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KALMAN-BUCY FILTERING FOR LINEAR SYSTEMS DRIVEN BY THE COX PROCESS WITH SHOT NOISE INTENSITY AND ITS APPLICATION TO THE PRICING OF REINSURANCE CONTRACTS.

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Abstract

In practical situations, we observe the number of claims to an insurance portfolio but not the claim intensity. It is therefore of interest to try to solve the'filtering problem', that is to obtain the best estimate of the claim intensity on the basis of reported claims. In order to use the Kalman-Bucy filter, based on the Cox process incorporating a shot noise process as claim intensity, we need to approximate it by a Gaussian process. We demonstrate that if the primary event arrival rate of the shot noise process is reasonably large, we can then approximate the intensity, claim arrival and aggregate loss processes by a three-dimensional Gaussian process. We establish weak convergence results. We then use the Kalman-Bucy filter and we obtain the price of reinsurance contracts involving high frequency events.

Keywords: The Kalman-Bucy filter; Gaussian processes; the Cox process; Shot noise process; Piecewise deterministic Markov processes theory; stop-loss reinsurance contract.

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1. Introduction

In insurance modelling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in Cramér (1930), Cox & Lewis (1966), Bühlmann (1970), Cinlar (1975), Gerber (1979) and Medhi (1982). However there has been a significant volume of literature that questions the suitability of the Poisson process in insurance modelling (Seal 1983 and Beard *et al.* 1984). From a practical point of view, there is no doubt that the insurance industry needs a more suitable claim arrival process than the Poisson process that has deterministic intensity.

As an alternative point process to generate the claim arrivals, we can employ the Cox process or a doubly stochastic Poisson process (Cox 1955; Bartlett 1963; Haight 1967; Serfozo 1972; Grandell 1976, 1991, 1997; Brémaud 1981; Consul 1989 and Lando 1994). An important book on Cox processes is the book by Benning and Korolev 2002, where various limit theorems as well as applications in both insurance and finance are discussed. The Cox process provides us with the flexibility to allow the intensity not only to depend on time but also to be a stochastic process. In a recent paper (Dassios and Jang 2003), we demonstrated how the Cox process with shot noise intensity can be used in the pricing of catastrophe reinsurance and derivatives.

As the claim intensity function within the Cox process is not observable, it implies that it can only be observed on the basis of an observed process of reported claims. Thus, we consider the 'filtering problem' to obtain the best estimate of the claim intensity on the basis of the observed process of reported claims or observed accidents (Dassios and Jang 1998).

We start by defining the quantities of interest; these are the doubly stochastic (with a shot-noise intensity) point process of claim arrivals and the aggregate loss process. In section 3, we prove a weak convergence result for the three dimensional process consisting of the intensity, claim arrival and aggregate loss processes. In section 4, we obtain the Kalman-Bucy filter result which is then used in section 5 to price a reinsurance contract.

We employ piecewise deterministic Markov processes, whose theory was developed by Davis (1984), to obtain the original moments of our processes. The piecewise deterministic Markov processes theory is a powerful mathematical tool for examining non-diffusion models. For details, we refer the reader to Davis (1984), Dassios (1987), Dassios and Embrechts (1989), Jang (1998, 2004), Rolski *et al.* (1999) and Dassios and Jang (2003).

For similar results to the ones derived in this paper we refer the reader to Gnedenko and Kolmogorov 1954; Snyder 1975; Kruglov 1976; Davis 1977; Lipster and Shiryayev 1977, 1978; Ahmed 1998 and Benning and Korolev 2002.

2. The Cox process and the shot noise process

The Cox process (or a doubly stochastic Poisson process) can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process). Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Brémaud (1981).

Definition 1. Let $(\Omega,\, F,\, P)$ be a probability space with information structure given by $F = \{\Im_t,\, t \in [0,T]\}$. Let N_t be a point process adapted to F. Let λ_t be a non-negative process adapted to F such that

$$\int_0^t \lambda_s ds < \infty \; \textit{almost surely (no explosions)}.$$

If for all $0 \leq t_1 \leq t_2$ *and* $u \in \mathbb{R}$

$$\mathbb{E}\left\{ e^{iu\left(N_{t_2}-N_{t_1}\right)} | \Im_{t_2}^{\lambda} \right\} = \exp\left\{ \left(e^{iu}-1\right) \int_{t_1}^{t_2} \lambda_s ds \right\} \tag{1}$$

then N_t *is called a* $\Im_t$*-doubly stochastic Poisson process with intensity* λ_t *where* $\Im_t^{\lambda}$ *is the* σ*-algebra generated by* λ *up to time* t*, i.e.* $\Im_t^{\lambda} = \sigma\left\{\lambda_s;\, s \leq t\right\}$.

Equation (1) gives us

$$\Pr\left\{N_{t_2} - N_{t_1} = k | \lambda_s;\, t_1 \leq s \leq t_2\right\} = \frac{\exp\left(-\int_{t_1}^{t_2} \lambda_s ds\right)\left(\int_{t_1}^{t_2} \lambda_s ds\right)^k}{k!}. \tag{2}$$

One of the processes that can be used to measure the impact of primary events is the shot noise process (Cox & Isham, 1980, 1986 and Klüppelberg & Mikosch, 1995). The shot noise process is particularly useful within the claim arrival process as it measures the frequency, magnitude and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of doubly stochastic Poisson process to measure the number of claims due to primary events, i.e. we will use it as a claim intensity function to generate the Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta(t-S_i)}$$

where:

- λ_0 is the initial initial value of λ_t
- $\{Y_i\}_{i=1,2,\ldots}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ $(y > 0)$ and $\mathbb{E}(Y_i) = \mu_1$.
- $\{S_i\}_{i=1,2,\ldots}$ is the sequence representing the event times of a Poisson process M_t with constant intensity ρ.
- δ is the rate of exponential decay.

We also define the aggregate loss process

$$C_t = \sum_{i=1}^{N_t} \aleph_i,$$

where N_t is defined earlier and $\{\aleph_i\}_{i=1,2,\ldots}$ is a sequence of independent and identically distributed random variables representing the claim sizes with distribution function $H(u)$ $(u > 0)$ and $m_1 = \int_0^\infty u dH(u)$.

We also make the additional assumption that the Poisson process M_t and the sequences $\{Y_i\}_{i=1,2,\ldots}$ and $\{\aleph_i\}_{i=1,2,\ldots}$ are independent of each other.

As λ_t is a Markov process, the generator of the process (λ_t, t) acting on a function $f(\lambda, t)$ belonging to its domain is given by

$$A\ f(\lambda, t) = \frac{\partial f}{\partial t} - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho\{\int_0^\infty f(\lambda + y, t)dG(y) - f(\lambda, t)\}. \tag{3}$$

For $f(\lambda, t)$ to belong to the domain of the generator A, it is sufficient that $f(\lambda, t)$ is differentiable w.r.t. λ, t for all λ, t and that $\left| \int_0^\infty f(\lambda + y, t) dG(y) - f(\lambda, t) \right| < \infty$.

Now let us derive the mean and variance of λ_t assuming that λ_0 is given.

Theorem 1. *The expectation of claim intensity process λ_t, assuming that we know λ_0, is given by*

$$\mathbb{E}\left(\lambda_t \mid \lambda_0\right) = \frac{\mu_1 \rho}{\delta} + (\lambda_0 - \frac{\mu_1 \rho}{\delta}) e^{-\delta t}. \tag{4}$$

Proof. Set $f(\lambda) = \lambda$ in (3), then

$$A\lambda = -\delta\lambda + \mu_1 \rho.$$

From $\mathbb{E}(\lambda_t \mid \lambda_0) - \lambda_0 = \mathbb{E}[\int_0^t \{A\ f(\lambda_s) \mid \lambda_0\} ds]$,

$$\mathbb{E}\left(\lambda_t \mid \lambda_0\right) = \lambda_0 - \delta \int_0^t \mathbb{E}\left(\lambda_s \mid \lambda_0\right) ds + \int_0^t \mu_1 \rho ds$$

and differentiate w.r.t. t

$$\frac{d\mathbb{E}\left(\lambda_t \mid \lambda_0\right)}{dt} = -\delta \mathbb{E}\left(\lambda_t \mid \lambda_0\right) + \int_0^t \mu_1 \rho ds.$$

Solving the differential equation, we have

$$\mathbb{E}\left(\lambda_t \mid \lambda_0\right) = \frac{\mu_1 \rho}{\delta} + (\lambda_0 - \frac{\mu_1 \rho}{\delta}) e^{-\delta t}.$$

Lemma 1. *The second moment of claim intensity process λ_t is given by*

$$\mathbb{E}\left(\lambda_t^2 \mid \lambda_0\right) = \lambda_0^2 e^{-2\delta t} + \frac{2\mu_1 \rho}{\delta}\left(\lambda_0 - \frac{\mu_1 \rho}{\delta}\right)\left(e^{-\delta t} - e^{-2\delta t}\right) + \left(\frac{\mu_1^2 \rho^2}{\delta^2} + \frac{\mu_2 \rho}{\delta}\right)(1 - e^{-2\delta t}) \tag{5}$$

where $\mu_2 = \int_0^\infty y^2 dG(y)$.

Proof. Set $f(\lambda) = \lambda^2$ in (3), then from the proof of the previous theorem the result follows immediately.

Corollary 1. *The variance of claim intensity process λ_t is given by*

$$\mathrm{Var}\left(\lambda_t \mid \lambda_0\right) = (1 - e^{-2\delta t}) \frac{\mu_2 \rho}{2\delta}. \tag{6}$$

Proof. From $\mathrm{Var}\left(\lambda_t \mid \lambda_0\right) = \mathbb{E}\left(\lambda_t^2 \mid \lambda_0\right) - \{\mathbb{E}\left(\lambda_t \mid \lambda_0\right)\}^2$, the result follows immediately.

Similarly, the asymptotic (stationary) mean and variance of λ_t can be obtained from Theorem 1 and Corollary 1.

Corollary 2. *If λ_t is stationary, that is λ_0 has the stationary distribution, then*

$$\mathbb{E}\left(\lambda_t\right) = \frac{\mu_1 \rho}{\delta} \tag{7}$$

and

$$\operatorname{Var}\left(\lambda_t\right) = \frac{\mu_2 \rho}{2\delta}. \tag{8}$$

Proof. Let $t \to \infty$ in (4) and (6) then the results follow immediately.

From (2), we have

$$\mathbb{E}(N_t) = \mathbb{E}\left(\int_0^t \lambda_s ds\right) = \mathbb{E}(X_t) \tag{9}$$

where $X_t = \int_0^t \lambda_s ds$ (*the aggregated process*). Hence, assuming that λ_t is stationary, the expectation of claim number process, N_t is given by

$$\mathbb{E}(N_t) = \frac{\mu_1 \rho}{\delta} t. \tag{10}$$

Similarly, assuming that we know λ_0, we can also obtain that the variance of the aggregated process X_t,

$$\operatorname{Var}\left(\int_0^t \lambda_s ds \mid \lambda_0\right) = \operatorname{Var}\left(X_t \mid \lambda_0\right) = \left\{\frac{\mu_2}{\delta^2} t - \frac{2\mu_2}{\delta^3}\left(1 - e^{-\delta t}\right) + \frac{\mu^2}{2\delta^3}\left(1 - e^{-2\delta t}\right)\right\} \rho \tag{11}$$

and assuming that λ_t is stationary, we have

$$\operatorname{Var}\left(\int_0^t \lambda_s ds\right) = \operatorname{Var}\left(X_t\right) = \left(\frac{\mu_2}{\delta^2} t - \frac{\mu_2}{\delta^3} e^{-\delta t} - \frac{\mu_2}{\delta^3}\right) \rho. \tag{12}$$

The reason for the derivation of the variance of X_t will become apparent later when we transform and approximate the Cox and shot noise processes.

Based on small ρ, that is the rate of primary event arrival, Dassios and Jang (2003) used the shot noise process as an intensity function from catastrophic events. However if the rate of primary event arrival, ρ, is large, it implies that the primary events are not catastrophes anymore. Therefore we can consider it as an intensity function to generate the number of claims from common events with high frequency, such as car accidents or accidents from a large collective insurance portfolio, not catastrophic events.

3. Convergence results

We start by introducing the following linear transformations of the processes λ_t, N_t and C_t:

$$Z_t^{(\rho)} = \frac{\lambda_t - \frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \quad \text{i.e.} \quad \lambda_t = \frac{\mu_1 \rho}{\delta} + Z_t^{(\rho)} \sqrt{\frac{\mu_2 \rho}{2\delta}} \tag{13}$$

$$W_t^{(\rho)} = \frac{N_t - \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \quad \text{i.e.} \quad N_t = \frac{\mu_1 \rho}{\delta} t + W_t^{(\rho)} \sqrt{\frac{\mu_2 \rho}{2\delta}} \tag{14}$$

and

$$U_t^{(\rho)} = \frac{C_t - m_1 \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \quad \text{i.e.} \quad C_t = m_1 \frac{\mu_1 \rho}{\delta} t + U_t^{(\rho)} \sqrt{\frac{\mu_2 \rho}{2\delta}}. \tag{15}$$

Let us continue with a proposition by Ethier & Kurtz (1985).

Proposition 1. *For $n = 1, 2, ...,$ let $\{\Im_t^n\}$ be a filtration and let M_n be an $\{\Im_t^n\}$-local martingale with sample paths in $D_{\Re^d}[0,\infty)$ and $M_n(0) = 0$. Let $A_n = \left(\left(A_n^{ij}\right)\right)$ be symmetric $d \times d$ matrix-valued processes such that A_n^{ij} has sample paths in $D_{\Re^d}[0,\infty)$ and $A_n(t) - A_n(s)$ is nonnegative definite for $0 \le s < t$. Assume that*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\le T} \left|A_n^{ij}(t) - A_n^{ij}(t-)\right|\right] = 0, \tag{16}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\le T} \left|M_n(t) - M_n(t-)\right|^2\right] = 0, \tag{17}$$

and for $i,\ j = 1, 2, ..., d$,

$$M_n^i(t)\, M_n^j(t) - A_n^{ij}(t) \tag{18}$$

is an $\{\Im_t^n\}$-local martingale. If for each $t \ge 0$ and $i,\ j - 1, 2, ...d$,

$$A_n^{ij}(t) \to c_{ij}(t) \tag{19}$$

in probability where $C = ((c_{ij}))$ *is a continuous, symmetric,* $d \times d$ *matrix-valued function, defined on* $[0, \infty)$*, satisfying* $C(0) = 0$ *and* $\sum (c_{ij}(t) - c_{ij}(s)) \xi_i \xi_j \geq 0$, $\xi \epsilon \Re^d$*. Then*

$$M_n \Rightarrow X \tag{20}$$

in law where X *is a process with independent Gaussian increments such that* $X_i X_j - c_{ij}$ *are (local) martingales with respect to* $\{\Im_t^n\}$.

Let us now define $V_t^{(\rho)} = \frac{J_t - \mu_1 \rho t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$, $L_t^{(\rho)} = \frac{N_t - \int_0^t \lambda_s ds}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} = \frac{N_t - X_t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$ and $Q_t^{(\rho)} = \frac{C_t - m_1 N_t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$, where $J_t = \sum_{i=1}^{M_t} Y_i$.

Lemma 2. *Assuming that* $\rho \to \infty$,

$$\begin{bmatrix} V_t^{(\rho)} \\ L_t^{(\rho)} \\ Q_t^{(\rho)} \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{2\delta} B_t^{(1)} \\ \sqrt{\frac{2\mu_1}{\mu_2}} B_t^{(2)} \\ \sqrt{k_2 \frac{2\mu_1}{\mu_2}} B_t^{(3)} \end{bmatrix} \tag{21}$$

in law where $B_t^{(1)}, B_t^{(2)}$ *and* $B_t^{(3)}$ *are three independent standard Brownian motions and* $k_2 = \int_0^\infty u^2 dH(u) - \left(\int_0^\infty u dH(u) \right)^2$ *(the variance of claim sizes).*

Proof. The generator of the process $V_t^{(\rho)}$ acting on a function $f(v)$ is given by

$$A\ f(v) = -\frac{\mu_1 \rho}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \frac{\partial f}{\partial v} + \rho \left\{ \int_0^\infty f(v + \frac{y}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}) dG(y) - f(v) \right\}. \tag{22}$$

Setting $f(v) = v^2$ in (22) then

$$Av^2 = 2\delta.$$

The generator of the process $(X_t, N_t, C_t, \lambda_t, J_t, t)$ acting on a function $f(x, n, c, \lambda, j, t)$ is given by

$$\begin{aligned} A\ f(x,n,c,\lambda,j,t) = {} & \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} + \lambda \left\{ \int_0^\infty f(x, n+1, c+u, \lambda, j, t)\, dH(u) - f(x,n,c,\lambda,j,t) \right\} \\ & - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho \left\{ \int_0^\infty f(x,n,c,\lambda+y, j+y, t)\, dG(y) - f(x,n,c,\lambda,j,t) \right\}. \end{aligned} \tag{23}$$

Clearly, for $f(x,n,c,\lambda,j,t)$ to belong to the domain of the generator A, it is essential that $f(x,n,c,\lambda,j,t)$ is differentiable w.r.t. x,c,λ,t for all x,n,c,λ,j,t and that $\left|\int_0^\infty f(\cdot,\lambda+y,\cdot)\,dG(y) - f(\cdot,\lambda,\cdot)\right| < \infty$ and $\left|\int_0^\infty f(\cdot,c+u,\cdot)\,dH(u) - f(\cdot,c,\cdot)\right| < \infty$.

Setting $f(x,n,c,\lambda,j,t) = \left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)^2$ and $f(x,n,c,\lambda,j,t) = \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)^2$ in (23), then

$$A\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)^2 = \frac{2\delta}{\mu_2}\frac{\lambda}{\rho} \text{ and } A\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)^2 = k_2\frac{2\delta}{\mu_2}\frac{\lambda}{\rho}$$

where $m_1 = \int_0^\infty u\,dH(u)$, $m_2 = \int_0^\infty u^2 dH(u)$ and $k_2 = m_2 - m_1^2$.

$f(X_t) - \int_0^t A\ f(X_s)\,ds$ is a martingale therefore $A\ f$ is the solution to the 'martingale problem'. Hence from Proposition 3.1, $\left\{V_t^{(\rho)}\right\}^2 - 2\delta t$, $\left\{L_t^{(\rho)}\right\}^2 - \int_0^t \frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds$ and $\left\{Q_t^{(\rho)}\right\}^2 - \int_0^t k_2\frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds$ are martingales.

It is trivial to check the condition of (16) as $2\delta t$, $\int_0^t \frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds$ and $\int_0^t k_2\frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds$ are continuous (they are proportional to either t or $\int_0^t \lambda_s \mathrm{d}s$). For the condition of (17), we have to look at the jumps of the processes $V_t^{(\rho)}$, $L_t^{(\rho)}$ and $Q_t^{(\rho)}$. Firstly, $L_t^{(\rho)}$ satisfies the condition rather trivially since its jumps are the jumps of $\frac{N_t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}$ and they are of course always of size $\frac{1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}$ and so converge to 0 as $\rho\to\infty$. For $V_t^{(\rho)}$ and $Q_t^{(\rho)}$, we have to check the jumps of $\frac{J_t^2}{\frac{\mu_2\rho}{2\delta}}$ and $\frac{C_t^2}{\frac{\mu_2\rho}{2\delta}}$. The jumps have finite expectation since we have assumed second moments exist for our jumps. We need to prove that for a sequence of non-negative *i.i.d.* random variables $Z_1, Z_2, \ldots$ with finite mean

$$\lim_{n\to\infty}\frac{\mathbb{E}(U_n)}{n} = 0,$$

where $U_n = \max(Z_1, Z_2, \cdots, Z_n)$. In fact, we can prove that $\frac{U_n}{n}\to 0$ in probability. Let F be the distribution function of Z_i. Then for any $\varepsilon > 0$,

$$\Pr\left(\frac{U_n}{n} > \varepsilon\right) = \Pr(U_n > n\varepsilon) = 1 - (F(n\varepsilon))^n = 1 - \left(1 - \bar{F}(n\varepsilon)\right)^n,$$

but since the random variables have a finite expectation, $n\bar{F}(n\varepsilon)\to 0$, so $\left(1-\bar{F}(n\varepsilon)\right)^n \to 1$. Now secondly, for $\frac{C_t^2}{\frac{\mu_2\rho}{2\delta}}$, consider $M^{(\rho)}$ the number of jumps of the Poisson process M_t in the interval $[0,T]$. From above $\frac{C_t^2}{M^{(\rho)}}\to 0$ and since $\frac{M^{(\rho)}}{\rho}\to T$ we are fine.

Lastly, for $\frac{J_t^2}{\frac{\mu_2\rho}{2\delta}}$, consider $N^{(\rho)}$, the number of jumps of the process N_t. Clearly, from above $\frac{J_t^2}{N^{(\rho)}} \to 0$ and also $\frac{N^{(\rho)}}{\rho} \to \frac{\mu_1}{\delta}T$.

As can be seen from (11) (see also (12)) $\text{Var}\left(\int_0^t \lambda_s ds\right) = K(t)\rho$. Therefore, by Chebyschev's inequality, as $\rho \to \infty$

$$\Pr\left\{\left|\int_0^t \frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds - \frac{2\mu_1}{\mu_2}t\right| > \varepsilon\right\} \leq \frac{\left(\frac{2\delta}{\mu_2}\right)^2 \text{Var}\left(\int_0^t \lambda_s ds\right)}{\rho^2\varepsilon^2} = \frac{\left(\frac{2\delta}{\mu_2}\right)^2 K(t)\rho}{\rho^2\varepsilon^2} \to 0 \quad (24)$$

and

$$\Pr\left\{\left|k_2\int_0^t \frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds - k_2\frac{2\mu_1}{\mu_2}t\right| > \varepsilon\right\} \leq \frac{k_2^2\left(\frac{2\delta}{\mu_2}\right)^2 \text{Var}\left(\int_0^t \lambda_s ds\right)}{\rho^2\varepsilon^2} = \frac{k_2^2\left(\frac{2\delta}{\mu_2}\right)^2 K(t)\rho}{\rho^2\varepsilon^2} \to 0. \quad (25)$$

Therefore from (24) and (25)

$$\int_0^t \frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds \to \frac{2\mu_1}{\mu_2}t$$

and

$$\int_0^t k_2\frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds \to k_2\frac{2\mu_1}{\mu_2}t$$

in probability.

Set $f(x,n,c,\lambda,j,t) = \left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)$, $f(x,n,c,\lambda,j,t) = \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)$ and $f(x,n,c,\lambda,j,t) = \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)$ in (23). Then

$$A\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) = 0,\ A\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) = 0 \text{ and } A\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) = 0. \quad (26)$$

Therefore from Proposition 1,

$$V_t^{(\rho)} = \frac{J_t - \mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \Rightarrow \sqrt{2\delta}B_t^{(1)}$$

$$L_t^{(\rho)} = \frac{N_t - \int\limits_0^t \lambda_s ds}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \Rightarrow \sqrt{\frac{2\mu_1}{\mu_2}} B_t^{(2)}$$

and

$$Q_t^{(\rho)} = \frac{C_t - m_1 N_t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \Rightarrow \sqrt{k_2 \frac{2\mu_1}{\mu_2}} B_t^{(3)}$$

in law where $B_t^{(1)}, B_t^{(2)}$ and $B_t^{(3)}$ are three independent standard Brownian motions.

Let us prove the main result of this section.

Theorem 2. *Assuming that $\rho \to \infty$ and that λ_0 is a random variable that is independent of everything else such that $\frac{\lambda_0 - (\mu_1 \rho/\delta)}{\mu_2 \rho / 2\delta}$ converges in distribution to Z_0, $Z_t^{(\rho)}, W_t^{(\rho)}$ and $U_t^{(\rho)}$ converge in law to Z_t, W_t and U_t where*

$$dZ_t = -\delta Z_t dt + \sqrt{2\delta} dB_t^{(1)} \tag{27}$$

$$dW_t = Z_t dt + \sqrt{\frac{2\mu_1}{\mu_2}} dB_t^{(2)} \tag{28}$$

$$dU_t = m_1 dW_t + \sqrt{k_2 \frac{2\mu_1}{\mu_2}} dB_t^{(3)} = m_1 Z_t dt + \sqrt{m_2 \frac{2\mu_1}{\mu_2}} dB_t^{(4)} \tag{29}$$

where $B_t^{(1)}, B_t^{(2)}, B_t^{(3)}$ are three independent standard Brownian motions and $B_t^{(4)} = \frac{m_1 \sqrt{\frac{2\mu_1}{\mu_2}} B_t^{(2)} + \sqrt{k_2 \frac{2\mu_1}{\mu_2}} B_t^{(3)}}{\sqrt{(m_1^2 + k_2)\frac{2\mu_1}{\mu_2}}}$ (also a standard Brownian motion).

Proof. $Z_t^{(\rho)}, W_t^{(\rho)}$ and $U_t^{(\rho)}$ can be written as follows:

$$\begin{aligned} Z_t^{(\rho)} = \frac{\lambda_t - \frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} &= \frac{\lambda_0 - \frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} e^{-\delta t} - \frac{\frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \left(1 - e^{-\delta t}\right) + \frac{J_t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} - \delta \int_0^t e^{-\delta(t-u)} \frac{J_u}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} du \\ &= \frac{\lambda_0 - \frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} e^{-\delta t} + \frac{J_t - \mu_1 \rho t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} - \delta \int_0^t e^{-\delta(t-u)} \frac{J_u - \mu_1 \rho u}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} du \end{aligned} \tag{30}$$

since

$$\delta \int_0^t u e^{-\delta(t-u)} du = t - \frac{1 - e^{-\delta t}}{\delta}$$

$$W_t^{(\rho)} = \frac{N_t - \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} = \frac{N_t - \int\limits_0^t \lambda_s ds}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} + \int\limits_0^t \frac{\lambda_s - \frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} ds \tag{31}$$

and

$$U_t^{(\rho)} = \frac{C_t - m_1 \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} = \frac{C_t - m_1 N_t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} + m_1 \left(\frac{N_t - \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right). \tag{32}$$

Therefore by continuous mapping theorem (Billingsley, 1968) and lemma 2, (30), (31) and (32) converge to

$$Z_t = Z_0 e^{-\delta t} + \sqrt{2\delta} \left(B_t^{(1)} - \int_0^t e^{-\delta(t-s)} B_s^{(1)} ds \right) = Z_0 e^{-\delta t} + \sqrt{2\delta} \int_0^t e^{-\delta(t-s)} dB_s^{(1)} \tag{33}$$

$$W_t = \int_0^t Z_s ds + \sqrt{\frac{2\mu_1}{\mu_2}} B_t^{(2)} \tag{34}$$

and

$$U_t = m_1 W_t + \sqrt{k_2 \frac{2\mu_1}{\mu_2}} B_t^{(3)}. \tag{35}$$

From (34) and (35), we have

$$dU_t = m_1 dW_t + \sqrt{k_2 \frac{2\mu_1}{\mu_2}} B_t^{(3)} = m_1 Z_t dt + m_1 \sqrt{\frac{2\mu_1}{\mu_2}} dB_t^{(2)} + \sqrt{k_2 \frac{2\mu_1}{\mu_2}} dB_t^{(3)} \tag{36}$$

Since the sum of two independent standard Brownian motions is also a standard Brownian motion this completes the proof of the theorem.

Theorem 2 implies that Z_t, W_t and U_t are normally distributed. Therefore we can define $\tilde{\lambda}_t$, $\tilde{N}_t$ and $\tilde{C}_t$ as Gaussian approximations of λ_t, N_t and C_t;

$$\tilde{\lambda}_t = \frac{\mu_1 \rho}{\delta} + Z_t \sqrt{\frac{\mu_2 \rho}{2\delta}} \quad \text{i.e.} \quad Z_t = \frac{\tilde{\lambda}_t - \frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \tag{37}$$

$$\tilde{N}_t = \frac{\mu_1 \rho}{\delta} + W_t \sqrt{\frac{\mu_2 \rho}{2\delta}} \quad \text{i.e.} \quad W_t = \frac{\tilde{N}_1 - \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \tag{38}$$

and

$$\tilde{C}_t = m_1 \frac{\mu_1 \rho}{\delta} + U_t \sqrt{\frac{\mu_2 \rho}{2\delta}} \quad \text{i.e.} \quad U_t = \frac{\tilde{C}_t - m_1 \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}. \tag{39}$$

4. The Kalman-Bucy filter and the distribution of $\mathbf{Z_t}$

Let us derive the conditional distribution of Z_t, given $\{W_s; 0 \le s \le t\}$ by the Kalman-Bucy filter where

$$dZ_t = -\delta Z_t dt + \sqrt{2\delta} dB_t^{(1)} \tag{40}$$

and

$$dW_t = Z_t dt + \sqrt{\frac{2\mu_1}{\mu_2}} dB_t^{(2)}. \tag{41}$$

To do so, we begin with a proposition used by Øksendal (1992) (see theorem 6.10 in chapter IV).

Proposition 2. *The solution $\hat{Z}_t = \mathbb{E}(Z_t | W_s; 0 \leq s \leq t)$ of the 1-dimensional linear filtering problem*

$$dZ_t = F(t) Z_t dt + C(t) dB_t^{(1)};\ F(t),\ C(t) \epsilon \Re \tag{42}$$

$$dW_t = G(t) Z_t dt + D(t) dB_t^{(2)};\ G(t), D(t) \epsilon \Re \tag{43}$$

satisfies the stochastic differential equation

$$d\hat{Z}_t = \left\{ F(t) - \frac{G^2(t) S(t)}{D^2(t)} \right\} \hat{Z}_t dt + \frac{G(t) S(t)}{D^2(t)} dW_t;\ \hat{Z}_0 = \mathbb{E}(Z_0) \tag{44}$$

where $S(t) = \mathbb{E}\left\{ \left(Z_t - \hat{Z}_t \right)^2 \right\}$ *satisfies the Riccati equation*

$$\frac{dS}{dt} = 2F(t) S(t) - \frac{G^2(t)}{D^2(t)} S^2(t) + C^2(t), \quad S(0) = \mathbb{E}\left[\{ Z_0 - \mathbb{E}(Z_0) \}^2 \right] = \mathrm{Var}(Z_0). \tag{45}$$

Theorem 3. *Let (Z_t, W_t) be a two-dimensional normal process satisfying the system of equations of (40) and (41). Then the estimate of Z_t based on the observed $\{W_s; 0 \leq s \leq t\}$ is*

$$\hat{Z}_t = \mathbb{E}(Z_t \mid W_s; 0 \leq s \leq t) = \exp\{\int_0^t \Psi(s) ds\} \hat{Z}_0 + \frac{\mu_2}{2\mu_1} \int_0^t \exp\{\int_s^t \Psi(u) du\} S(s) dW_s \tag{46}$$

where

$$S(s) = \frac{\xi(1+\eta)}{\eta - 1} - 2\delta \frac{\mu_1}{\mu_2} \tag{47}$$

and

$$\Psi(s) = -\frac{\xi(1+\eta)}{\frac{2\mu_1}{\mu_2}(\eta - 1)} \tag{48}$$

where

$$\xi = \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)},\ \eta = \frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s\right)$$

and $S(0) = a^2$.

Proof. Let $S(0) = a^2$. Then from (45) the Riccati equation has the solution

$$S(t) = \frac{\xi(1+\varphi)}{\varphi-1} - 2\delta\frac{\mu_1}{\mu_2} \tag{49}$$

where $\xi = \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}$ and $\varphi = \frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}t\right)$.

Therefore, from (44), (49) offers the solution for $\hat{Z}_t$ of the form

$$\hat{Z}_t = \mathbb{E}(Z_t \mid W_s; 0 \le s \le t) = \exp\{\int_0^t \Psi(s)ds\}\hat{Z}_0 + \frac{\mu_2}{2\mu_1}\int_0^t \exp\{\int_s^t \Psi(u)du\}S(s)dW_s$$

where

$$\Psi(s) = -\frac{\xi(1+\eta)}{\frac{2\mu_1}{\mu_2}(\eta-1)}$$

where $\eta = \frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}\exp\left(\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s\right)$.

Now we can easily obtain the conditional distribution of Z_t, given $\{W_s; 0 \le s \le t\}$ as we have obtained $\mathbb{E}(Z_t \mid W_s; 0 \le s \le t) = \hat{Z}_t$.

Corollary 3. *Let Z_t, W_t, $\hat{Z}_t$ and $S(t)$ be as defined. Then the conditional distribution of Z_t, given $\{W_s; 0 \le s \le t\}$ is given by*

$$\mathbb{E}\left(e^{-\gamma Z_t} \mid W_s; 0 \le s \le t\right) = \exp\left\{-\gamma\hat{Z}_t + \frac{1}{2}\gamma^2 S(t)\right\}. \tag{50}$$

Proof. From theorem 3 and the fact that $\mathrm{Var}(Z_t \mid W_s; 0 \le s \le t) = S(t)$ and Z_t is normally distributed, the result follows immediately.

It will be interesting to examine 'the filtering problem' for the Cox process with shot noise intensity when the rate of primary event arrival, ρ, is small. It should be necessary for us to remove the linearity and derive a non-linear filter to obtain premiums for primary events with low frequency, using numerical framework.

5. Pricing of a reinsurance contract using the Kalman-Bucy filter

We have transformed and approximated λ_t and N_t as normal variables Z_t and W_t from which we have obtained the conditional distribution of Z_t, given $\{W_s; 0 \leq s \leq t\}$. Now let us derive the pricing model for stop-loss reinsurance contract using normal variables Z_t and W_t. As mentioned earlier, as we have assumed that $\rho \to \infty$, this approach can be used for the pricing of common events with high frequency such as car accidents or accidents from a large collective insurance portfolio.

Let $\aleph_i$, $i = 1, 2, \cdots$, be the claim amounts, which are assumed to be independent and identically distributed with distribution function. The actuarial stop-loss reinsurance premium at time t is

$$\mathbb{E}\left[\left(\sum_{i=1}^{N_T - N_t} \aleph_i - b\right)^+ \mid N_s; 0 \leq s \leq t\right] \tag{51}$$

where b is a suitably large retention limit. In particular we set

$$b = \sqrt{\frac{\mu_2 \rho}{2\delta}}\beta + m_1 \frac{\mu_1 \rho}{\delta}(T - t). \tag{52}$$

Let $C_T - C_t$ be the total amount of claims between time T and t. Then from (51), the stop-loss reinsurance premium at time t becomes

$$\mathbb{E}\left[\{(C_T - C_t) - b\}^+ | N_s; 0 \leq s \leq t\right]. \tag{53}$$

Since we have obtained $\tilde{C}_t$ and $\tilde{N}_t$ which are Gaussian approximations of C_t and N_t, we will use these approximations (see (38) and (39)). Therefore set $\tilde{C}_t = m_1 \frac{\mu_1 \rho}{\delta} t + U_t \sqrt{\frac{\mu_2 \rho}{2\delta}}$ in (53) then

$$\mathbb{E}\left[\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+ \mid \tilde{N}_s; 0 \leq s \leq t\right] = \sqrt{\frac{\mu_2 \rho}{2\delta}}\mathbb{E}\left[\{U_T - U_t - \beta\}^+ \mid W_s; 0 \leq s \leq t\right] \tag{54}$$

Let us derive the expectation and variance of $U_T - U_t$ as they need to be determined to obtain stop-loss reinsurance premium based on (54).

Lemma 3. *The expectation of $U_T - U_t$ is given by*

$$\mathbb{E}\left(U_T - U_t \mid W_s; 0 \leq s \leq t\right) = m_1 \frac{1 - e^{-\delta(T-t)}}{\delta} \hat{Z}_t \tag{55}$$

and the variance of $U_T - U_t$ *is given by*

$$\mathrm{Var}\left(U_T - U_t \mid W_s; 0 \le s \le t\right) =$$

$$\left(\frac{m_1}{\delta}\right)^2 \left[\left\{1 - e^{-\delta(T-t)}\right\}^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right] + 2\left(\frac{m_1^2}{\delta} + \frac{m_2\mu_1}{\mu_2}\right)(T-t). \tag{56}$$

Proof. From (29)

$$U_T - U_t = m_1 \int_t^T Z_s ds + \sqrt{m_2 \frac{2\mu_1}{\mu_2}} \int_t^T dB_s^{(4)} \tag{57}$$

Set (33) in (57) then

$$U_T - U_t = m_1 \frac{1 - e^{-\delta(T-t)}}{\delta} Z_t + m_1\sqrt{2\delta} \int_t^T \frac{1 - e^{-\delta(T-u)}}{\delta} dB_u^{(1)} + \sqrt{m_2 \frac{2\mu_1}{\mu_2}} \int_t^T dB_s^{(4)} \tag{58}$$

Take expectation in (58) then (55) follows immediately. Also from

$$\mathrm{Var}\left(Z_t \mid W_s; 0 \le s \le t\right) = S(t), \tag{59}$$

(56) follows.

We can now easily find the stop-loss reinsurance premium at time t based on the observations $\{W_s; 0 \le s \le t\}$.

Theorem 4. *The stop-loss reinsurance premium at time t based on the observations* $\{W_s; 0 \le s \le t\}$ *is given by*

$$\mathbb{E}\left[\left\{\left(\tilde{C}_T - \tilde{C}_t\right) - b\right\}^+ \mid W_s; 0 \le s \le t\right] = \sqrt{\frac{\mu_2 \rho \Sigma}{4\delta\pi}} e^{-\frac{1}{2}L^2} + \sqrt{\frac{\mu_2 \rho}{2\delta}} (\Omega - \beta)\, \Phi(-L) \tag{60}$$

where

$$\Omega = \mathbb{E}\left(U_T - U_t \mid W_s; 0 \le s \le t\right) = m_1 \frac{1 - e^{-\delta(T-t)}}{\delta} \hat{Z}_t, \tag{61}$$

$$\Sigma = \mathrm{Var}\left(U_T - U_t \mid W_s; 0 \le s \le t\right) = \tag{62}$$

$$\left(\frac{m_1}{\delta}\right)^2\left[\left\{1-e^{-\delta(T-t)}\right\}^2 S(t)-e^{-2\delta(T-t)}+4e^{-\delta(T-t)}-3\right]+2\left(\frac{m_1^2}{\delta}+\frac{m_2\mu_1}{\mu_2}\right)(T-t), \tag{63}$$

$L=\frac{\beta-\Omega}{\sqrt{\Sigma}}$ *and* $\Phi(\cdot)$ *is the cumulative normal distribution function.*

Proof. From (54), we have

$$\mathbb{E}\left[(U_T-U_t-\beta)^+ \mid W_s; 0\leq s\leq t\right]=\int_\beta^\infty (\upsilon-\beta)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2}\frac{(\upsilon-\Omega)^2}{\Sigma}}d\upsilon \tag{64}$$

where

$$\Omega=\mathbb{E}\left(U_T-U_t \mid W_s; 0\leq s\leq t\right)=m_1\frac{1-e^{-\delta(T-t)}}{\delta}\hat{Z}_t,$$

$$\Sigma=\text{Var}\left(U_T-U_t \mid W_s; 0\leq s\leq t\right)=$$

$$\left(\frac{m_1}{\delta}\right)^2\left[\left\{1-e^{-\delta(T-t)}\right\}^2 S(t)-e^{-2\delta(T-t)}+4e^{-\delta(T-t)}-3\right]+2\left(\frac{m_1^2}{\delta}+\frac{m_2\mu_1}{\mu_2}\right)(T-t).$$

Set $y=\frac{\upsilon-\Omega}{\sqrt{\Sigma}}$ in (64) and put $L=\frac{\beta-\Omega}{\sqrt{\Sigma}}$ then multiply both sides by $\sqrt{\frac{\mu_2\rho}{2\delta}}$ and (60) follows.

The following example illustrates the calculation of premiums for stop-loss reinsurance contract for high frequency events using the pricing model derived.

Example The numerical values used to simulate the claim arrival process are $\delta=0.5$, $\lambda_0=200$. We will assume that $\rho=100$, i.e. the interarrival time between jumps is exponential with mean 0.01 and that the jump size follows exponential with mean 1, i.e. $y \sim Exponential(1)$. *S-Plus* was used to generate random values and to simulate the claim arrival process. The numerical values used to calculate (46) and (60) are

$$\begin{aligned}\hat{Z}_0 &= 0,\ S(0)=0,\ \theta=0.1,\ \mu_1=1,\ \mu_2=2,\ m_1=1,\ m_2=3,\ t=1,\ T=2,\\ b &= 0,\ 180,\ 190,\ 200,\ 210,\ 220\end{aligned}$$

where from (10) and the compound Poisson, we have

$$\mathbb{E}=(C_T-C_t)=\mathbb{E}(N_T-N_t)\,\mathbb{E}(\aleph)=\frac{\mu_1\rho}{\delta}m_1=200.$$

By computing (46) and (60) using *MAPLE* and *S-Plus*, where $\hat{Z}_1=0.5579152$, the calculation of stop-loss reinsurance premiums for high frequency events at each retention level b, with/without a relative security loading factor θ, are shown in Table 5.1.

Table 5.1

Retention level b	Net reinsurance premium ($\theta = 0$)	Risk reinsurance premium ($\theta = 0.1$)
0	206.21	226.83
180	26.58	29.24
190	18.06	19.87
200	11.00	12.10
210	5.77	6.35
220	2.41	2.66

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