Incomplete-Market Prices for Real Estate

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Abstract: This paper reconsiders the predictions of the standard real options models for real estate in the context of incomplete markets. We value vacant land as a European call option on a building that could be built on that land. Since we relax the completeness assumption of the Black-Scholes (1973) model, it is no longer possible to construct a replicating portfolio to price the option. To mitigate this problem, we use the good-deal bounds technique and arrive at closed-form solutions for the price of the real option. We determine an upper and a lower bound for the price of this option and find that, contrary to standard option theory, increasing the volatility of the underlying asset doesn’t necessarily increase the option value (the value of the vacant land). The lower bound option prices are always a decreasing function of the volatility of the underlying asset, which cannot be explained by a Black-Scholes (1973) type of argument. This is however consistent with the presence of unhedgeable risk on the incomplete market, additional to the hedgeable component found on the complete market of the standard real options models.

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1. Introduction

The purpose of this paper is to show that real-estate projects are better priced as real options in the setting of incomplete markets. Specifically, we assume that a project carries both hedgeable and unhedgeable risk and use the good-deal bounds technique to price the option and bring new insights into the predictions of the real options models.

We value vacant land as a European call option on a building that could potentially be built on that land. The characteristics of this real option are as follows: the underlying asset is the building, the strike price is the construction cost and the exercise time is at the start of the development. Following Miao and Wang (2007), we distinguish between a lump-sum and a flow payoff case corresponding, respectively, to the decision to sell the building once the construction is finalised or to keep it as a rental. The result of the option pricing mechanism is a modified Black-Scholes (1973) semi-closed form solution for each type of payoff. We find that, contrary to the predictions of the standard real options models, the good-deal bounds prices don’t always display an increasing pattern when the volatility of the underlying asset increases.

The first paper to ever introduce the idea of pricing vacant land as a call option on a building was the theoretical paper of Titman (1985). Subsequent works from Quigg (1993), Cunningham (2006) and Clapp and Salavei (2010) try to prove Titman’s assertion that there is option value in land prices by empirically testing the predictions of a real options model applied to real-estate. Unfortunately, the evidence is contradictory. The option value is neither consistent, nor always present over different time periods. The usual culprit for this inconsistency is deemed the hedonic pricing methodology, which is subject to a certain degree of measurement error, but perhaps the problem comes from somewhere else.

The main assumptions of Titman (1985) are that the market on which the real option exists is frictionless and that the price of the option can be calculated by means of replicating arguments. In reality, the real estate market is an incomplete market, because real estate assets are illiquid assets; they are not traded continuously, which makes it impossible to construct a replicating portfolio as in the Black-Scholes option pricing model.

Henderson (2007) and Miao and Wang (2007) take the analysis a step further and move the setting to an incomplete market. They derive semi-closed form solutions for options on real estate using the utility indifference pricing technique. Utility indifference pricing assumes a utility function for a representative agent who maximizes his utility of wealth, where wealth is
influenced by an investment in the option. Both Henderson (2007) and Miao and Wang (2007) show that market incompleteness, the degree of risk aversion in particular, leads to different predictions according to the analysed type of payoff. They conclude that market incompleteness reduces the option value for the lump-sum payoff case. Additionally, Miao and Wang (2007) find that for flow payoffs the effect can be the opposite.

The utility indifference pricing approach might seem like a good candidate for a pricing mechanism. Unfortunately, the results go only as far as a partial differential equation that the option price must satisfy and only for the exponential utility. Furthermore, it is impossible to price short call positions analytically, because the prices converge to infinity (Henderson and Hobson, 2004).

The methodology we propose for pricing the real options on real estate assets in incomplete markets combines Titman's (1985) idea of valuing land as a call option on a potential real-estate project with a slightly modified version of the good-deal bounds pricing approach of Cochrane and Saa-Requejo (2000). We say slightly modified version, because we make use of a change of measure to calculate the option price instead of solving for the stochastic discount factor and then taking the expectation of the stochastic discount factor multiplied by the option payoff. To our knowledge, the good-deal bounds have never been suggested as an application for real estate, but they have been successfully used to price other types of options, like index options or options on non-traded events (see Cochrane and Saa-Requejo, 2000).

Whenever we want to price a general claim, we calculate the expectation of a stochastic discount factor times the payoff of that claim. This is straightforward on a complete market, where all assets are assumed to be traded which means that we can observe their market price of risk. The volatility term of the stochastic discount factor is nothing else than the Sharpe ratio of the asset we are trying to price. Unfortunately, on an incomplete market, we cannot observe the market price of risk, because here there are also illiquid assets, say buildings. We can however distinguish between hedgeable and unhedgeable risk and express the market price of risk for the unhedgeable component in terms of what we already know: the Sharpe ratio of a traded asset, for instance a REIT\(^1\), correlated with the underlying asset of the option (the building). This Sharpe ratio is an essential tool in determining the expression for the overall volatility of the stochastic discount factor, such that we can restrict the set of all possible discount factors to obtain the option price.

\(^1\)A Real Estate Investment Trust (REIT) is a capital asset product which offers publicly traded common stock shares in companies which own income-producing properties and mortgages (Geltner et al., 2007).
The parameters which ultimately determine the value of vacant land are the volatility of the underlying asset, the restriction on the volatility of the stochastic discount factor, the correlation coefficient between the underlying and a traded risky asset and the expected return of the investment.

Interestingly, unlike in the standard real options models, the good-deal bounds option prices don’t always increase as the volatility of the underlying asset increases. The lower bound prices are actually decreasing with increasing volatility of the underlying. This is a reflection of the additional uncertainty coming from the presence of unhedgeable risk, which is present on an incomplete market but not in the Black-Scholes setting.

The advantage of the good-deal bounds over other incomplete market techniques is that it arrives at semi-closed form solutions for the option prices and, more importantly, comparable to the Black-Scholes (1973) price. For very low values of the volatility of the underlying (the non-traded asset), the only source of uncertainty comes from the traded asset and we are back in the Black-Scholes (1973) framework. Similarly, for very high values of the correlation coefficient, the good-deal bounds prices approach the Black-Scholes (1973) price. Furthermore, when the restriction on the volatility of the stochastic discount factor is exactly equal to the Sharpe ratio of the traded asset, we again exit the incomplete market setting and the good-deal bounds prices converge to the Black-Scholes (1973) price. In other words, we generalise the market setting to the incomplete market and bring it closer to real life, but, at the same time, maintain a reference point, which is the Black-Scholes (1973) result.

The paper is organised as follows: Section 2 presents the mathematics of deriving the good-deal bounds option prices, Section 3 deals with the intuition behind the proposed methodology and Section 4 concludes.

2. Real estate and incomplete markets – a closer look

There are a series of frictions which can render a market incomplete: transaction costs, the presence of non-traded assets on that market or portfolio constraints, like no short-selling or a predetermined allocation to a particular asset or asset group in the portfolio.

Real-estate assets are illiquid assets. Even if we observe a trade for a particular property, it might never be traded again or, at best, at large intervals of time. In other words, it is even difficult to observe prices for real estate, let alone the market price of risk. Furthermore, real
Estate assets are heterogeneous assets: each property is unique, which makes it impossible to trade units of this kind of assets the way we do with liquid assets like, say, stocks. Given all these, we can at best obtain a partial hedge for a real estate asset, so the valuation of an option on such an asset should be done in the context of incomplete markets. To better understand why let’s go back to a complete-market setting, specifically the Black-Scholes (1973) option pricing model.

The Black-Scholes (1973) model relies on the following assumptions: the price process for the underlying asset follows a geometric Brownian motion, with a constant drift and a constant volatility, the underlying asset is traded continuously and it pays no dividends, short selling is allowed, there are no transaction costs or taxes, there are no riskless arbitrage opportunities and the risk-free interest rate is constant (Hull, 2002).

The continuous trading assumption is the one that makes our case. The whole idea of the Black-Scholes is that we can construct a replicating portfolio of stocks and bonds. Next, we apply the following investment strategy: we take a long position in the option and a short position in the replicating portfolio. The payoff of this strategy is zero. In the absence of arbitrage opportunities, the initial price of this strategy must also be equal to zero, making the price of the option equal to the cost of the replicating portfolio. However, this is only possible because we can continuously trade in the underlying. Since real-estate assets are illiquid, we cannot construct this replicating portfolio anymore to conveniently perform the valuation in the Black-Scholes complete market setting.

So far, we have established that, to price an option on a real-estate asset, we should start from the assumption of an incomplete market. The question is how to proceed from here.

The problem with pricing in incomplete markets is that imposing the no-arbitrage condition is no longer sufficient to arrive at a unique price for a general contingent claim (financial derivative). Take for instance the example of a non-traded underlying asset. It is impossible to exactly replicate a claim on such an asset, so we can expect to be confronted with several price systems for this claim, all of them consistent with the absence of arbitrage. In fact, if we just impose absence of arbitrage, the price will be situated within the arbitrage bounds. For a call option, the lower arbitrage bound is zero and the upper arbitrage bound is the price of the underlying. Such an interval is not very informative, because it is too wide to be useful. The solution is to make additional assumptions about the risk preferences of the investors on the market (Björk, 2009).
The good-deal bounds (GDB) pricing is an incomplete-market approach to pricing, which uses a restriction on the volatility of the stochastic discount factor as an additional restriction to arrive at tighter and more informative bounds for the option price. With the good-deal bounds we are in the context of partly hedgeable partly unhedgeable risk. We assume that we can find on the market a risky asset (for instance, a REIT), correlated with the real estate project, with which to hedge the illiquid real estate asset at least partly. The advantage over the complete-market model is that the option price reflects both the hedgeable and unhedgeable risk. What is more, it is even possible to derive semi-closed form solutions for the price of the real options, as Cochrane and Saa-Requejo (2000) show in their paper and as we will see later in the next sub-section. The GDB prices resemble the Black-Scholes prices, which makes it even easier to see the effect of adding unhedgeable risk.

Before going over the details of the good-deal bounds pricing we should be clear on what the objective is. We want to obtain a price for a call option on a real estate project. The option price is in fact the value of the vacant piece of land which an investor owns and on which he can construct a building. The potential building is our underlying asset, with a market value denoted $V$. The strike price of this kind of option is the construction cost of the building, denoted $K$. The option is considered to have been exercised when the development starts.

It is important to realise that the value $V$ can be obtained from two different sources. Once the construction is finalised, the investor can either sell the property, in which case $V$ will be the market sale price and the option will have a lump-sum payoff, or he can start renting the property, turning $V$ into a discounted value of future rents and the option payoff into a flow payoff. The results are the same regardless of the type of payoff used. However, we believe it is useful to derive closed-form solutions for both types of payoffs depending on what information we have available to perform the valuation of the vacant land.

### 2.1. Market setting

Assume that the market is incomplete and that there are only three assets on this market: an illiquid risky asset $V$, a traded risky asset $S$, correlated with $V$, and a riskless asset $B$ (a bond). The dynamics of these assets are as follows:

$$dS = \mu_S S dt + \sigma_S S dz$$  \hspace{1cm} (1)

where: $dz$ – Brownian motion
\[
dV = \mu_V V dt + \sigma_V V \left( \rho dz + \sqrt{1 - \rho^2} dw \right) \quad (2)
\]

where: \( \rho \) – correlation coefficient between the assets \( V \) and \( S \)

\( dz, dw \) – independent Brownian motions

\[
\quad dB = r B dt \quad (3)
\]

where: \( r \) – deterministic short interest rate

The asset \( V \) cannot be continuously traded. The asset \( S \) is assumed to be continuously traded and correlated with \( V \) with a correlation coefficient \( \rho \).

We want to price a European call option on the asset \( V \). In our case, the underlying asset \( V \) is a building and the asset \( S \), correlated with \( V \), can be a REIT. The strike price, denoted \( K \), is the construction cost and is assumed to be constant. The option is considered to have been exercised once the construction of the building starts. Eventually, we want to equate the price of the call option with the price of a vacant piece of land on which we could construct the building (the underlying).

It is important to mention that we assume a very simplistic context in which the entire value of the land is given by the option value; the intrinsic value of the land is assumed to be zero. An intrinsic value different from zero would only complicate the calculations without significantly changing the results.

Like any other contingent claim, our call option can be priced as the discounted expected value of its payoff (Björk, 2009):

\[
C_t = E_t \left[ \Lambda_T \max(V_T - K, 0) \right] \quad (4)
\]

where

\begin{align*}
E_t & \quad \text{the expectation at time } t \\
C & \quad \text{the price of the call option} \\
\Lambda & \quad \text{the stochastic discount factor} \\
V_T & \quad \text{the value of the real estate project at time } T \\
K & \quad \text{the strike price (the construction cost)}
\end{align*}

On the real estate market, \( V_T \) can be either in the form of a lump-sum, if the investor decides to sell the building immediately after the construction is finished, or in the form of a stream of cash-flows (rental rates), if the investor wants to rent out the newly constructed building.
We assume that the day the construction is finalised coincides with the capital time \( T \).

Notice that if \( V \) and \( \Lambda \) were both deterministic, we would end up at an NPV valuation\(^2\) (Geltner et al., 2006). \( V_T - K \) would be a constant and thus would come out of the expectation as itself and we would be left with the expected value of the discount factor, similar to a risk-adjusted discount rate in an NPV type of formula. So, we could say that the good-deal bounds pricing is an extension of the traditional NPV valuation, for a stream of uncertain cash-flows at a stochastic discount rate.

### 2.2. Pricing with the good-deal bounds

Cochrane and Saa-Requejo (2000) start from the idea that investors will always trade in assets with very high Sharpe ratios (which they call “good deals”) and pure arbitrage opportunities (or “ridiculously good deals”, according to Björk and Slinko (2006)), so such investments would be immediately traded in and would disappear from the market. What we want is a Sharpe ratio high enough to induce trade, but without including the good-deals which are too good to be true.

In incomplete markets we need additional restrictions on the risk preferences, apart from the no-arbitrage condition, to construct tighter bounds for the price of a contingent claim (Björk, 2009). If we restrict the volatility of the stochastic discount factor as an additional constraint, we arrive at the so-called good-deal bounds, which are much tighter than the arbitrage bounds resulting from simply imposing the no-arbitrage condition.

On a complete market, where we assume the underlying asset of an option to be continuously traded, we can observe its market price of risk, \((\mu - r)/\sigma\). So, the process for the stochastic discount factor, which we will denote \( \Lambda \), is simply:

\[
d\Lambda = -r\Lambda dt - \frac{\mu - r}{\sigma} \Lambda dz
\]

\(^2\)NPV = \[
\left\{ \frac{E_0[CF_1]}{1 + E_0[r]} + \frac{E_0[CF_2]}{(1 + E_0[r])^2} + ... + \frac{E_0[CF_T]}{(1 + E_0[r])^T} \right\} - \text{sale price}, \quad \text{for the buyer}
\]

\[
\text{Sale price} - \left\{ \frac{E_0[CF_1]}{1 + E_0[r]} + \frac{E_0[CF_2]}{(1 + E_0[r])^2} + ... + \frac{E_0[CF_T]}{(1 + E_0[r])^T} \right\}, \quad \text{for the seller}
\]

where:
- \( E_0 \) – expectation at time 0
- \( CF_t \) – the cash-flow generated at time \( t \)
- \( E_d[t] \) – the risk-adjusted discount rate
- \( T \) – the end of the investment holding period such that \( CF_T \) includes the resale value of the property at time \( T \)
The volatility term of the stochastic discount factor is actually a Sharpe ratio hence the justification for the good-deal bounds pricing mechanism. Restricting the volatility of the discount factor is thus equivalent to restricting the Sharpe ratio of a risky asset (Hansen and Jagannathan, 1991). The minus sign in front of the volatility component shows that the stochastic discount factor assigns more weight to the bad outcomes of the value of the underlying: whenever \( z \) decreases (determining a bad outcome for the underlying), the discount factor increases.

We can use the same logic for incomplete markets. Even though our underlying (i.e. the building) is not liquidly traded and we cannot observe its market price of risk, we can still express it in terms of what we know, for instance the market price of risk of a traded asset correlated with the non-traded underlying. This leads to a partial hedge of the focus derivative (using the traded asset \( S \) correlated with the illiquid asset \( V \)), so we are confronted with both hedgeable and unhedgeable sources risk.

Consequently, the stochastic discount factor on an incomplete market should look like:

\[
d\Lambda = -r\Lambda dt - \kappa_1 \Lambda dz - \kappa_2 \Lambda dw
\]

where:
- \( dz \) – hedgeable component
- \( dw \) – unhedgeable component

Restricting the total volatility of the discount factor to be at most \( k \), Cochrane and Saas-Requejo (2000) write:

\[
\kappa_1^2 + \kappa_2^2 \leq k^2
\]

Fixing \( \kappa_1 = (\mu - r)/\sigma \) as the market price of the hedgeable risk, \( \kappa_2 \) has the solution:

\[
\kappa_2 \in \left[-\sqrt{k^2 - \left(\frac{\mu-r}{\sigma}\right)^2}, \sqrt{k^2 - \left(\frac{\mu-r}{\sigma}\right)^2}\right]
\]

\( \kappa_2 \) can take any value in the interval above. To each \( \kappa_2 \) in this interval corresponds an option price, so, eventually, the option price (i.e. the price of the vacant land) will also be within an interval.
Remember the expression for the price of the call option in (4). Instead of calculating the expectation of the stochastic discount factor in equation (6) times the option payoff, the way Cochrane and Saa-Requejo (2000) do, we could equivalently perform a Girsanov transformation (Björk, 2009) on the process V and simply calculate the expectation of the resulting process, which will have a new Brownian motion and a new drift term.

To understand why that is the case, let’s start from the physical (real-world) measure P:

\[ C_0 = E^P \left[ \frac{\Lambda_T}{\Lambda_0} \max(V_T - K, 0) \right] \]  

(9)

The stochastic discount factor (continuous-time pricing kernel) \( \frac{\Lambda_T}{\Lambda_0} \) is the product of a risk-free rate discount factor and a Radon-Nikodym derivative (Björk, 2009):

\[ \frac{\Lambda_T}{\Lambda_0} = e^{-rT} \frac{dQ_{GDB}}{dP} \]  

(10)

\[ C_0 = E^P \left[ e^{-rT} \frac{dQ_{GDB}}{dP} \max(V_T - K, 0) \right] \]  

(11)

The Radon-Nikodym derivative of \( Q_{GDB} \) with respect to \( P \), \( \frac{dQ_{GDB}}{dP} \), performs a change in the probability measure, from the physical measure \( P \) to a new probability measure. This change of measure is done using the Girsanov Theorem, which says that one can modify the drift of a Brownian motion process by interpreting that process under a new probability distribution (Björk, 2009). In the end, we will be evaluating the option payoff under the new probability measure \( Q_{GDB} \), equivalent to the initial probability measure \( P \):

\[ C_0 = e^{-rT} E^{Q_{GDB}}[\max(V_T - K, 0)] \]  

(12)

2.2.1. The lump-sum payoff case

We are in the situation in which the value \( V \) at time \( T \) is the sale price of the building (the underlying asset).

Take the illiquid asset \( V \) in equation (2) and the traded risky asset \( S \) in equation (1), which is correlated with \( V \) and with which we can hedge part of the total risk of the project. The stochastic discount factor is the one in equation (6).
Fix the volatility part corresponding to the hedgeable risk to be equal to the Sharpe ratio of the risky asset $S$:

$$\kappa_1 = \frac{\mu_S - r}{\sigma_S}$$  \hfill (13)

The stochastic discount factor in this context becomes:

$$d\Lambda = -r\Lambda dt - \frac{\mu_S - r}{\sigma_S} \Lambda dz - \kappa_2 \Lambda dw$$  \hfill (14)

Next, apply the following Girsanov transformations on the process $V$ for the Brownian motions $z$ and $w$:

$$d z = dz^* - \frac{\mu_S - r}{\sigma_S} dt$$  \hfill (15)

$$d w = dw^* - \kappa_2 dt$$  \hfill (16)

The process for $V$ becomes:

$$d V = (\mu_V - \sigma_V p \frac{\mu_S - r}{\sigma_S} - \sigma_V \sqrt{1 - \rho^2 \kappa_2}) V dt + \sigma_V V (\rho dz^* + \sqrt{1 - \rho^2} dw^*)$$  \hfill (17)

Notice that the Girsanov transformations only affect the drift term and the adjustment they make is equal to the volatility of the stochastic discount factor. The expected return is now lowered by the market price of each type of risk, but proportionately to how much can be hedged and how much is left unhedged ($\rho$ and $\sqrt{1 - \rho^2}$, respectively).

We can now finally understand the intuition behind the chosen process for the stochastic discount factor. Remember our goal: pricing a call option. For any such contract there exists a buyer and a seller, each with his reservation price. The buyer’s reservation price shows the buyer’s maximum valuation and the seller’s reservation price, the seller’s minimum valuation of the contract. For any price lower than his maximum valuation, the buyer will decide to buy. Similarly, for any price higher than his minimum valuation, the seller will agree to sell. Otherwise, the transaction will no longer take place.

The question is: how can we derive such prices? The answer lies in the payoff structure. The call price is positively determined by the value of the underlying at terminal time $T$, $V_T$. 
However, $V$ is a stochastic process positively determined by the drift term $\mu_V$. So, the call price will ultimately be positively determined by $\mu_V$. The only way we can minimise the call price to arrive at the buyer’s reservation price is if we take the lowest possible value of $\mu_V$ and that occurs only when $\kappa_2 = \sqrt{k^2 - \left(\frac{\mu S - r}{\sigma_s}\right)^2}$. Similarly, the only way the call price is maximised to derive the seller’s reservation price is if $\mu_V$ takes the highest value possible, meaning that $\kappa_2 = -\sqrt{k^2 - \left(\frac{\mu S - r}{\sigma_s}\right)^2}$.

We are then dealing with a generic interval $[\text{call}_{\text{min}}, \text{call}_{\text{max}}]$ for the call price, which, in terms of the good-deal bounds pricing, is given by the stochastic discount factors in the interval $\left[-\sqrt{k^2 - \left(\frac{\mu S - r}{\sigma_s}\right)^2}, \sqrt{k^2 - \left(\frac{\mu S - r}{\sigma_s}\right)^2}\right]$. Call$_{\text{min}}$ is the lower bound and call$_{\text{max}}$ is the upper bound for the option price.

Bear in mind though that these are individual transactions. What we are modelling here is not a market for a homogeneous good for which there are numerous buyers and sellers bidding and asking prices at the same time (like a stock market), but a market for individual properties, where occasionally there exists an interested buyer or a seller. Under these conditions, we can only specify the likely interval for the price of vacant land. Eventually, by making additional assumptions about the type of market and whether or not we own the land (i.e. which counterparty we are: the buyer or the seller), we could uniquely determine the value of the vacant land.

The process for $V$ can be re-written as:

$$dV = (r - q_1)Vdt + \sigma_V \left( \rho dz + \sqrt{1 - \rho^2} dw \right)$$

where: $$q_1 = r - \mu_V + \sigma_V \rho \frac{\mu S - r}{\sigma_s} + \sigma_V \sqrt{1 - \rho^2} \kappa_2$$

If $\mu_V$, $\sigma_V$ and $\kappa_2$ are constants, then $V$ follows a lognormal distribution. Remembering equation (12) and noticing that the process for $V$ looks like the process for a stock paying a dividend yield equal to $q_1$, we can express the option price as the Black-Scholes price of a call option on a dividend paying stock:

$$C_0 = V_0 e^{-q_1 T} N(d_1) - K e^{-r T} N(d_2)$$
\[ C_0 = e^{-rT} \left[ V_0 e^{\mu_V - \sigma_V \rho \left( \frac{\mu_S - r}{\sigma_S^2} \right) - \sigma_V \sqrt{1 - \rho^2} \kappa_2} N(d_1) - K N(d_2) \right] \]  

(20)

where:  
\[ d_1 = \frac{\ln \left( \frac{V_0}{K} \right) + \left( \mu_V - \sigma_V \rho \left( \frac{\mu_S - r}{\sigma_S^2} \right) \right) T}{\sigma_V \sqrt{T}} \]
\[ d_2 = d_1 - \sigma_V \sqrt{T} \]
\[ V_0 - \text{forward value} \]
\[ \kappa_2 = -\sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S^2} \right)^2}, \text{ for the upper bound} \]
\[ \kappa_2 = \sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S^2} \right)^2}, \text{ for the lower bound} \]

Following Geltner et al. (2006), \( V_0 \) can be interpreted as the observable value of a newly developed property of the type that would be optimal on the analysed vacant piece of land.

The final option price is clearly a modified version of the Black-Scholes option pricing formula. The modification reflects exactly the adjusted drift term that was used to describe the process \( V \) under the new probability measure \( Q_{GB} \) and which appears in equation (17). The drift is adjusted downwards to reflect the higher degree of uncertainty which exists on an incomplete market compared to a complete market due to the part of the total risk which remains unhedged.

2.2.2. The flow payoff case

We are now in the situation in which the value \( V \) at time \( T \) is no longer the sale price of the building, but the present value of a stream of uncertain cash-flows (rental rates). So, \( V_T \) is an expression of the cash-flows cashed in from time \( T \) onwards, which, similarly to the previous case, we have to discount back to time 0 to find the current value of the vacant land.

Assume a rent process \( R \):

\[ dR = \mu_R dt + \sigma_R R \left( \rho_R dz + \sqrt{1 - \rho_R^2} dw \right) \]  

(21)

where: \( \rho_R \) – correlation coefficient between the rental rates and the traded risky asset
\( dz, dw \) – independent Brownian motions
The value of the building at the maturity of the option will be dependent on the potential rental rates that the investor could charge, meaning that it will be a discounted expected value of all the future cash-flows from rents (i.e. the continuous-time equivalent of a present value expression):

\[ V_T = E^P \left[ \sum_{i=1}^{n} \frac{\Lambda_{T+i}}{\Lambda_T} R_{T+i}/R_T \right] \]  \hspace{1cm} (22)

where: \( n \) – the total number of rental rates (yearly basis)

The stochastic discount factor from time \( T \) onwards, the product of the risk-free rate discount factor and the Radon-Nikodym derivative, is now dependent on the number of time periods ‘\( i \)’ that the building will be rented out:

\[ \frac{\Lambda_{T+i}}{\Lambda_T} = e^{-r(T+i-T)} \frac{dQ_{GDB}}{dP} = e^{-r_i} \frac{dQ_{GDB}}{dP} \]  \hspace{1cm} (23)

Plugging equation (23) into (22), we get the expression for the value of the building at time \( T \) under the new probability measure \( Q_{GDB} \):

\[ V_T = \sum_{i=1}^{n} e^{-r_i} E^{Q_{GDB}} [R_{T+i}/R_T] \]  \hspace{1cm} (24)

Since the value \( V_T \) is expressed in terms of \( R_{T+i} \), it is accurate to say that both \( V \) and \( R \) is driven by the same Brownian motion, so we can choose the same risky asset \( S \) in equation (1) as correlated with the rental rates to partially hedge them with.

The stochastic discount factor to discount the rent process back to time \( T \) is:

\[ d\Lambda = -r\Lambda dt - \frac{\mu_S - r}{\sigma_S} \Lambda dz - \kappa(t)\Lambda dw \]  \hspace{1cm} (25)

Notice that the restriction on the volatility of the stochastic discount factor is not constant anymore, but it is time dependent. The reason is that with a flow type of payoff, the investment horizon is actually split into the time period before \( T \) and after \( T \). From time 0 to time \( T \), the analysis is exactly the same as in the lump-sum payoff case. Once we go beyond \( T \), we add another dimension to the analysis by replacing the market value of the building with a value from a stream of uncertain cash-flows. The same way, the discounting can be divided between the time periods 0 -> \( T \) and \( T \) onwards. However, since we can use the same risky asset \( S \) for the partial hedge throughout the whole investment period (i.e. before
and after $T$) and the restriction $\kappa$ is an expression of the Sharpe ratio of this risky asset, $\kappa(t)$ will turn out to be constant.

We will proceed as in the previous case; first apply the two Girsanov transformations to the process $R$:

$$dz = dz^* - \frac{\mu_S - r}{\sigma_S} dt$$

$$dw = dw^* - \kappa(t) dt$$

and then calculate the new process for $R$:

$$dR = \left[\mu_R - \sigma_R \rho_R \frac{\mu_S - r}{\sigma_S} - \sigma_R \sqrt{1 - \rho_R^2} \kappa(t)\right] R dt + \sigma_R \left(\rho_R dz^* + \sqrt{1 - \rho_R^2} dw^*\right)$$

Assume that $R$ is lognormal (i.e. $\mu_R$ and $\sigma_R$ are constant). Then:

$$E^{Q, GDB}[R_{T+i}/R_T] = R_T e^{\left[\mu_R - \sigma_R \rho_R \left(\frac{\mu_S - r}{\sigma_S}\right) - \sigma_R \sqrt{1 - \rho_R^2} \kappa(t)\right] (T+i-T)}$$

The process for the value of the underlying asset at time $T$ becomes:

$$V_T = R_T \sum_{i=1}^{n} e^{-2ri \left[\mu_R - \sigma_R \rho_R \left(\frac{\mu_S - r}{\sigma_S}\right) - \sigma_R \sqrt{1 - \rho_R^2} \kappa(t)\right]}$$

The sum in equation (31) is $\sum_{i=1}^{n} e^{iq} = \frac{e^{q(1-e^nq)}}{1-e^q}$, which leads to a modified option payoff compared to the lump-sum case, a payoff that is dependent on the rental rates:

$$C_0 = e^{-rT} E^{Q, GDB}[\max(R_T - A, 0)]$$

where:

$$A = \frac{e^{q(1-e^nq)}}{1-e^q}$$

$$q = -2r \left[\mu_R - \sigma_R \rho_R \left(\frac{\mu_S - r}{\sigma_S}\right) - \sigma_R \sqrt{1 - \rho_R^2} \kappa(t)\right]$$

Fortunately, this is again the payoff of an option on a dividend paying stock, so, following equation (19), we can immediately write the closed-form solution to the price of the call option on the real estate asset as:
\[ C_0 = e^{-rT} \left\{ R_0 A e^{\left[ \mu_R - \sigma_R \rho_R \left( \frac{\mu_S - r}{\sigma_S} \right) \sqrt{1 - \rho_R^2} \kappa(t) \right]} N(d_1) - KN(d_2) \right\} \]  

where:  
\[ d_1 = \frac{\ln \left( \frac{R_0}{K} \right) + \left[ \mu_R - \sigma_R \rho_R \left( \frac{\mu_S - r}{\sigma_S} \right) - \sigma_R \sqrt{1 - \rho_R^2} \kappa(t) + \frac{\sigma^2}{2} \right]}{\sigma_R \sqrt{T}} \]  
\[ d_2 = d_1 - \sigma_R \sqrt{T} \]  

\[ R_0 - \text{forward value} \]  

\[ \kappa(t) = -\sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2}, \text{ for the upper bound} \]  

\[ \kappa(t) = \sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2}, \text{ for the lower bound} \]

We will continue with a sensitivity analysis only for the lump-sum payoff case. The results are the same regardless of the type of payoff used, because, for a comparison, the input parameters would have to be calibrated such that the two types of prices are equal.

3. The implications of the good-deal bounds on real-estate assets

3.1. Calibrating the upper and the lower bound

The difficulties of the good-deal bounds pricing mechanism are the choice of the appropriate correlated risky asset and the calibration of \( k \).

The selection of \( k \): Once we have found the correlated risky asset, we can express the restriction on the volatility of the stochastic discount factor in terms of the Sharpe ratio of this asset. Mathematically speaking, \( k \) must be at least equal to the Sharpe ratio of the risky traded asset \( S \) in order for \( \kappa_2 = \left[ -\sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2}, \sqrt{k^2 - \left( \frac{\mu_S - r}{\sigma_S} \right)^2} \right] \) to be defined.

Cochrane and Saa-Requejo (2000) suggest that we set the bound \( k \) equal to twice the market price of risk on the stock market. In other words, we relate the unknown \( k \) to something that we can find out, the Sharpe ratio of a traded asset. In the context of real estate, a good benchmark for imposing the bound \( k \) would be a REIT. As a rule though, the larger the difference between the restriction \( k \) and the Sharpe ratio of the traded risky asset is, the wider the option price bounds become.
Another way to determine $k$ is empirically, by estimating the growth rate of the asset $V$, $\mu_V$ in our notation, from historical data. The confidence interval around this estimate is $\mu_V \pm \frac{2\sigma_V}{\sqrt{T}}$.

Since the drift term for asset $V$ is $\mu_V - \sigma_V \frac{\mu S - R}{\sigma_S} - \sigma_V \sqrt{1 - \rho^2} \kappa_2$ from equation (17), $k$, the restriction on the total volatility of the stochastic discount factor, becomes $\frac{2}{\sqrt{T}}$. Imagine that $T = 25$ years of data. Then $k$ would have a value of 0.4.

**Interpretation of the good-deal bounds:** Cochrane and Saa-Requejo (2000) point towards a nice interpretation of the bounds as a bid-ask spread. The lower bound would correspond to the bid price and the upper bound, to the ask price. The bid and ask prices also relate to our interpretation of the good-deal bounds prices in terms of reservation prices. The buyer’s reservation price shows the buyer’s maximum valuation of an asset and the seller’s reservation price, the seller’s minimum valuation of that asset. For any price lower than his maximum valuation, the buyer will agree to buy and, for any price higher than his minimum valuation, the seller will want to sell. Otherwise, no transaction occurs.

It can be shown that, mathematically, the good-deal bounds are equivalent to the coherent risk measure of Artzner et al. (1999), which searches for the infimum$^3$ of all risk measures over an acceptance set. Placing an upper bound $k$ on the total volatility of the stochastic discount factor ($\kappa_1^2 + \kappa_2^2 \leq k^2$) exactly translates into a minimization over the set of all risk measures. We are actually choosing all measures $Q_{GDB}$ lower than or equal to $k^2$, such that equation (12) can be re-written as:

$$C_0 = \min_{Q_{GDB} \leq k^2} e^{-rT} E^{Q_{GDB}}[(V_T - K)^+]$$

(33)

In this case, the minimum of the payoff leads to the lower bound and the maximum (i.e. the minimum of the negative payoff) leads to the upper bound.

3.2. **The influential parameters**

The standard (complete-market) real options models predict that an increase in the volatility of the underlying asset (the building) always leads to an increase in the value of the option (the value of the vacant land). The good-deal bounds technique, which prices the assets starting from the assumption that the market is incomplete, shows that this is not always the

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$^3$ The infimum (i.e. inf) returns the greatest element of $T$ that is less than or equal to all elements of $S$, where $S$ is a subset of $T$. 
case. This can best be seen graphically in Figure 1, where, all else equal, the volatility of the underlying asset increases from 1% to as much as 35%, but the option prices on the lower bound no longer follow an increasing pattern.

Figure 1 was constructed using the following parameter values: \( \sigma_S = 16\% \), \( \mu_S = 8\% \), \( r = 4\% \), Sharpe ratio asset \( S = 0.25 \), \( k = 0.5 \), \( \rho = 0.8 \), \( V_0 = 100 \), \( K = 70 \), \( T = 1 \) year. Note that we cannot increase \( \sigma_V \) without adjusting \( \mu_V \), which is why we used a CAPM type of approach to calculate the drift term: \( \mu_V = r + \rho \frac{\sigma_V}{\sigma_S} (\mu_S - r) \).

\[ Figure 1: \text{The sensitivity of the call price with respect to the volatility of the underlying – good-deal bounds prices vs. Black-Scholes prices} \]

By fixing the market price of risk and all other parameters except the volatility of the underlying asset, we see that for increasing values of this volatility the lower bound prices display a decreasing pattern, instead of an increasing one as we would expect. This means that, when we don’t own the land (the lower bound prices are the buyer’s reservation prices), we are willing to pay less and less for it as uncertainty increases. This feature cannot be explained by complete-market models, because these take only hedgeable sources of risk into account, not unhedgeable sources as well. However, the results are consistent with the findings of Henderson (2007) and Miao and Wang (2007), who also conclude that market incompleteness can decrease the option value.

For very low values of \( \sigma_V \), the prices converge, because, if we eliminate all the uncertainty in the underlying asset (the illiquid asset), the only source of uncertainty left comes from the
traded asset and we are back in the Black-Scholes framework. As $\sigma_V$ increases though, the effect of the unhedged risk also increases and that is reflected in the steady widening of the bounds.

The graph also shows that the replicating portfolio argument performs poorly when we introduce unhedgeable risk. The higher the uncertainty in the price evolution of the underlying, the farther we get from the Black-Scholes price. What is more, the difference in GDB and BS prices is not negligible. The documented annual volatility for commercial real estate (individual properties) in the US is between 10% and 25% (Geltner et al., 2007). If we look at Figure 1, at 10% volatility, the difference in prices is 2.6, whereas at 25% volatility, it is as high as 6.

A very important parameter for the option price is the restriction $k$ on the total volatility of the stochastic discount factor. We know that the stochastic discount factor is the product of a risk-free rate discount factor and a Radon-Nikodym derivative (Björk, 2009). Its expected value is a constant and it is equal to the risk-free rate discount factor. It is the variance of the stochastic discount factor that changes and that we restrict via $k$. The variance of the stochastic discount factor shows the “distance” between the physical (real world) probability measure $P$ and the new probability measure $Q_{GDB}$. If these two probability measures are very close to each other, the variance of the stochastic discount factor is low and the good-deal bounds are tight. The farther the probability measures get from each other, the higher the variance of the stochastic discount factor is and the wider the bounds become.

The effect of the volatility restriction $k$ on the real option price is presented in Figure 2. When the restriction is exactly equals to the Sharpe ratio of the traded asset, we exit the incomplete market setting and the GDB price converges to the Black-Scholes price. Afterwards, as $k$ increases, the bounds widen. The prices on the lower bound decrease rapidly and the ones on the upper bound experience a sharp increase.

Figure 2 was constructed using the following values: $V_0 = 100$, $K = 70$, $T = 1$ year, $r = 4\%$, $\sigma_V = 15\%$, $\sigma_S = 16\%$, $\mu_S = 8\%$, $\rho = 0.8$, Sharpe ratio $S = 0.25$. The drift term is still given by:

$$\mu_V = r + \rho \frac{\sigma_V}{\sigma_S} (\mu_S - r).$$
The correlation coefficient $\rho$ between the underlying asset and the traded risky asset also plays a role in determining the behaviour of the option prices. Figure 3 shows that as the correlation coefficient increases and the hedge improves, the GDB prices approach the Black-Scholes price more and more. In fact, for a perfect (negative or positive) correlation, the two types of prices equalise. The largest price difference can be observed at the other extreme, when $\rho = 0$, because here we cannot hedge any part of the risk and we deal only with unhedgeable sources of risk.

The interesting part about Figure 3 is the fact that there is a big gap between the prices on an almost complete market (see, for instance, the values at $\rho = 0.9$) and the Black-Scholes price at $\rho = 1$. In fact, the GDB prices approach the Black-Scholes price at the speed of $\sqrt{1 - \rho^2}$. This is consistent with the results of Davis (2006), who also compares the prices for a high correlation coefficient with the complete-market prices via a utility indifference approach.

The example in Figure 3 is based on: $V_0 = 100$, $K = 70$, $T = 1$ year, $r = 4\%$, $\sigma_V = 15\%$, $\sigma_S = 16\%$, $\mu_S = 8\%$, Sharpe ratio $S = 0.25$ and $\mu_V = r + \rho \frac{\sigma_V}{\sigma_S} (\mu_S - r)$. 

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*Figure 2: The sensitivity of the call price with respect to the restriction on the volatility of the stochastic discount factor*
Figure 3: The sensitivity of the call price with respect to the correlation coefficient between the underlying asset and the correlated traded risky asset

One also has the opportunity to analyse the investment from the perspective of the expected return, because, unlike in the Black-Scholes model, on the incomplete market, the expected return $\mu_V$ is still present in the expression for the option price (see equations (20) and (32)). In fact, we can talk about a risk/return trade-off: the higher the risk of a project, the lower the return and option value. This effect is visible in equation (17), where we express the process $V$ under the new probability measure. The drift term is adjusted downwards, more than it would be on a complete market, precisely to account for the presence of both hedgeable and unhedgeable risk on the incomplete market.

Even if we were able to find a traded asset perfectly correlated with our underlying making $\rho$ equal to 1, the pricing formula would still depend on the expected return of the real estate asset (the building). This is because the underlying and the risky asset correlated with it have different expected returns and both have to be taken into account in the pricing mechanism.

4. Conclusion

This paper presents a pricing mechanism for real options in incomplete markets with an application to real estate. We price vacant land as a European call option on a building that could be built on that land using the good-deal bounds pricing technique. By relaxing the assumption of a complete market, we not only price in a more realistic market setting, but we
are also able to distinguish between a lump-sum and a flow payoff and gain further insights into the implications of the real options models for real estate.

The good-deal bounds pricing mechanism establishes a narrow and informative interval for the call option and reveals that land prices behave in a way that contradicts standard real options theory. On an incomplete market, an increase in the volatility of the underlying asset (i.e. an increase in the unheadgeable sources of risk) leads to a widening of the bounds and, consequently, more possible values for the price of vacant land than the Black-Scholes (1973) would predict, due to the presence of the un hedgeable sources of risk additional to the hedgeable risk that we find on a complete market. Specifically, the lower bound prices (the buyer’s prices) decrease as the volatility of the underlying asset increases. This means that, when uncertainty increases, the buyer is willing to pay less and less to acquire the piece of land.

The advantage of the described methodology over other incomplete market techniques is that it arrives at semi-closed form solutions for the option prices and, more importantly, comparable to the Black-Scholes (1973) prices. Even though we generalise the market setting to an incomplete market and bring it closer to real life, we still maintain a reference point, which is the Black-Scholes (1973) result.

The nice thing about analytical formulas is that there is no need to know all the intermediate mathematical calculations, making the technique accessible and easy to implement in practice. All we have to understand is which parameters influence the final price of the option we are trying to evaluate. The difficulties of this approach remain the choice of the appropriate risky asset, which has to be traded and correlated with the underlying, and the calibration of k, the restriction on the stochastic discount factor volatility. However, once we find such a risky asset, for instance a REIT, we can use the Sharpe ratio of that asset as a guideline in setting the restriction on the volatility of the stochastic discount factor.
5. References


