as the value of the protection leg of the CDS contingent claim written on the loss process of the investor.

### 2.6.1 Pricing under Rank $n$ Convention

Practical arguments justify the assumption that only a limited number of potential ‘equivalent’ counterparties exist for a given contract. Also, the required simulations under conventions $a’), b), c)$ and $c’) make it necessary to take a view on the value of the $n$th equivalent contract for some $n \geq 1$, where by the $n$th equivalent contract we mean an ‘equivalent’ contract used to replace the $(n-1)$th ‘equivalent’ contract after the default of the $(n-1)$th counterparty. Using this concept, we may introduce the notion of the rank $n$ convention. It can be considered as either a computational approximation or a realistic feature of the market reflecting the fact that the number of potential equivalent counterparties is limited.

**Definition 2.18.** For any convention i) and for any natural number $n \geq 1$, the price of a contract with bilateral counterparty risk under rank $n$ convention is equal to the price computed under the assumption that the $n$th equivalent contract is either given by some predetermined value or by a risk-free contract.

Of course, the definition above is only aimed to convey the general idea of the rank $n$ convention. For explicit computations, we can use one of the following alternative definitions, whichever is found to be practically appealing and/or suitable for an efficient numerical procedure.

**Definition 2.19.** For any convention i and for any natural number $n \geq 1$, the price of a contract with bilateral counterparty risk under investor rank $n$ (resp. counterparty rank $n$) convention is equal...
to the price derived under the assumption that the \( n \)th equivalent counterparty (resp. investor) contract is either given by some predetermined value or by a risk-free contract.

**Definition 2.20.** For any convention \( i \) and for any natural number \( n \geq 1 \), the price of a contract with bilateral counterparty risk under risk-free rank \( n \) convention is equal to the price that is derived under the assumption that the \( n \)th equivalent contract equals the risk-free contract.

**Definition 2.21.** For any convention \( i \) and for any natural number \( n \geq 1 \), the price of a contract with bilateral counterparty risk under zero rank \( n \) convention is equal to the price that is derived under the assumption that the \( n \)th equivalent contract has value zero.

Recall that \( S_1^i \) and \( S_2^i \) are functions of the mark-to-market value \( M_i^1 \) and recovery processes \( R_i^1 \), so that we can write \( S_1^i = S_1^i(M_i^1) \) and \( S_2^i = S_2^i(M_i^1) \) (see Definition 2.4).

**Definition 2.22.** The price of a contract with the bilateral counterparty risk under convention \( i \) and under rank \( n \) equals \( \hat{P}_{i}^{i,n} = \mathbb{E}_Q[\Pi^{i,n}(t,T) | \mathcal{G}_t] \), for any \( n \geq 1 \) and \( i = a, a', b, c, c' \), where the discounted cash flows \( \Pi^{i,n}(t,T) \) are given by

\[
\Pi^{i,n}(t,T) = \mathbb{1}_{\tau^1 \land \tau^2 > T} \Pi(t,T) + \mathbb{1}_{\tau^1 \leq T, \tau^2 > \tau^1} \left[ \Pi(t,\tau^1) + \frac{B_{\tau^1}}{B_{\tau^2}} S_2^i(M_{\tau^1}) \right] \\
+ \mathbb{1}_{\tau^2 \leq T, \tau^1 > \tau^2} \left[ \Pi(t,\tau^2) + \frac{B_{\tau^2}}{B_{\tau^1}} S_2^i(M_{\tau^2}) \right]
\]

where in turn the quantities \( M_{\tau^1}^{i,n} \) and \( M_{\tau^2}^{i,n} \) are computed under the assumption that at most \( n \) equivalent contracts can default before maturity \( T \) and the value of the \( n \)th equivalent contract is either a predefined value or the value of an equivalent risk-free contract.

Note that the price of a contract with bilateral counterparty risk under any convention \( i \) for \( i = b, c, c' \), and under risk-free rank 1 is simply equal to the price of the contract under convention \( a \).

### 3 CVA for Credit Default Swaps

In this section, we propose a general procedure for computation of the credit value adjustment. For concreteness, we specify our study to the case of a credit default swap and we postulate that the credit qualities of the investor, the counterparty, and the reference entity are identical and depend on the level of systemic risk. It should be made clear that the generalization of the approach to the situation where the credit qualities are not equal is rather straightforward, since it suffices to modify the pricing engine accordingly. The contract under consideration may be seen, ahead of the 2007 credit crunch, as a stylized version of a credit default swap between Goldman Sachs, the buyer of protection on Lehman Brothers, and AIG, the protection seller. The possibility of a systemic crisis is accounted for in the pricing, by considering not only the systemic impact of the potential defaults of the latter companies, but also of other systemic companies, such as, for instance, Merrill Lynch, Bear Stearns, or Northern Rock. Consequently, this model prices the possibility of a scenario similar to the wrong-way risk situation that was experienced by Goldman Sachs during the financial crisis, namely, the bankruptcy of AIG at the time when Lehman Brothers, following on the path of major financial institutions such as Bear Stearns, was on the brink of default. Recall that, as it is much documented today, AIG stopped meeting its contractual obligations when, as the creditworthiness of Lehman Brothers kept on deteriorating, AIG could not meet the margin calls on its credit default swaps, on Lehman Brothers, especially. Obviously, AIG was ill prepared to a manage an abrupt increase in liquidity requirements.

#### 3.1 Risk-Free Credit Default Swap

Let \((\Omega, \mathcal{G}, \mathbb{Q})\) be a probability space endowed with a filtration \( \mathbb{G} = (\mathbb{G}_t)_{0 \leq t \leq T} \), where \( \mathbb{Q} \) represents the risk-neutral probability measure and \( T \) stands for the horizon date of trading activities. We
denote by \( r \) the short-term rate process and by \( B \) the savings account process. The next lemma is an immediate consequence of the definition of a stylized credit default swap (CDS).

**Lemma 3.1.** The discounted cash flows \( \Pi(t, T) \) as seen by the buyer at time \( t \) of a risk-free CDS of maturity \( T \) and notional amount \( 1 \) are

\[
\Pi(t, T) = 1_{t \leq \tau \leq T} \frac{B_t}{B_T} (1 - R_\tau) - \kappa \int_t^{\tau \wedge T} \frac{B_u}{B_T} du
\]

where \( \tau \) is the default time of the reference entity, \( R_\tau \) is the recovery rate, that is, the rate of recovery at time \( \tau \) on the assets of the reference entity after a credit event and \( \kappa \) is the fixed CDS spread.

For simplicity, we assume henceforth that the recovery rate \( R \) is constant and the interest rate \( r \) is null. By taking the conditional expectations under \( Q \), we obtain the following representation for the price of the risk-free CDS, on the event \( \{ \tau > t \} \),

\[
P_t = (1 - R) Q[\tau \leq T \mid \mathcal{G}_t] - \kappa E_Q[\tau \wedge T - t \mid \mathcal{G}_t].
\]

### 3.2 Contagion Effects

Contagion effects, that is, changes in the likelihood of default of surviving names, can be studied either at the level of individual credit names or for large portfolios.

#### 3.2.1 Contagion Effect at First Default

We first introduce the concept of *contagion effect at first default*.

**Definition 3.1.** If the default of a party tends to increase (resp. decrease) the probability of default of the other, we say that there is a negative (resp. positive) contagion effect at first default.

It is worth noting that this is precisely the contagion effect at first default which justifies the distinction between conventions \( c \) and \( c' \).

#### 3.2.2 Systemic Contagion

As a consequence of the recent financial crisis, the systemic risk, which is commonly interpreted as the risk of cascading defaults following the bankruptcy of a major player within the economical system, has been given increased scrutiny among scholars and practitioners alike. Some studies focus on details of the mechanism which has unfolded (see Adrian and Shin (2009), Allen and Carletti (2008), Brunnermeier (2009), Danielsson and Shin (2002), Ewerhart and Tapking (2008), Gorton and Metrick (2009), and Singh and Aitken (2009)). Other papers shed new perspectives on how to model complex relationships among players (see, for instance, Cont *et al.* (2009) and Duffie and Zhu (2009)).

**Definition 3.2.** The systemic contagion risk is the risk of a ‘collapse’ of the entire market (or its sector) resulting from the counterparty risk contagion.

The systemic contagion effect is frequently modeled by means of a Markov chain, which represents the successive defaults of any systemic company within a systemic basket of companies whose defaults increase the systemic risk. Typically, these systemic companies may be considered as leading financial institutions such as banks or trading exchanges. One may also consider, in lieu of a systemic company’s default, any event with systemic effects, such as a disruption within the repo market, a state credit event, and so on.

As in Jarrow and Yu (2001) and Laurent *et al.* (2008), we find it natural and convenient to tie the credit qualities of surviving obligors to the number of defaults \( N_t \) of the reference entities of the systemic basket. We underline this dependency by amending our notation: the credit qualities of the obligors in the presence of systemic risk are denoted by \( c_j^T(N_t) \) for \( j = 1, 2 \).
3.3 Time-Homogeneous Markovian Contagion Model

We postulate hereafter that the joint dynamics of defaults for a systemic portfolio of \( m \) firms, the investor, the counterparty, and the reference entity are represented by a continuous-time Markov chain \( \mathcal{M} = (\mathcal{M}_t)_{t \in [0,T]} \) (not to be confused with the MIM value \( M_t \)) with the state space

\[
E = \{(j_0, 0, 0, 0), (j_0, 1, 0, 0), (j_0, 0, 1, 0), (j_0, 0, 0, 1), (j_0, 1, 1, 0),
(j_0, 1, 0, 1), (j_0, 0, 1, 1), (j_0, 1, 1, 1), j_0 = 0, 1, \ldots, m\}.
\]

A generic state \( (j_0, j_1, j_2, j_3) \) of the Markov chain \( \mathcal{M} \) has the following interpretation:
(a) \( j_0 \) is the number of defaults in systemic portfolio,
(b) \( j_1 \) is the default indicator for the investor,
(c) \( j_2 \) is the default indicator for the counterparty,
(d) \( j_3 \) is the default indicator for the reference entity.

Suppose that we have specified the intensities of transitions between states and thus we are given the matrix of transition intensities, denoted as \( \mathcal{A}_\mathcal{M} \). We are then in a position to formally define the continuous-time Markov process \( M \) with the state space \( E \) and the infinitesimal generator \( \mathcal{A}_\mathcal{M} \). Note that \( \mathcal{M} \) admits the following representation \( \mathcal{M}_t = (N_t, \hat{H}_1^t, \hat{H}_2^t, \hat{H}_3^t) \), where the process \( N \) represents the number of systemic defaults and takes values in \( \{0, 1, \ldots, m\} \), whereas the default indicator processes \( \hat{H}_j, j = 1, 2, 3 \) take values in \( \{0, 1\} \). We will sometimes assume that \( M_0 = (0, 0, 0, 0) \), although, obviously, the choice of an initial value for the Markov chain \( \mathcal{M} \) is arbitrary.

Next, we specify the default times by setting \( \tau^i = \inf \{ t \in \mathbb{R}_+ : \hat{H}_i^t = 1 \} \). Let us stress that \( \tau^3 = \tau \) is the default time of the reference entity. Finally, we define the default counting process \( N^*_t = N_t + \hat{H}_1^t + \hat{H}_2^t + \hat{H}_3^t \). It is clear that \( N^*_t \) represents the total number of defaults in the model up to time \( t \) and a generic value \( l \) of the process \( N^* \) satisfies \( l = j_0 + j_1 + j_2 + j_3 \).

We start by listing the desired properties of the model. Note that all intensities are assumed to be specified under the risk-neutral probability measure \( Q \).

(M.1) Following Laurent et al. (2008), we postulate that the default intensity of a systemic company is the same for every entity within the systemic basket and it is a function of time and the total number of defaults (including the defaults of the investor, the counterparty, and the reference entity).

(M.2) The systemic portfolio is assumed to be time-homogeneous and the reference name can be seen as an additional name in the systemic portfolio. The reference entity is formally isolated, however, since we need the exact knowledge of the default time of this name. By contrast, for the original systemic portfolio of \( m \) entities, it is sufficient for our purposes to know the number of defaults that occurred by time \( t \) in this portfolio; the identities of defaulting entities are not relevant.

(M.3) The default intensities of the investor and the counterparty depend on time parameter \( t \) and the total number of defaults (including the default of the investor and the counterparty).

In order to specify the generator of \( \mathcal{M} \), we introduce the following auxiliary quantities:
(i) \( \gamma(t, j_0) \) representing the intensity of a new default in the systemic portfolio of \( m + 1 \) entities (including the reference entity), given that the total number of defaults that occurred by time \( t \) equals \( j_0 \) (that is, on the event \( \{ N^*_t = j_0 \} \)),
(ii) \( \tilde{\gamma}_1(t, j_0) \) (resp. \( \tilde{\gamma}_2(t, j_0) \)), which represents the intensity of default of the investor (resp. the counterparty) given that the total number of defaults that occurred by time \( t \) equals \( j_0 \) (that is, on the event \( \{ N^*_t = j_0 \} \)).

**Definition 3.3.** By the time-inhomogeneous Markovian contagion model we mean the Markov chain \( \mathcal{M} \) on the state space \( E \) with the generator \( \mathcal{A}_\mathcal{M} \) specified by the transition intensities satisfying, for every \( j_0 = 0, \ldots, m \),

\[
\lambda((j_0, 0, 0, 0), (j_0 + 1, 0, 0, 0)) = a_{j_0} \gamma(t, j_0),
\lambda((j_0, 0, 0, 0), (j_0, 1, 0, 0)) = \tilde{\gamma}_1(t, j_0),
\lambda((j_0, 0, 0, 0), (j_0, 0, 1, 0)) = \tilde{\gamma}_2(t, j_0),
\]
Let us note that for the Markov chain

\[ \lambda((j_0, 0, 0), (j_0, 0, 0, 1)) = b_{j_0} \gamma(t, j_0), \]

\[ \lambda((j_0, 1, 0, 0), (j_0 + 1, 0, 0, 1)) = a_{j_0} \gamma(t, j_0 + 1), \]

\[ \lambda((j_0, 1, 0, 0), (j_0, 1, 1, 0)) = \gamma_2(t, j_0 + 1), \]

\[ \lambda((j_0, 0, 1, 0), (j_0, 0, 1, 0, 1)) = b_{j_0} \gamma(t, j_0 + 1), \]

\[ \lambda((j_0, 0, 0, 1), (j_0 + 1, 0, 0, 1)) = a_{j_0} \gamma(t, j_0 + 2), \]

\[ \lambda((j_0, 1, 1, 0), (j_0, 1, 1, 1)) = b_{j_0} \gamma(t, j_0 + 2), \]

\[ \lambda((j_0, 1, 0, 1), (j_0 + 1, 0, 1, 1)) = \gamma(t, j_0 + 2), \]

\[ \lambda((j_0, 0, 1, 1), (j_0, 1, 1, 1)) = \gamma_2(t, j_0 + 2), \]

\[ \lambda((j_0, 1, 1, 1), (j_0 + 1, 1, 1, 1)) = \gamma(t, j_0 + 3), \]

where

\[ a_{j_0} = \frac{m - j_0}{m + 1 - j_0}, \quad b_{j_0} = \frac{1}{m + 1 - j_0}. \]

In what follows, the filtration \( \mathcal{G} \) is assumed to be the natural filtration of the Markov chain \( \mathcal{M} \). Let us note that for the Markov chain \( \mathcal{M} \) satisfying Definition 3.3, the process \( N_t = N_t + \hat{N}_t^t \) does not follow a Markov chain, in general, unless the equalities \( \hat{\gamma}_1(t, j_0) = \hat{\gamma}_2(t, j_0) = 0 \) hold. By the same token, the default counting process \( \lambda^* \) is not necessarily a Markov chain. We have, however, the following result, which corresponds to the case when the investor and the counterparty can be seen as belonging to the systemic portfolio. Observe, in particular, that equality (17) means that the intensity of a new default in the original systemic portfolio (including the reference name) is proportional to the number of surviving names in this portfolio. Note also that a number \( l \in \{0, 1, \ldots, m + 3\} \) represents the total number of defaults, that is, a generic value of the process \( \lambda^* \).

Recall that for a generic state \((j_0, j_1, j_2, j_3)\) of the Markov chain \( \mathcal{M} \) we have that \( l = j_0 + j_1 + j_2 + j_3 \).

**Lemma 3.2.** (i) Let \( \hat{\gamma}(t, l), t \in [0, T], l = 0, 1, \ldots, m + 2 \) be a strictly positive function. Assume that, for every \( l = 0, 1, \ldots, m + 2 \),

\[ \hat{\gamma}_1(t, l) = \hat{\gamma}_2(t, l) = \hat{\gamma}(t, l) \]

and the equality

\[ \lambda((j_0, j_1, j_2, j_3), (j_0 + 1, j_1, j_2, j_3)) = (m - j_0) \hat{\gamma}(t, l) \]  

(17)

holds for \( j_0 = 0, 1, \ldots, m - 1 \) and \( j_1, j_2 = 0, 1 \), where \( l = j_0 + j_1 + j_2 + 1 \). Then the default counting process \( \lambda^* \) is a time-inhomogeneous Markov chain with state space \( E^* = \{0, 1, \ldots, m + 3\} \) and the generator

\[
\mathcal{A}_{\lambda^*} = \begin{bmatrix}
-\lambda^*(t, 0) & \lambda^*(t, 0) & 0 & \ldots & 0 & 0 \\
0 & -\lambda^*(t, 1) & \lambda^*(t, 1) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \lambda^*(t, m + 1) & 0 \\
0 & 0 & 0 & \ldots & -\lambda^*(t, m + 2) & \lambda^*(t, m + 2) \\
0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

where \( \lambda^*(t, l) = (m + 3 - l) \hat{\gamma}(t, l) \).
(ii) In particular, if \( \tilde{\gamma}(t,l) = \gamma(l) \) for some strictly positive function \( \tilde{\gamma} \), then the default counting process \( N^* \) is a time-homogeneous Markov chain with state space \( E^* = \{0,1,\ldots,m+3\} \) and the following generator

\[
A_{N^*} = \begin{bmatrix}
-(m+3)\tilde{\gamma}(0) & (m+3)\tilde{\gamma}(0) & 0 & \ldots & 0 & 0 \\
0 & -(m+2)\tilde{\gamma}(1) & (m+2)\tilde{\gamma}(1) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2\tilde{\gamma}(m+1) & 0 \\
0 & 0 & 0 & \ldots & -\tilde{\gamma}(m+2) & \tilde{\gamma}(m+2) \\
0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}.
\]

Proof. We will first show that assumption (17), combined with Definition 3.3, uniquely specifies the intensities \( \lambda((j_0,j_1,j_2,0),(j_0+1,j_1,j_2,0)) \) and \( \lambda((j_0,j_1,j_2,0),(j_0+1,j_1,j_2,1)) \). To this end, we note that equality (17) and Definition 3.3 yield

\[
\lambda((j_0,j_1,j_2,1),(j_0+1,j_1,j_2,1)) = (m-j_0)\tilde{\gamma}(t,l) = \gamma(t,l)
\]

where \( l = j_0 + j_1 + j_2 + 1 \). Using Definition 3.3, we get

\[
\lambda((j_0+1,j_1,j_2,0),(j_0+2,j_1,j_2,0)) = a_{j_0+1} \gamma(t,l) = \frac{m-j_0-1}{m-j_0} (m-j_0)\tilde{\gamma}(t,l) = (m-j_0-1)\tilde{\gamma}(t,l).
\]

Similarly, by applying once again Definition 3.3, we also obtain

\[
\lambda((j_0+1,j_1,j_2,0),(j_0+1,j_1,j_2,1)) = b_{j_0+1} \gamma(t,l) = \frac{1}{m+1-j_0} \gamma(t,l) = \tilde{\gamma}(t,l).
\]

It is now clear that under the assumptions of part (i), the choice of a non-negative function \( \tilde{\gamma} \) fully specifies the generator of a time-inhomogeneous Markov contagion model \( M \) of Definition 3.3.

We also note that in that case the intensity of default depends only on the total number of defaults up to time \( t \) and the size of sub-portfolio. Hence it is rather obvious that the process \( N^* \) has the Markov property and its generator \( A_{N^*} \) is represented by the matrix given in the statement of the lemma. Part (ii) is an immediate consequence of part (i). \( \square \)

It is obvious that we can easily compute the default counting process \( N^* \) from the Markov chain \( M \). The converse does not hold, however, that is, the process \( M \) cannot be recovered from \( N^* \), in general. This is intuitively clear, since \( N^* \) does not convey any information about the identities of defaulted names. By contrast, the Markov chain \( M \) specifies not only the total number of defaulted names, but also the default status of the investor, the counterparty, and the reference entity.

Let us now comment briefly on the most pertinent features of the model which is examined in detail in the next section, i.e., the time-homogeneous Markovian contagion model put forward under the hypothesis of Lemma 3.2 (ii). This model allows for the examination of not only systemic and at first default contagions, but also of right- or wrong-way risks. As a matter of fact, on the event of the default of an obligor, the systemic risk increases (systemic contagion), which automatically impacts the creditworthiness of the non-defaulting obligor (at first default contagion) as well as the creditworthiness of the reference entity of the credit default swap. Hence the value of the credit default swap jumps on the event of a party’s default (right- or wrong-way risks). It is important to notice that the at first default contagion effect cannot be accounted for under convention a), but only under other conventions introduced in Section 2. Also, given that all of the companies bear systemic risk, it makes practical sense to assume that the intensities of default of the non-defaulting companies increase on the event of a default. Consequently, \( \tilde{\gamma}(l) \) is chosen to be an increasing function of \( l \). This choice corresponds to the case of a wrong-way risk with a negative at first default contagion effect for the investor, which is consistent with our illustrative case study of the CDS between Goldman Sachs, the buyer of protection on Lehman Brothers, and AIG, the seller of protection.
3.4 CVA Computation in a Markovian Contagion Model

In what follows, we work within the setup of the time-homogeneous Markovian contagion model of Lemma 3.2 (ii). From the Markov property of $\mathcal{M}$ and formula (16), it is easy to deduce that the pre-default price of the risk-free credit default swap (that is, the price on the event $\{\tau^1 \land \tau^2 \land \tau > t\}$) can be represented as a function of $N_t$. Specifically, equation (16) becomes, on the event $\{\tau^1 \land \tau^2 \land \tau > t\}$,

$$P_t = P_t(N_t) = (1 - R) Q[\tau \leq T | N_t] - \kappa E_Q[\tau \land T - t | N_t].$$

We will now derive a convenient expression for the CVA in a general Markovian contagion model. For the ease of explanation and conciseness, we assume henceforth that the settlement value, as seen by the investor on the event of his default, is null (that is, $S^i_t = 0$). The following result applies under any counterparty rank $n$ convention (see Section 2.6.1) under the assumption that $M^i = P^i_t$. More specifically, given the value $N^*_t$ of the default counting process, the price at time $t$ of the counterparty equivalent contract is now denoted as either $P^i_t(N^*_t)$ or $P^{i,\Pi}_t(N^*_t)$. Similarly, $P_t(N^*_t)$ is the price of default-free contract, given the value of $N^*_t$. As usual, $\tau = \tau^3$ is the default time of the reference entity. Note that on the event $\{\tau^1 \land \tau^2 \land \tau > t\}$ the equality $N^*_t = N_t$ holds.

**Proposition 3.1.** The CVA of a credit default swap under convention $i$ for $i = a, a', b, c, c'$ equals, for every $t < T$ on the event $\{\tau^1 \land \tau^2 \land \tau > t\}$,

$$CVA^i_t = \int_t^T \left( \sum_{v=0}^{m-N_t} P_u(N_t + v + 1) Q_{B^1_u}[N_u = N_t + v | N_t] \right. \right.$$  

$$\left. + \sum_{v=0}^{m-N_t} P_u(N_t + v + 1) Q_{B^2_u}[N_u = N_t + v | N_t] \right. \right.$$  

$$\left. \left[ S^2_{v} (P^{r, i}_u(N_t + v + 1)) \right] Q_{B^2_u}[N_u = N_t + v | N_t] \right) dQ[\tau^1 \leq u, \tau^2 \land \tau > \tau^1 | N_t]$$

where $B^1_u = \{\tau^1 = u, \tau^2 \land \tau > \tau^1\}$ and $B^2_u = \{\tau^2 = u, \tau^1 \land \tau > \tau^2\}$ are two families of events from $\mathcal{G}_T$ indexed by the date $u \in [t, T]$.

**Proof.** Let us first explain the notation, which is used in the statement and the proof of the proposition. For any event $B \in \mathcal{G}_T$ and any $Q$-integrable random variable $X$, we define the conditional expectation $E_{Q_B}[X | N_t]$, where the probability measure $Q_B$ is defined as follows

$$Q_B(D) = \frac{Q(D \cap B)}{Q(B)}, \quad \forall D \in \mathcal{G}_T,$$

that is, $Q_B$ is the conditional probability measure on $(\Omega, \mathcal{G}_T)$ given the event $B$. It is known that the following version of the Bayes formula holds (see, e.g., Lemma A.3 in Collin-Dufresne et al. [6])

$$E_Q[I_{B} X | N_t] = Q[B | N_t] E_{Q_B}[X | N_t].$$

Let us now fix some $t < T$. We start the derivation of formula (18) by observing that, under convention $i$ and the assumption that the settlement value on the event of the investor’s default is null, the cash flows of a general bilateral contract are

$$\Pi^i(t, T) = \Pi_t^{\tau^1 \land \tau^2 > T} \Pi(t, T) + \Pi_{\tau^1 \leq T, \tau^2 > \tau^1} \Pi(t, \tau^1) + \Pi_{\tau^2 \leq T, \tau^1 > \tau^2} [\Pi(t, \tau^2) + S^2_{\tau^2}].$$

$$= \Pi(t, T) - \Pi_{\tau^1 \leq T, \tau^2 > \tau^1} \Pi(\tau^1, T) - \Pi_{\tau^2 \leq T, \tau^1 > \tau^2} \Pi(\tau^2, T) + \Pi_{\tau^2 \leq T, \tau^1 > \tau^2} S^2_{\tau^2},$$

where the settlement value $S^2_{\tau^2} = S^2_{\tau^2}(P^{r, i}_t(N^*_t))$ is given by formula (10). Note that

$$\Pi(\tau^2, T) = \Pi_{\tau^2 \leq \tau} \Pi(\tau^2, T) + \Pi_{\tau^2 > \tau} \Pi(\tau^2, T) = \Pi_{\tau^2 \leq \tau} \Pi(\tau^2, T),$$
since the second term is null; similarly, \( \Pi(\tau^1, T) = 1_{\tau^1 < \tau} \Pi(\tau^1, T) \). We thus obtain
\[
\Pi^i(t, T) = \Pi(t, T) - 1_{\tau^1 \leq T, \tau^2 \land \tau > \tau^1} \Pi(\tau^1, T) - 1_{\tau^2 \leq T, \tau^1 \land \tau > \tau^2} \Pi(\tau^2, T)
\]
\[+ 1_{\tau^2 \leq T, \tau^1 \land \tau > \tau^2} S_{2\tau}^2.\]

By taking the conditional expectation under the risk-neutral probability measure \( Q \), we obtain, on the event \( \{\tau^1 \land \tau^2 \land \tau > t\} \),
\[
\hat{P}_i^t = E_Q[\Pi^i(t, T) | N_t] = \hat{P}_i^t(N_t).
\]
More explicitly,
\[
\hat{P}_i^t(N_t) = E_Q[\Pi(t, T) | N_t] - E_Q[1_{t < \tau^1 \leq T, \tau^2 \land \tau > \tau^1} \Pi(\tau^1, T) | N_t] - E_Q[1_{t < \tau^2 \leq T, \tau^1 \land \tau > \tau^2} \Pi(\tau^2, T) | N_t]
\]
\[+ E_Q[1_{t < \tau^2 \leq T, \tau^1 \land \tau > \tau^2} S_{2\tau}^2 | N_t].\]

Using the law of total probability and formula (20), we can represent \( \hat{P}_i^t(N_t) \) as follows
\[
\hat{P}_i^t(N_t) = P_i(N_t) - \int_t^T E_{Q_{B_u}}[\Pi(u, T) | N_t] dQ[\tau^1 \leq u, \tau^2 \land \tau > \tau^1 | N_t]
\]
\[+ \int_t^T E_{Q_{B_u}}[\Pi(u, T) | N_t] dQ[\tau^2 \leq u, \tau^1 \land \tau > \tau^2 | N_t]
\]
\[+ \int_t^T E_{Q_{B_u}}[S_{2\tau}^2(P_u^*(N_u^*)) | N_t] dQ[\tau^2 \leq u, \tau^1 \land \tau > \tau^2 | N_t].\]

To establish the desired formula, it remains to observe that the conditional expectations
\[
E_{Q_{B_u}}[\Pi(u, T) | N_u = N_t + v, B_u^1, N_t], \quad E_{Q_{B_u}}[\Pi(u, T) | N_u = N_t + v, \tau^1 \land \tau > u]
\]
only depend upon the number \( N_t \) of defaults and they can be represented more explicitly. For instance, for every \( u \in [t, T] \), the first conditional expectation can be computed by conditioning on the number of defaults in the systemic portfolio (excluding, of course, the reference name).

To be more specific, using (20) and the equality
\[
E_Q[\Pi(u, T) | N_u = N_t + v, B_u^1, N_t] = E_Q[\Pi(u, T) | N_u^* = N_t + v + 1, \tau^1 \land \tau > u]
\]
\[= P_u(N_u^*) = P_u(N_t + v + 1),\]
we obtain
\[
E_{Q_{B_u}}[\Pi(u, T) | N_t] = \frac{E_Q[1_{B_u^1} \Pi(u, T) | N_t]}{Q[B_u^1 | N_t]} = \frac{1}{Q[B_u^1 | N_t]} \sum_{v=0}^{m-N_t} E_Q[1_{N_u=N_t+v}1_{B_u^1} \Pi(u, T) | N_t]
\]
\[= \frac{1}{Q[B_u^1 | N_t]} \sum_{v=0}^{m-N_t} E_Q[\Pi(u, T) | N_u = N_t + v, B_u^1, N_t] Q[N_u = N_t + v, B_u^1 | N_t]
\]
\[= \sum_{v=0}^{m-N_t} P_u(N_t + v + 1) Q_{B_u^1}[N_u = N_t + v | N_t]
\]
and analogous representations for the other two conditional expectations. To complete the proof, it thus suffices to use the equality \( CVA_i^t = P_i - \hat{P}_i^t \) (see Definition 2.16).

It is worth noting that for every \( u \in [t, T] \) the quantity \( P_u(N_u^*) \) is a deterministic function of \( N_t^* \), and thus the random variable \( P_u(N_t + v + 1) \) is measurable with respect to \( \sigma(N_t) \).

Our next goal is to establish a convenient approximation for the integral in the right-hand side of formula (18). To this end, we introduce a tenor of dates \( t_0 = 0 < t_1 < \cdots < t_M = T \) and we formally postpone a default from any date \( u \in [t_j, t_{j+1}] \) to the date \( t_j \). This leads to the following useful approximation result for the CVA.
Proposition 3.2. The CVA of a credit default swap under convention i) for \( i = a, a', b, c, c' \) can be approximated in a tenor of dates \( t = t_0, t_1, \ldots, t_M \) by

\[
\text{CVA}_t \simeq \sum_{j=0}^{M-1} \left( \sum_{v=0}^{m-N_t} P_{t,j+1} (N_t + v + 1) Q_{B_{j+1,1}}^{i} \left[ N_{t,j+1} = N_t + v \mid N_t \right] Q[\tau^1 \in [t_j, t_{j+1}], \tau^2 \wedge \tau > \tau^1 \mid N_t] \right. \\
+ \sum_{v=0}^{m-N_t} P_{t,j+1} (N_t + v + 1) Q_{B_{j+1,1}}^{i} \left[ N_{t,j+1} = N_t + v \mid N_t \right] Q[\tau^2 \in [t_j, t_{j+1}], \tau^1 \wedge \tau > \tau^2 \mid N_t] \\
- \left. \sum_{v=0}^{m-N_t} S^2_{t,j+1} \left( P^{*,i}_{t,j+1} (N_t + v + 1) \right) Q_{B_{j+1,1}}^{i} \left[ N_{t,j+1} = N_t + v \mid N_t \right] Q[\tau^2 \in [t_j, t_{j+1}], \tau^1 \wedge \tau > \tau^2 \mid N_t] \right) \\
\text{where we denote } B_{j+1,1}^i = \{ \tau^1 \in [t_j, t_{j+1}], \tau^2 \wedge \tau > \tau^1 \} \text{ and } B_{j+1,1}^2 = \{ \tau^2 \in [t_j, t_{j+1}], \tau^1 \wedge \tau > \tau^2 \}.
\]

Proof. Formula (18) yields

\[
\text{CVA}_t = \sum_{j=0}^{M-1} \int_{t_j}^{t_{j+1}} \left( \sum_{v=0}^{m-N_t} P_u (N_t + v + 1) Q_{B_{j+1,1}}^{i} \left[ N_u = N_t + v \mid N_t \right] dQ[\tau^1 \leq u, \tau^2 \wedge \tau > \tau^1 \mid N_t] \right. \\
+ \sum_{v=0}^{m-N_t} P_u (N_t + v + 1) Q_{B_{j+1,2}} \left[ N_u = N_t + v \mid N_t \right] dQ[\tau^2 \leq u, \tau^1 \wedge \tau > \tau^2 \mid N_t] \\
- \left. \sum_{v=0}^{m-N_t} \left[ S^2_{\tau^2} \left( P^{*,i}_{u} (N_t + v + 1) \right) \right] Q_{B_{j+1,2}} \left[ N_u = N_t + v \mid N_t \right] dQ[\tau^2 \leq u, \tau^1 \wedge \tau > \tau^2 \mid N_t] \right).
\]

Therefore, by assuming that defaults may only be observed at tenor dates \( t = t_1, \ldots, t_M \) (i.e., by formally postponing a default from any time \( u \in [t_j, t_{j+1}] \) to the date \( t_{j+1} \)), we obtain the desired formula. \( \Box \)

3.5 CVA Algorithm for a CDS in a Markovian Contagion Model

Our final goal is to develop a fairly general algorithm for the CVA of a stylized credit default swap under any settlement convention and margin agreements. Let us first emphasize that we consider the credit default swap under the usual convention that the contract value is null at contract inception. In other words, we assume that the obligors agree on a spread value which is such that the value of the contract is null at time 0. Under this convention, the CVA, which is equal to the difference between the price of the risk-free contract and the price of the contract with bilateral counterparty risk, is simply equal to the price of the risk-free contract.

3.5.1 Algorithm without Systemic Risk

Let us first examine the procedure in the absence of systemic risk. We start be noting that, with the exception of convention a), the computation of the CVA introduces a convolution. This convolution is manifested in the formulae for the CVA established in Propositions 3.1 and 3.2 by the presence of the settlement value in the event of the counterparty’s default, that is, \( S^2(P^{*,i}) \). In effect, no matter which specification in terms of margin agreements and segregation is chosen, that is, for either of formulae (2), (4), (7), (9), and (10), the settlement value \( S^2(P^{*,i}) \) is a function of the price \( P^{*,i} \) of the equivalent contract in the event of the counterparty’s default, where \( P^{*,i} \) is in turn a function of the price \( P^{*,i,2} \) of the second counterparty equivalent contract in the event of default of the first equivalent counterparty, and so on.

In the present set-up, the computational challenge due to this convolution can be tackled by means of a recursive procedure performed on a suitably defined tree. Within this tree, the node \( (k+1, k+1) \) is used for storing the price of the \( k \)th equivalent contract at time \( t_k \). For instance, the
price of the first equivalent contract at time \( t_1 \), on the event of default of the counterparty between \([t_0,t_1]\) (or ‘at time \( t_1 \), for short), is stored at the node \((2,2)\), whilst the price of the \( n \)th equivalent contract at time \( T = t_M \), on the event of default of the \((n-1)\)th equivalent contract ‘at time \( t_M \)’, is stored at the node \((n+1,M+1)\). In each step of the recursion procedure, the tree is constructed by making use of the values of the \( k \)th equivalent contract price in the event of default of the \((k-1)\)th equivalent contract at time \( t_k, t_{k+1}, \ldots, t_M \), which are stored at nodes

\[(k+1,k+1), (k+1,k+2), \ldots, (k+1,M+1),\]

so as to compute the price of the \((k-1)\)th equivalent contract in the event of default of the \((k-2)\)th equivalent contract at time \( t_{k-1} \). The latter value is then stored at the node \((k,k)\). The values, which are calculated through a similar step, at the nodes

\[(k,k), (k,k+1), \ldots, (k,M+1),\]

are subsequently used for the computation of the \((k-2)\)th equivalent contract price in the event of default of the \((k-3)\)th counterparty at time \( t_{k-2} \). The latter value is then stored at the node \((k-1,k-1)\), and so on until the price of the bilateral credit default swap at time 0, at node \((1,1)\).

Clearly, this computation must be initiated by first calculating the values of the price of the \( n \)th counterparty equivalent contract at nodes

\[(n+1,n+1), (n+1,n+2), \ldots, (n+1,M+1),\]

which not only depend on the chosen rank \( n \) convention, but also on the spread value. Consequently, under the convention that the CDS is worthless at its inception, the previously described iteration is run until a spread value for which the price of the credit default swap is sufficiently close to zero is found. This can be easily achieved, by adjusting the spread value all along the numerical procedure. Once a satisfactory spread value is found, it suffices to compute the value of the risk-free contract at inception, given the final value of the spread, in order to obtain the CVA. Note that the recursive steps require the knowledge of conditional probabilities of default, which, in practice, are computed ahead of the tree construction. This is an important feature of the pricing engine because it shows that it is readily applicable to any models of correlated default times, such as: copula-based models or affine models.

### 3.5.2 Algorithm with Systemic Risk

For the CVA in the presence of systemic risk, the algorithm is essentially the same as the one just previously described. However, we now need to account for the dependency of the settlement values on the level of systemic risk, that is, in our setting, on the number of systemic defaults. A simple and natural way of extending the previous procedure relies on adding to the tree a third dimension, characterizing the number of systemic defaults. In addition, a significant improvement of computational efficiency can be achieved by restricting the summation with respect to \( v \) in approximate formula of Proposition 3.2 to a reduced number of possible systemic defaults. To this end, for every \( j = 0,1,\ldots,M-1 \) and \( x > j \), we denote by \( N^{(j)}_{j,x+1} \) a set of numbers of defaults such that the random variable \( N_{t_j}^{(j)} \) takes values in this set with a high probability, given that \( N_{t_j} = l \). Then the iterations are run by calculating the relevant quantities on the tree, say at node \((k,j,l)\), as a function of the quantities computed at nodes \((k+1,x,v)\) for every \( x > j \) and any number \( v \in N^{(j)}_{j,x+1} \) of systemic defaults.

We now present the procedure more formally in the case of the time-homogeneous Markovian contagion model of Lemma 3.2(ii). We recall that we refer to the creditworthiness of a company as a function of its default intensity, which in turn is a function of the number of systemic defaults. Hence we will write \( c_{t_j} = \hat{\gamma}(N_{t_j}^{*}) \) and \( c_{t_j}^2 = \hat{\gamma}^2(N_{t_j}^{*}) \) for the creditworthiness of the investor and the counterparty at time \( t_j \), respectively. In addition, we will denote by \( \tau^{2,k} \) the moment of default of the \( k \)th counterparty equivalent contract, and by \( \gamma_{t_j}^{2,k} \) the creditworthiness of the \( k \)th counterparty.
at time \( t_j \). Also, the price at time \( t_j \) of the \( k \)th counterparty equivalent contract, given the number \( l \) of systemic defaults, will be denoted as \( P_{t_j}^{i,k,l} \).

It is important to note that the creditworthiness of the \( k \)th counterparty equivalent contract depends not only on the choice of a particular convention \( i \), but also on whether or not the equivalent counterparties trigger systemic effects. Under the Markov chain specification of Section 3.4, it is implicitly assumed that it is not the case (see also Remark 3.1 for a more sophisticated modeling approach). The random time \( \tau_{2,k} \) can be obtained by running an independent copy of model of Definition 3.3, started at time \( \tau_{2,k-1} \) of default of the \((k-1)\)th counterparty and with the initial number of defaults in the systemic portfolio equal to the observed number of defaults in this portfolio at time \( \tau_{2,k-1} \). To be more specific, let represent the joint dynamics of defaults for the systemic portfolio, the investor and the \( k \)th counterparty by a continuous-time Markov chain \( \mathcal{M} = (\mathcal{M}_t)_{t \in [\tau_{2,k-1},T]} \) with the state space

\[
E = \{(j_0, 0, 0, 0), (j_0, 1, 0, 0), (j_0, 0, 1, 0), (j_0, 0, 0, 1), (j_0, 1, 1, 0),
(j_0, 1, 0, 1), (j_0, 0, 1, 1), (j_0, 1, 1, 1), j_0 = 0, 1, \ldots, m\},
\]

where the generic state \((j_0, j_1, j_2, j_3)\) has the following interpretation:

(a) \( j_0 \) is the number of defaults in systemic portfolio \((j_0 \geq N_{2,k-1})\),
(b) \( j_1 \) is the default indicator for the investor,
(c) \( j_2 \) is the default indicator for the \( k \)th counterparty,
(d) \( j_3 \) is the default indicator for the reference entity.

For instance, under convention \( b \) we will have that

\[
\lambda((j_0, 0, 0, 0), (j_0 + 1, 0, 0, 0)) = a_{j_0} \gamma(t, j_0 + 1),
\lambda((j_0, 0, 0, 0), (j_0, 1, 0, 0)) = \gamma_1(t, j_0 + 1),
\lambda((j_0, 0, 0, 0), (j_0, 0, 0, 1)) = b_{j_0} \gamma(t, j_0 + 1),
\lambda((j_0, 0, 0, 0), (j_0, 0, 1, 0)) = \gamma_2(l_0),
\]

with \( \gamma_1(t, j_0 + 1) = \gamma_2(t, j_0 + 1) = \gamma(j_0 + 1) = b_{j_0} \gamma(t, j_0 + 1) \) under the specification of Lemma 3.2(ii), and \( l_0 \) stands for the initial number of systemic defaults, \( N_0^* = l_0 \).

We are in a position to describe in some detail our pricing algorithm. We assume convention \( i \) for some \( i = b, c, c' \) and the investor’s rank \( n \) convention. Recall that the approximation for the CVA of a credit default swap in a tenor of dates \( t = t_0, t_1, \ldots, t_M \) was established in Proposition 3.2. We now argue that the following algorithm yields a numerical implementation of the CVA formula of Proposition 3.2.

**Step 1.** Take as input an initial value for the spread \( \hat{\kappa}_0^* \). Run the following loop until a spread value \( \hat{\kappa}_0^* \) such that the contract price is sufficiently small (i.e., as close to zero as required) is found:

- **Step 1.1.** Compute the price of the \( n \)th equivalent contract for every \( t_j, j = n, \ldots, M - 1 \) and for each number \( l = 0, \ldots, m + 2 \) of systemic defaults observed by time \( t_j \), that is, at nodes \((n + 1, n + 1, l + 1), (n + 1, n + 2, l + 1), \ldots, (n + 1, M + 1, l + 1)\).

For this purpose:

- under risk-free investor rank \( n \) convention, use the approximation

\[
P_{t_j}^{i,n,l} \simeq (1 - R) \mathbb{Q}[\tau \leq T | N_{t_j}^* = l, \tau > t_j] - \hat{\kappa}_0^* \mathbb{E}_Q[\tau \wedge T - t_j | N_{t_j}^* = l, \tau > t_j],
\]

- under zero rank \( n \) convention, set \( P_{t_j}^{i,n,l} = 0 \).

- **Step 1.2.** Compute the price of the \( k \)th equivalent contract for every \( t_j, j = k, \ldots, M - 1 \) and for each number \( l = 0, \ldots, m + 2 \) of systemic defaults observed by time \( t_j \), that is, at nodes \((k + 1, k + 1, l + 1), (k + 1, k + 2, l + 1), \ldots, (k + 1, M + 1, l + 1)\).
To this end, use the following equation at each iteration with respect to \( k \), where \( 1 \leq k \leq j \)

\[
P_{t_j}^{*,i,k|l} \simeq (1 - R) \mathbb{Q}\left[ \tau \leq T, \tau^1 \land \tau^{2,k+1} > \tau \mid A_j^{i,k|l} \right] - \hat{\kappa}_0^i \mathbb{E}_\mathbb{Q}\left[ \tau \land \tau^{1} \land \tau^{2,k+1} \land T - t_j \mid A_j^{i,k|l} \right]
\]

\[
+ \sum_{x=0}^{M-1} \sum_{v \in N_{t_{x+1}}^l} S_{t_{x+1}}^2(P_{t_{x+1}}^{*,i,k|l}|v) \mathbb{Q}\left[ B_{t_{x+1}}^{*,i,k|l} \right] \mathbb{Q}\left[ t_x < \tau^2, \tau^1 \land \tau > \tau^{2,k+1} \mid A_j^{i,k|l} \right]
\]

(22)

where we denote \( \hat{B}_{t_{x+1}}^{*,i,k|l} = \{ t_x < \tau^2, \tau^1 \land \tau > \tau^{2,k+1} \} \) and

\[
A_j^{u,k|l} = \{ t_{j-1} < \tau^2, \tau^1 \land \tau > \tau^{2,k}, N_{t_j}^* = l, c_{t_j}^1 = \hat{\gamma}(l), c_{t_j}^{2,k} = \infty \},
\]

\[
A_j^{b,k|l} = \{ t_{j-1} < \tau^2, \tau^1 \land \tau > \tau^{2,k}, N_{t_j}^* = l, c_{t_j}^1 = \hat{\gamma}(l), c_{t_j}^{2,k} = \hat{\gamma}(l) \},
\]

\[
A_j^{c,k|l} = \{ t_{j-1} < \tau^2, \tau^1 \land \tau > \tau^{2,k}, N_{t_j}^* = l, c_{t_j}^1 = \hat{\gamma}(l), c_{t_j}^{2,k} = \hat{\gamma}(l-1) \},
\]

\[
A_j^{c',k|l} = \{ t_{j-1} < \tau^2, \tau^1 \land \tau > \tau^{2,k}, N_{t_j}^* = l, c_{t_j}^1 = \hat{\gamma}(l-1) = c_{t_j}^{2,k} \},
\]

and where \( N_{j,x+1}^l \) denotes a set of number of defaults such that \( N_{t_j}^* \) takes values in this set with a high probability given that \( N_{t_j}^* = l \). Recall also that \( l_0 = N_0^{*} \).

- **Step 1.3.** Compute the price of the contract at time 0 from the formula

\[
\hat{P}_0^i \simeq (1 - R) \mathbb{Q}\left[ \tau \leq T, \tau^1 \land \tau^2 > \tau \right] - \hat{\kappa}_0^i \mathbb{E}_\mathbb{Q}\left[ \tau \land \tau^1 \land \tau^2 \land T \right]
\]

\[
+ \sum_{x=0}^{M-1} \sum_{v \in N_{t_{x+1}}^0} S_{t_{x+1}}^2(P_{t_{x+1}}^{*,i,k|l}|v) \mathbb{Q}\left[ B_{t_{x+1}}^{*,i,k|l} \right] \mathbb{Q}\left[ t_x < \tau^2, \tau^1 \land \tau > \tau^2 \right]
\]

where \( \hat{B}_{t_{x+1}}^{*,i,k|l} = \{ t_x < \tau^2, \tau^1 \land \tau > \tau^2 \} \) and \( N_{t_{x+1}}^{0} \) denotes a set of number of defaults such that \( N_{t_{x+1}}^{*} \) belongs to this set with a high probability given that \( N_{t_{x+1}}^{*} = l_0 \).

**Step 2.** Compute the CVA at time 0, which is given by \( P_0(\hat{\kappa}_0^i) \).

**Algorithm under convention b).** We will now justify the algorithm by postulating, for the sake of concreteness, the convention b). Note that Step 2 is a straightforward consequence of the definition of the CVA (recall that \( \hat{P}_0^i(\hat{\kappa}_0^i) = 0 \))

\[\text{CVA}_0 = P_0(\hat{\kappa}_0^i) - \hat{P}_0^i(\hat{\kappa}_0^i) = P_0(\hat{\kappa}_0^i).\]

For Step 1, we will only focus on derivation of equation (22), since formula (23) can be obtained using analogous arguments and formula (21) is clear. Equality (22) hinges on the following two simplifications:

(i) the assumption that defaults can only occur at the tenor dates,
(ii) the summation of conditional expectations of settlement values over a reduced set of possible defaults.

Let us write

\[
A_j^{b,k|l} = \{ t_{j-1} < \tau^2, \tau^1 \land \tau > \tau^{2,k}, N_{t_j}^* = l, c_{t_j}^1 = \hat{\gamma}(l), c_{t_j}^{2,k} = \hat{\gamma}(0) \}.
\]

We start by noting that the price \( P_{t_j}^{*,b,k|l} \) of the kth equivalent investor contract given that \( N_{t_j}^* = l \),
equals

\[
P_{t_j}^{*,b,k|l} = \mathbb{E}_Q \left[ \mathbb{I}_{\tau \leq T, \tau^1 \wedge \tau^2, k, l > \tau} (1-R) - \int_{t_j}^{T \wedge \tau^1 \wedge \tau^2, k, l} \hat{\kappa}^b \, du \right.
\]

\[+ \mathbb{I}_{\tau^2, k, l \leq T, \tau^1 \wedge \tau^2, k, l} \mathbb{E} \left[ P_{t_j}^{*,b,k+1} | A_j^{b,k|l} \right] \mathbb{I}_{\tau^2, k, l} \right]
\]

\[= (1-R) \mathbb{E} \left[ \mathbb{I}_{\tau \leq T, \tau^1 \wedge \tau^2, k, l > \tau} | A_j^{b,k|l} \right]
\]

\[+ \mathbb{E} \left[ \mathbb{I}_{\tau^1 \wedge \tau^2, k, l \leq T} S_{\tau^2, k, l} \mathbb{I}_{\tau^2, k, l} | A_j^{b,k|l} \right].
\]

This means that \( P_{t_j}^{*,b,k|l} = (1-R)J - \hat{\kappa}^b I + K \) where we denote

\[I = \mathbb{E}_Q \left[ \mathbb{I}_{\tau \leq T, \tau^1 \wedge \tau^2, k, l > \tau} | A_j^{b,k|l} \right]
\]

and

\[J = \mathbb{E}_Q \left[ \mathbb{I}_{\tau \leq T, \tau^1 \wedge \tau^2, k, l > \tau} | A_j^{b,k|l} \right].
\]

We thus need to approximate \( K \). For this purpose, we observe that

\[K = \mathbb{E}_Q \left[ \mathbb{I}_{\tau^1 \wedge \tau^2, k, l > \tau} S_{\tau^2, k, l} \mathbb{I}_{\tau^2, k, l} | A_j^{b,k|l} \right]
\]

\[= \int_{t_j}^{T} \mathbb{E}_Q \left[ S_{\tau^2, k, l} (P_{u}^{*,b,k+1}) | A_j^{b,k|l} \right] du \mathbb{E}_Q \left[ \mathbb{I}_{\tau^1 \wedge \tau^2, k, l > \tau} | A_j^{b,k|l} \right]
\]

\[= \sum_{x=0}^{M-1} \int_{t_j}^{t_{x+1}} \mathbb{E}_Q \left[ S_{\tau^2, k, l} (P_{u}^{*,b,k+1}) | A_j^{b,k|l} \right] du \mathbb{E}_Q \left[ \mathbb{I}_{\tau^1 \wedge \tau^2, k, l > \tau} | A_j^{b,k|l} \right]
\]

where \( B_u^{*,b,k+1} = \{ \tau^2, k, l = u, \tau^1 \wedge \tau > \tau^2, k, l \} \). If we assume that defaults can only occur at the tenor dates (that is, equivalently, if we forcibly postpone default from any time \( u \in [t_{i-1}, t_i] \) to the date \( t_j \)), we obtain the following approximation

\[K \approx \sum_{x=0}^{M-1} \int_{t_j}^{t_{x+1}} \mathbb{E}_Q \left[ S_{\tau^2, k, l} (P_{u}^{*,b,k+1}) | A_j^{b,k|l} \right] du \mathbb{E}_Q \left[ \mathbb{I}_{\tau^1 \wedge \tau^2, k, l > \tau} | A_j^{b,k|l} \right]
\]

\[\approx \sum_{x=0}^{M-1} \mathbb{E}_Q \left[ S_{\tau^2, k, l} (P_{t_{x+1}}^{*,b,k+1}) | A_j^{b,k|l} \right] \mathbb{E}_Q \left[ \mathbb{I}_{\tau^1 \wedge \tau^2, k, l > \tau} | A_j^{b,k|l} \right].
\]

Finally, the conditional expectations of settlement values can be approximated as follows

\[\mathbb{E}_Q \left[ S_{\tau^2, k, l} (P_{t_{x+1}}^{*,b,k+1}) | A_j^{b,k|l} \right] \approx \sum_{v \in N_{t_{x+1}}^*} \mathbb{E}_Q \left[ S_{\tau^2, k, l} (P_{t_{x+1}}^{*,b,k+1} | v) | N_{t_{x+1}}^* = v \right] A_j^{b,k|l}
\]

where \( N_{t_{x+1}}^* \) denotes a set of number of defaults such that \( N_{t_{x+1}}^* \) belongs to this set with a high probability (say 95%) given that the number of systemic defaults at time \( t_j \) equals \( l \). Modifications of this algorithm from convention \( b \) to conventions \( c \) and \( c' \) are straightforward.

**Remark 3.1.** Under the Markov chain specification, as described in Section 3.4, it is implicitly assumed that the defaults of the successive equivalent contracts do not trigger systemic effects. One can nevertheless account for such systemic defaults by means of a rather straightforward extension of the model. Indeed, it suffices to postulate the existence of a systemic portfolio of counterparties, containing \( n - 1 \) potential counterparties, and to assume that the equivalent counterparty is always
the first-to-default among this basket. In other words, if, at time $t$, the counterparty has defaulted, and $k - 1$ subsequent equivalent counterparties went bankrupt as well, then the $k$th equivalent counterparty is the first-to-default entity among the remaining $n - 1 - k$ entities within the systemic basket of potential counterparties. In this setting, the joint dynamics of defaults for the systemic portfolio, the systemic portfolio of potential counterparties, the investor, the counterparty, and the reference entity can be represented by a continuous-time Markov chain $\mathcal{M} = (\mathcal{M}_t)_{t \in [0,T]}$ with the state space $E$ given by

$$E = \{(i, k, 0, 0, 0), (i, k, 1, 0, 0), (i, k, 0, 1, 0), (i, k, 0, 0, 1), (i, k, 1, 1, 0), (i, k, 1, 0, 1),$$

$$(i, k, 1, 1, 1), (i, k, 1, 1, 0), i \in \{0, 1, \ldots, m\}, k \in \{0, 1, \ldots, n - 1\}\}

with the following interpretation of a generic state $(i, k, j_1, j_2, j_3)$:

(a) $i$ is the number of defaults in some systemic portfolio of entities,

(b) $k$ is the number of defaults in a systemic portfolio of potential counterparties,

(c) $j_1$, $j_2$ and $j_3$ are the default indicators for the investor, the counterparty and the reference entity. Hence $l = i + k + j_1 + j_2 + j_3$ stands for the total number of defaults. Note that the sum $i + k$ formally corresponds to the variable $j_0$ in Section 3.3.

Let us now explicitly write the transition intensities in the event that the counterparty has defaulted. We only consider conventions $b)$, $c)$ and $c')$, since under conventions $a)$ and $a')$ the contagion effects at first default are absent.

**Convention b).** Under convention $b)$, the defaults within the systemic portfolio of potential counterparties impact the credit rating transitions of each firm in our model, to the exception of the potential counterparties, of which transition intensities are constant and equal to the credit quality of the counterparty at the contract’s inception. We have

$$\lambda((i, k, 0, 1, 0), (i + 1, k, 0, 1, 0)) = a_i \gamma(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k, 1, 1, 0)) = \gamma_1(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k, 0, 1, 1)) = b_i \gamma(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k + 1, 0, 1, 0)) = \gamma_2(l_0),$$

where $l_0 \in \{0, 1, \ldots, m\}$ is the initial value of $N^*$, that is, $l_0 = N_0^*$.

**Convention c).** Under this convention, the $k$th equivalent counterparty is chosen in such a way that its credit quality equals the credit quality of the $(k - 1)$th equivalent counterparty just before default. This implies that the credit qualities of potential counterparties only depend on the number of defaults $i$ within the systemic portfolio. We obtain

$$\lambda((i, k, 0, 1, 0), (i + 1, k, 0, 1, 0)) = a_i \gamma(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k, 1, 1, 0)) = \gamma_1(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k, 0, 1, 1)) = b_i \gamma(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k + 1, 0, 1, 0)) = \gamma_2(t, i).$$

**Convention c').** Finally, under convention $c')$, we take as a reference the price of an equivalent contract between two firms with credit qualities are the same as the credit qualities of the obligors of the original contract just before default. We thus establish the credit rating transition of the investor by discarding the first, as well as the subsequent, ‘contagion effects at first default’

$$\lambda((i, k, 0, 1, 0), (i + 1, k, 0, 1, 0)) = a_i \gamma(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k, 1, 1, 0)) = \gamma_1(t, i),$$

$$\lambda((i, k, 0, 1, 0), (i, k, 0, 1, 1)) = b_i \gamma(t, i + k + 1),$$

$$\lambda((i, k, 0, 1, 0), (i, k + 1, 0, 1, 0)) = \gamma_2(t, i).$$
3.5.3 Numerical Illustration

We conclude the paper by presenting a stylized example of explicit computations of a credit default swap, with maturity $T = 5$ years, and under rank $n = 3$ convention. Let us emphasize that accounting for systemic risk gives, of course, more credence to the possibility of successive defaults. The margin period of risk is set to 15 days, which is standard practice and gives a total number of discretization steps $M$ equal to 120. The conditional probabilities are obtained by means of Monte Carlo simulations, for each total number of systemic defaults $N$. The intensities of default of the investor, the counterparty, and the systemic entities are derived from the CDS market. Notice that the calibration of the model to the latter is essentially a matter of calibrating the associated Markov chain that accounts for both the systemic and at first default contagion effects (hence the wrong-way risk). Such calibration procedure is not uncommon for practitioners, which supports our view that the model should prove attractive to the financial industry.

Exhibit 1 displays the spreads of credit default swaps under convention $b)$, as a percentage of the related risk-free spreads in the absence of collateralization and for various initial values of the systemic risk factor $N_0 = 1, 2, 3, 4, 5$.

Figure 1: Example of computations of spreads of credit default swaps under convention $b)$, as a percentage of the risk free spreads, under systemic risk and without margin agreements.

More generally, our results show significant variations in the pricing of the credit default swap for different choices of conventions, systemic risk and margin agreements. They make the case that by accounting for the counterparty risk and systemic risk, as well as various settlement conventions, the CVA of a financial instrument may significantly differ from the one obtained through more traditional pricing methods. Needless to say that fully practical implementations of conventions and results presented in this paper are yet to be examined, preferably within a professional environment.

References


