The distribution of the interval of the Cox process with shot noise intensity for insurance claims and its moments

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Abstract. Applying piecewise deterministic Markov processes theory, the probability generating function of the Cox process, incorporating with shot noise process as the claim intensity, is obtained. We also derive the Laplace transform of the distribution of the shot noise process at claim jump times, using stationary assumption of the shot noise process at any times. Based on this Laplace transform and from the probability generating function of the Cox process with shot noise intensity, we obtain the distribution of the interval of the Cox process with shot noise intensity for insurance claims and its moments, i.e. mean and variance.

Key words: Piecewise deterministic Markov processes theory; martingale; stopping time; the distribution of the interval the Cox process with shot noise intensity; insurance claims.

1. Introduction

In insurance modeling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in Cramér (1930), Cox and Lewis (1966), Bühlmann (1970), Cinlar (1975), Gerber (1979) and Medhi (1982). However there has been a significant volume of literature that questions the suitability of the Poisson process in insurance modeling (Seal 1983 and Beard et al. 1984). From a practical point of view, there is no doubt that the insurance industry needs a more suitable claim arrival process than the Poisson process that has deterministic intensity.

As an alternative point process to generate the claim arrivals, we can employ the Cox process or a doubly stochastic Poisson process (Cox 1955; Bartlett 1963; Haight 1967; Serfozo 1972; Grandell 1976, 1991, 1997; Brémaud 1981 and Lando 1994). An important book on Cox processes is the book by Benning and Korolev 2002, where the applications in both insurance and finance are discussed. The Cox process provides us with the flexibility to allow the intensity not only to depend on time but also to be a stochastic process. In a recent paper (Dassios and Jang 2003), we demonstrated how the Cox process with shot noise intensity could be used in the pricing of catastrophe reinsurance and derivatives.

It is important to measure the time interval between the claims in insurance. Thus in this paper, we examine the distribution of the interval of the Cox process with shot noise intensity for insurance claims. The result of this paper can be used or easily modified in computer science/telecommunications modeling, electrical engineering and queueing theory.

We start by defining the quantity of interest; this is the doubly stochastic (with a shot-noise intensity) point process of claim arrivals. Then we derive the probability generating function of the Cox process with shot noise intensity using piecewise deterministic Markov processes (PDMP) theory, whose theory was developed by Davis (1984). The piecewise deterministic Markov processes theory is a powerful mathematical tool for examining non-diffusion models. For details, we refer the reader to Davis (1984), Dassios (1987), Dassios and Embrechts (1989), Jang (1998, 2004), Rolski et al. (1999), Dassios and Jang (2003) and Jang and Krvavych (2004). In section 3, we derive the Laplace transform of the distribution of the shot noise process at claim times, using stationary assumption of the shot noise process at any times. Using this Laplace transform within the probability generating function of the Cox process with shot noise intensity, we derive the distribution of the interval of the Cox process with shot noise intensity for insurance claims and its moments, i.e. mean and variance. In section 4 concludes.

2. The Cox process and the shot noise process

The Cox process (or a doubly stochastic Poisson process) can be viewed as a two-step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process). Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Brémaud (1981).

Definition 2.1 *Let* (Ω, F, P) *be a probability space with information structure given by* $F = \{\Im_t,\ t \in [0,\ T]\}$. *Let* N_t *be a point process adapted to F. Let* λ_t *be a non-negative process adapted to F such that*

$$\int_0^t \lambda_s ds < \infty \quad \textit{almost surely (no explosions).}$$

If for all $0 \le t_1 \le t_2$ *and* $u \in \Re$

$$E\left\{e^{iu\left(N_{t_2} - N_{t_1}\right)} \middle| \Im^{\lambda}_{t_2}\right\} = \exp\left\{\left(e^{iu} - 1\right)\int_{t_1}^{t_2} \lambda_s ds\right\} \tag{2.1}$$

then N_t *is called a* $\Im_t$*-doubly stochastic Poisson process with intensity* λ_t *where* $\Im^{\lambda}_t$ *is the* $\sigma -$ *algebra generated by* λ *up to time t, i.e.* $\Im^{\lambda}_t = \sigma\{\lambda_s;\ s \le t\}$.

Equation (2.1) gives us

$$\Pr\left\{N_{t_2} - N_{t_1} = k \middle| \lambda_s; t_1 \le s \le t_2\right\} = \frac{\exp\left(-\int_{t_1}^{t_2} \lambda_s ds\right)\left(\int_{t_1}^{t_2} \lambda_s ds\right)^k}{k!} \tag{2.2}$$

and

$$\Pr\left\{\tau_2 > t \middle| \lambda_s; t_1 \le s \le t_2\right\} = \Pr\left\{N_{t_2} - N_{t_1} = 0 \middle| \lambda_s; t_1 \le s \le t_2\right\} = \exp\left(-\int_{t_1}^{t_2} \lambda_s ds\right) \tag{2.3}$$

where τ_k denotes the length of the time interval between the $(k-1)^{\text{th}}$ and the k^{th} claims. Therefore from (2.3), we can easily find that

$$\Pr(\tau_2 \le t) = E\left\{\lambda_{t_2} \exp(-\int_{t_1}^{t_2} \lambda_s ds)\right\}. \tag{2.4}$$

If we consider the process $\Lambda_t = \int_0^t \lambda_s ds$ (*the aggregated process*), then from (2.2) we can also easily find that

$$E(\theta^{N_{t_2} - N_{t_1}}) = E\left\{e^{-(1-\theta)(\Lambda_{t_2} - \Lambda_{t_1})}\right\}. \tag{2.5}$$

The equation (2.5) suggests that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of Λ_t, the aggregated process. It means that we just have to find the p.g.f. (probability generating function) of N_t to retrieve the m.g.f. (moment generating function) of Λ_t and vice versa.

One of the processes that can be used to measure the impact of primary events is the shot noise process (Cox and Isham, 1980, 1986 and Klüppelberg and Mikosch, 1995). The shot noise process is particularly useful within the claim arrival process as it measures the frequency, magnitude and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of doubly stochastic Poisson process to measure the number of claims due to primary events, i.e. we will use it as a claim intensity function to generate the Cox process. We will adopt the shot noise process used by Cox and Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta(t-S_i)}$$

where

- λ_0 is initial value of λ_t.
- $\{Y_i\}_{i=1,2,\cdots}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ $(y>0)$, where $E(Y_i)=\mu_1$.
- $\{S_i\}_{i=1,2,\cdots}$ is the sequence representing the event times of a Poisson process M_t with constant intensity ρ.
- δ is rate of exponential decay.

We assume that the Poisson process M_t and the sequences $\{Y_i\}_{i=1,2,\cdots}$ are independent of each other.

The generator of the process $(\Lambda_t, \lambda_t, t)$ acting on a function $f(\Lambda, \lambda, t)$ belonging to its domain is given by

$$\mathrm{A}\, f(\Lambda, \lambda, t) = \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} - \delta\lambda \frac{\partial f}{\partial \lambda} + \rho [\int_0^\infty f(\Lambda, \lambda + y, t) dG(y) - f(\Lambda, \lambda, t)]. \qquad (2.6)$$

For $f(\Lambda, \lambda, t)$ to belong to the domain of the generator A, it is sufficient that $f(\Lambda, \lambda, t)$ is differentiable w.r.t. Λ, λ, t for all Λ, λ, t and that $\left| \int_0^\infty f(\cdot, \lambda + y, \cdot) dG(y) - f(\cdot, \lambda, \cdot) \right| < \infty$.

Let us find a suitable martingale in order to derive the p.g.f. (probability generating function) of N_t at time *t* (Dassios 1987).

Theorem 2.2 *Let us assume that Λ_t and λ_t evolve up to a fixed time t^*. Considering constants k_1 and k_2 are such that $k_1 \geq 0$ and $k_2 \geq -k_1 e^{-\delta t^*}$,*

$$\exp\left(-k_1 \delta \Lambda_t\right) \exp\left\{-(k_1 + k_2 e^{\delta t}) \lambda_t\right\} \exp\left[\rho \int_0^t \{1 - \hat{g}(k_1 + k_2 e^{\delta s})\} ds\right] \qquad (2.7)$$

is a martingale.

Proof

Define $W_t = \delta \Lambda_t + \lambda_t$ and $Z_t = \lambda_t e^{\delta t}$, then the generator of the process (W_t, Z_t, t) acting on a function $f(w, z, t)$ is given by

$$\mathrm{A}\, f(w, z, t) = \frac{\partial f}{\partial t} + \rho [\int_0^\infty f(w + y, z + y e^{\delta t}, t) dG(y) - f(w, z, t)]. \qquad (2.8)$$

and $f(w, z, t)$ has to satisfy $\mathrm{A} f = 0$ for $f(W_t, Z_t, t)$ to be a martingale. Setting $e^{-k_1 w} e^{-k_2 z} h(t)$ we get the equation

$$h'(t) - \rho [1 - \hat{g}(k_1 + k_2 e^{\delta t})] h(t) = 0. \qquad (2.9)$$

$e^{-k_1 w} e^{-k_2 z} h(t)$ belongs to the domain of the generator because of our choice of k_1, k_2; the function is bounded for all $t \leq t^*$ and our process evolves up to time t^* only. Solving (2.9)

$$h(t) = \mathrm{K} e^{\rho \int_0^t \{1 - \hat{g}(k_1 + k_2 e^{\delta s})\} ds} \qquad (2.10)$$

where K is an arbitrary constant. Therefore

$$e^{-k_1 W_t} e^{-k_2 Z_t} e^{\rho \int_0^t \{1 - \hat{g}(k_1 + k_2 e^{\delta s})\} ds}$$

is a martingale and hence the result follows.

Corollary 2.3 *Let* $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\nu \geq 0$, $0 \leq \theta \leq 1$ *and* t_1, t_2 *be fixed times. Then*

$$E\{e^{-\nu_1(\Lambda_{t_2}-\Lambda_{t_1})}e^{-\nu_2\lambda_{t_2}} \mid \Lambda_{t_1}, \lambda_{t_1}\}$$

$$= \exp\left[-\{\tfrac{\nu_1}{\delta} + (\nu_2 - \tfrac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}\right]\exp\left[-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\tfrac{\nu_1}{\delta} + (\nu_2 - \tfrac{\nu_1}{\delta})e^{-\delta s}\}]ds\right] \quad (2.11)$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}} \mid N_{t_1}, \lambda_{t_1}\}$$

$$= \exp\left[-\{\tfrac{1-\theta}{\delta} + (\nu - \tfrac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}\right]\exp\left[-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\tfrac{1-\theta}{\delta} + (\nu - \tfrac{1-\theta}{\delta})e^{-\delta s}\}]ds\right]. \quad (2.12)$$

Proof

We set $k_1 = \dfrac{\nu_1}{\delta}$, $k_2 = (\nu_2 - \dfrac{\nu_1}{\delta})e^{-\delta t_2}$, $t^* \geq t_2$ in Theorem 2.2 and (2.11) follows immediately. (2.12) follows from (2.11) and (2.5).

Now we can easily derive the p.g.f. (probability generating function) of N_t and the Laplace transform of λ_t using Corollary 2.3.

Corollary 2.4 *The probability generating function of* N_t *is given by*

$$E\{\theta^{(N_{t_2}-N_{t_1})} \mid \lambda_{t_1}\} = \exp\left[-\tfrac{1-\theta}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}\right]\exp\left[-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\tfrac{1-\theta}{\delta}(1-e^{-\delta s}\}]ds\right], \quad (2.13)$$

the Laplace transform of the distribution of λ_t *is given by*

$$E\left\{e^{-\nu\lambda_t} \mid \lambda_0\right\} = \exp\left(-\nu\lambda_0 e^{-\delta t}\right)\exp\left[-\rho\int_0^t\{1-\hat{g}(\nu e^{-\delta s})\}ds\right] \quad (2.14)$$

and if λ_t *is asymptotic* (*stationary*), *it is given by*

$$E(e^{-\nu\lambda_t}) = \exp\left[-\rho\int_0^\infty\{1-\hat{g}(\nu e^{-\delta s})\}ds\right] \quad (2.15)$$

which can also be written as

$$E(e^{-\nu\lambda_t}) = \exp\left\{-\frac{\rho}{\delta}\int_0^\nu \hat{\boldsymbol{G}}(u)du\right\} \quad (2.16)$$

where $\hat{\boldsymbol{G}}(u) = \dfrac{1-\hat{g}(u)}{u}$.

Proof

If we set $\nu = 0$ in (2.12) then (2.13) follows. (2.14) follows if we either set $\nu_1 = 0$ in (2.11) or set $\theta = 1$ in (2.12). Let $t \to \infty$ in (2.14) and the result follows immediately.

If we differentiate (2.14) and (2.16) with respect to θ and put $\theta = 1$, we can easily obtain the first moments of λ_t, i.e.

$$E(\lambda_t | \lambda_0) = \frac{\mu_1 \rho}{\delta} + (\lambda_0 - \frac{\mu_1 \rho}{\delta}) e^{-\delta t} \tag{2.17}$$

and

$$E(\lambda_t) = \frac{\mu_1 \rho}{\delta} \ . \tag{2.18}$$

The higher moments can be obtained by differentiating them further, i.e.

$$Var(\lambda_t | \lambda_0) = (1 - e^{-2\delta t}) \frac{\mu_2 \rho}{2\delta} \tag{2.19}$$

and

$$Var(\lambda_t) = \frac{\mu_2 \rho}{2\delta} \tag{2.20}$$

where $\mu_2 = E(Y^2) = \int_0^\infty y^2 dG(y)$.

3. The distribution of the interval of the Cox process with shot noise intensity and its moment

Let us examine the Laplace transform of the distribution of the shot noise intensity at claim times. To do so, let us denote the time of the n^{th} claim of N_t by τ_n and denote the value of λ_t, when N_t takes the value *n* for the first time by λ_{τ_n}. Since a claim occurs at time τ, this implies that the intensity at claim times, λ_τ, should be higher than the intensity at any times λ_t. Therefore the distribution of λ_τ should not be the same as the distribution of λ_t.

Let us start with the following lemma in order to obtain the Laplace transform of the distribution of the shot noise intensity at claim times. We assume that the claims and jumps (or primary events) in shot noise intensity do not occur at the same time.

Lemma 3.1 *Suppose that $f(\lambda)$ is a function belonging to its domain and furthermore that it satisfies*

$$\lim_{t \to \infty} E\{ f(\lambda_t) \exp(-\int_0^t \lambda_s ds) | \lambda_0 \} = 0 \ . \tag{3.1}$$

If $h(\lambda)$ is such that

$$\lambda \{ h(\lambda) - f(\lambda) \} + \mathrm{A} f(\lambda) = 0 \tag{3.2}$$

then

$$E\{ h(\lambda_{\tau_1}) | \lambda_0 \} = f(\lambda_0) . \tag{3.3}$$

Proof

From (3.2)

$$f(\lambda_t) + \int_0^t [\lambda_s\{h(\lambda_s) - f(\lambda_s)\}]ds$$

is a martingale and since $\tau_1 \wedge t$ is a stopping time, where $\Pr(\tau_1 \le s) = \Pr(N_s > 0)$ and N_s is λ_s-measurable, we have

$$E\{f(\lambda_{\tau_1 \wedge t}|\lambda_0) + E[\int_0^{\tau_1 \wedge t}[\lambda_s\{h(\lambda_s) - f(\lambda_s)\}]ds|\lambda_0] = f(\lambda_0)\,. \tag{3.4}$$

If we now place a condition on the realisation $\lambda_v;\ 0 \le v \le t$, then the first term of the left-hand side in (3.4) is

$$E\{f(\lambda_{\tau_1 \wedge t}|\lambda_0)\} = \int_\Omega E\{f(\lambda_{\tau_1 \wedge t})|\lambda_v; 0 \le v \le t\} \cdot dP(\lambda_v; 0 \le v \le t) \tag{3.5}$$

and the second term of the left-hand side in (3.4) is

$$E[\int_0^{\tau_1 \wedge t}[\lambda_s\{h(\lambda_s) - f(\lambda_s)\}]ds|\lambda_0] = \int_\Omega E[\int_0^{\tau_1 \wedge t}\lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds|\lambda_v; 0 \le v \le t] \cdot dP(\lambda_v; 0 \le v \le t) \tag{3.6}$$

where $dP(\lambda_v; 0 \le v \le t)$ is the probability differential of a particular realisation in Ω, the set of all possible realisations.

Since $\tau_1 \wedge t$ is distributed with density, $\lambda_r \exp(-\int_0^r \lambda_s ds)$ on $(0,t)$ and a mass, $\exp(-\int_0^t \lambda_s ds)$ at *t*, conditionally on $\lambda_r; 0 \le r \le t$, where N_t is the Cox process, we have

$$E\{f(\lambda_{\tau_1 \wedge t})|\lambda_v; 0 \le v \le t\} = \int_0^t \{f(\lambda_r)\lambda_r \exp(-\int_0^r \lambda_s ds)\}dr + f(\lambda_t)\exp(-\int_0^t \lambda_s ds) \tag{3.7}$$

and

$$E[\int_0^{\tau_1 \wedge t}\lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds|\lambda_v; 0 \le v \le t]$$

$$= \int_0^t [\int_0^r \lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds \Pr(\tau_1 = r)]dr + \int_0^t \lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds \Pr(\tau_1 > t)$$

$$= \int_0^t [\int_0^r \lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds \cdot \lambda_r \exp(-\int_0^r \lambda_s ds)]dr + \int_0^t \lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds \exp(-\int_0^t \lambda_s ds)$$

$$= \int_0^t [-\int_s^t -\lambda_r \exp(-\int_0^r \lambda_s ds)dr] \cdot \lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds + \int_0^t \lambda_s\{h(\lambda_s) - f(\lambda_s)\}ds \exp(-\int_0^t \lambda_s ds)$$

$$= \int_0^t \{\exp(-\int_0^s \lambda_s ds) - \exp(-\int_0^t \lambda_s ds)\} \cdot \lambda_s \{h(\lambda_s) - f(\lambda_s)\} ds + \int_0^t \lambda_s \{h(\lambda_s) - f(\lambda_s)\} ds \exp(-\int_0^t \lambda_s ds)$$

$$= \int_0^t \lambda_s \{h(\lambda_s) - f(\lambda_s)\} \exp(-\int_0^s \lambda_s ds) ds . \tag{3.8}$$

Put $s = r$ in (3.8), then we have

$$E[\int_0^{\tau_1 \wedge t} \lambda_s \{h(\lambda_s) - f(\lambda_s)\} ds \big| \lambda_v, 0 \le v \le t] = \int_0^t \lambda_r \{h(\lambda_r) - f(\lambda_r)\} \exp(-\int_0^r \lambda_s ds) dr . \tag{3.9}$$

Therefore (3.4) becomes

$$E\{ f(\lambda_{\tau_1 \wedge t} \big| \lambda_0) + E[\int_0^{\tau_1 \wedge t} [\lambda_s \{h(\lambda_s) - f(\lambda_s)\}] ds \big| \lambda_0]$$

$$= \int_\Omega [\int_0^t f(\lambda_r) \lambda_r \exp(-\int_0^r \lambda_s ds) dr + f(\lambda_t) \exp(-\int_0^t \lambda_s ds)] dP(\lambda_v; 0 \le v \le t)$$

$$+ \int_\Omega [\int_0^t \lambda_r \{h(\lambda_r) - f(\lambda_r)\} \exp(-\int_0^r \lambda_s ds) dr] dP(\lambda_v; 0 \le v \le t)$$

$$= \int_\Omega f(\lambda_t) \exp(-\int_0^t \lambda_s ds) dP(\lambda_v; 0 \le v \le t) + \int_\Omega [\int_0^t \lambda_r h(\lambda_r) \exp(-\int_0^r \lambda_s ds) dr] dP(\lambda_v; 0 \le v \le t)$$

$$= E\{ f(\lambda_t) \exp(-\int_0^t \lambda_s ds) \big| \lambda_0\} + \int_\Omega [\int_0^t h(\lambda_r) \lambda_r \exp(-\int_0^r \lambda_s ds) dr] dP(\lambda_v; 0 \le v \le t) \tag{3.10}$$

$$= f(\lambda_0).$$

Letting $t \to \infty$ in (3.10), then from (3.1), the first term in the left-hand side tends to 0 and the second term to $E\{h(\lambda_{\tau_1}) \big| \lambda_0\}$, as $\lambda_r \exp(-\int_0^r \lambda_s ds)$ is a density. Therefore we have $E\{h(\lambda_{\tau_1}) \big| \lambda_0\} = f(\lambda_0)$.

Assuming that the shot noise process λ_t is stationary, let us derive the Laplace transform of the distribution of the shot noise process at claim times, λ_τ.

Theorem 3.2 *If the shot noise process λ_t is stationary, the Laplace transform of the distribution of the shot noise process at claim times is given by*

$$E\left(e^{-\nu \lambda_{\tau_i}}\right) = \frac{\hat{\boldsymbol{G}}(\nu)}{\mu_1} \cdot \exp\left\{ -\frac{\rho}{\delta} \int_0^\nu \hat{\boldsymbol{G}}(u) du \right\}. \tag{3.11}$$

<u>Proof</u>

From Lemma 3.1, which implies that if $f(\lambda)$ and $h(\lambda)$ are such that

$$\lambda\{h(\lambda)-f(\lambda)\}-\delta\lambda f'(\lambda)+\rho\{\int_0^\infty f(\lambda+y)dG(y)-f(\lambda)\}=0 \tag{3.12}$$

and (3.1) is satisfied, we have

$$E\left\{h(\lambda_{\tau_{i+1}})\middle|\lambda_{\tau_i}\right\}=f(\lambda_{\tau_i}) \tag{3.13}$$

by starting the process from τ_i. Employing $f(\lambda)=\left\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\right\}e^{-\nu\lambda}$, the function $f(\lambda)$ clearly satisfies (3.1) and substituting into (3.12), then we have

$$\lambda\left\{h(\lambda)-\lambda e^{-\nu\lambda}+\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda}\right\}+\delta\nu\lambda\left\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\right\}e^{-\nu\lambda}-\delta\lambda e^{-\nu\lambda}$$

$$=-\rho\left[e^{-\nu\lambda}\left\{\lambda\hat{g}'(\nu)+\hat{g}'(\nu)-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\hat{g}(\nu)\right\}-\left\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\right\}e^{-\nu\lambda}\right]$$

$$=-\rho\lambda e^{-\nu\lambda}\left\{\hat{g}(\nu)-1\right\}.$$

Divide by λ and simplify then we have

$$h(\lambda)=\lambda e^{-\nu\lambda}(1-\delta\nu)+\delta e^{-\nu\lambda}-(1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda}+\rho e^{-\nu\lambda}\{1-\hat{g}(\nu)\}. \tag{3.14}$$

From (3.13), it is given that

$$E\left\{h(\lambda_{\tau_{i+1}})\right\}=E\left[E\left\{h(\lambda_{\tau_{i+1}})\middle|\lambda_{\tau_i}\right\}\right]=E\{f(\lambda_{\tau_i})\}. \tag{3.15}$$

So put (3.14) into (3.15), then

$$E[\lambda_{\tau_{i+1}}\exp(-\nu\lambda_{\tau_{i+1}})(1-\delta\nu)+\delta\exp(-\nu\lambda_{\tau_{i+1}})-(1-\delta\nu)\tfrac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\exp(-\nu\lambda_{\tau_{i+1}})+\rho\exp(-\nu\lambda_{\tau_{i+1}})\{1-\hat{g}(\nu)\}]$$

$$=E\{\lambda_{\tau_i}\exp(-\nu\lambda_{\tau_{i+1}})-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\exp(-\nu\lambda_{\tau_{i+1}})\}. \tag{3.16}$$

When the process λ_t is stationary, $\lambda_{\tau_{i+1}}$ and λ_{τ_i} have the same distribution whose Laplace transform we denote by $H(\nu)=E\left(e^{-\nu\lambda_{\tau_i}}\right)$. Therefore from (3.16), we have

$$-(1-\delta\nu)H'(\nu)-(1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}H(\nu)+[\delta+\rho\{1-\hat{g}(\nu)\}]H(\nu)$$

$$=-H'(\nu)-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}H(\nu). \tag{3.17}$$

Divide both sides of (3.17) by $\delta\nu$, then we have

$$H'(\nu)+\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}H(\nu)+\{\frac{1}{\nu}+\frac{\rho}{\delta}\frac{1-\hat{g}(\nu)}{\nu}\}H(\nu)=0. \tag{3.18}$$

Solving (3.18), subject to

$$H(0)=1 \tag{3.19}$$

then the Laplace transform of a distribution of the shot noise process at claim times is given by

$$H(\nu)=K\left(\frac{1-\hat{g}(\nu)}{\nu}\right)\exp\left\{-\frac{\rho}{\delta}\int_0^{\nu}\hat{\boldsymbol{G}}(u)du\right\}$$

where K is a constant. Therefore from (3.19), $K=\frac{1}{\mu_1}$ and

$$H(\nu)=\frac{1}{\mu_1}\frac{1-\hat{g}(\nu)}{\nu}\cdot\exp\left\{-\frac{\rho}{\delta}\int_0^{\nu}\hat{\boldsymbol{G}}(u)du\right\}=\frac{\hat{\boldsymbol{G}}(\nu)}{\mu_1}\cdot\exp\left\{-\frac{\rho}{\delta}\int_0^{\nu}\hat{\boldsymbol{G}}(u)du\right\}. \tag{3.20}$$

Equation (3.20) provides us with a very interesting result that this is the distribution of the sum of two random variables; one having the stationary distribution of λ_t (see Corollary 2.4) and the other, having density $\frac{\boldsymbol{G}(y)}{\mu_1}$, where $\boldsymbol{G}(y)=1-G(y)$. Comparing it with the distribution of the shot noise process, λ_t at any times, we can easily find that

$$\frac{\hat{\boldsymbol{G}}(\nu)}{\mu_1}\cdot\exp\left\{-\frac{\rho}{\delta}\int_0^{\nu}\hat{\boldsymbol{G}}(u)du\right\} \quad > \quad \exp\left\{-\frac{\rho}{\delta}\int_0^{\nu}\hat{\boldsymbol{G}}(u)du\right\}.$$

It is therefore the case that λ_τ is stochastically larger than λ_t. In other words, the intensity at claim times is higher than the intensity at any times.

Now let us derive the distribution of the interval of the Cox process with shot noise intensity for insurance claims using Theorem 3.2.

Corollary 3.3 *Assume that 0 is the time at which a claim of N_t has occurred and the stationary of λ_t has been achieved. Then the tail of the distribution of the interval of the Cox process with shot noise intensity is given by*

$$\Pr(\tau>t)=\frac{\hat{\boldsymbol{G}}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta t})}{\mu_1}\exp\left\{-\frac{\rho}{\delta}\int_0^{t}\hat{\boldsymbol{G}}(\tfrac{1}{\delta}-\tfrac{1}{\delta}e^{-\delta s})ds\right\}. \tag{3.21}$$

Proof

From (2.13), the probability generating function of N_t is given by

$$E\left(\theta^{N_t}\,|\lambda_0\right)=\exp\left\{-\tfrac{1-\theta}{\delta}(1-e^{-\delta t})\lambda_0\right\}\exp\left[-\rho\int_0^t[1-\hat{g}\{\tfrac{1-\theta}{\delta}(1-e^{-\delta s})\}]ds\right]. \tag{3.22}$$

Set $\theta=0$ in (3.22) and take expectation, then the tail of the distribution of τ is given by

$$\Pr(\tau>t)=\exp\left[-\rho\int_0^t\{1-\hat{g}(\frac{1-e^{\delta s}}{\delta})\}ds\right]E\left[\exp\left\{-\tfrac{(1-e^{-\delta t})}{\delta}\lambda_0\right\}\right]. \tag{3.23}$$

Substitute (3.11) into (3.23), then the result follows immediately as 0 is the time at which a claim has occurred and λ_t is stationary.

Corollary 3.4 *The expectation and variance of the interval between claims are given by*

$$E(\tau)=\int_0^\infty \Pr(\tau>t)dt=\frac{\delta}{\mu_1\rho} \tag{3.24}$$

and

$$Var(\tau)=2\int_0^\infty[u\cdot\frac{\hat{\boldsymbol{G}}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta u})}{\mu_1}\exp\{-\frac{\rho}{\delta}\int_0^u\hat{\boldsymbol{G}}(\tfrac{1}{\delta}-\tfrac{1}{\delta}e^{-\delta s})ds\}]du-\left(\frac{\delta}{\mu_1\rho}\right)^2. \tag{3.25}$$

Proof

Integrate (3.21), then (3.24) follows. (3.25) is obtained from

$$E(\tau^2)=\int_0^\infty t^2 f(t)dt=2\int_0^\infty[u\cdot\frac{\hat{\boldsymbol{G}}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta u})}{\mu_1}\exp\{-\frac{\rho}{\delta}\int_0^u\hat{\boldsymbol{G}}(\tfrac{1}{\delta}-\tfrac{1}{\delta}e^{-\delta s})ds\}]du\,.$$

An interesting result we can find from (3.24) and (2.18) is that the expected interval between claims is the inverse of the expected number of claims, where the number of claims follows the Cox process with shot noise intensity, which is also the case for a Poisson process.

4. Conclusion

We started with deriving the probability generating function of the Cox process with shot noise intensity, employing piecewise deterministic Markov processes theory. It was necessary to obtain the distribution of the shot noise process at claim times as it is not the same as the distribution of the shot noise process at any times, i.e. the intensity of claim times are higher than the intensity at any times. Assuming that the shot noise process is stationary, we derived the distribution of the interval of the Cox process with shot noise intensity for insurance claims and its moments from its probability generating function. The result of this paper can be used or easily modified in computer science/telecommunications modeling, electrical engineering and queueing theory as an alternative counting process of a Poisson process.

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