A Way of Hedging Mortality Rate Risks in Life Insurance Product Development

Changki Kim*

Abstract

Forecasting mortality improvements in the future is important and necessary for insurance business. An interesting observation is that mortality rates for a few ages are improving recently and that there may exist mortality rate risks. If the life table constructed from a mortality model predicts lower mortality rates than those actually experienced by the life insurance policy holders then the company will face losses from the sales of life insurance contracts.

As a hedging strategy, the insurance company may promote the sale of polices such as annuities or pure endowments to offset the losses from the life insurance sales. We present a way of hedging mortality rate risks in developing endowment policies using the hedge ratios of pure endowment to offset the losses from term life insurance. We also show a hedging strategy under a stochastic force of mortality model using the results from Malliavin Calculus.

Keywords: Mortality Rate Risks, Force of Mortality, Hedge Ratios, Stochastic Mortality rate Models, Malliavin Calculus

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1 Introduction

Forecasting mortality improvements in the future is important and necessary in life insurance product developments. An interesting observation is that mortality rates for a few ages are improving recently. We may refer to a few papers such as Friedland (1998), Gutterman and Vanderhoof (1998), Tuljapurkar and Boe (1998), Tuljapurkar (1998), and Goss et al. (1998), which discuss the trends in mortality changes and forecasting.

For more recent investigations on the trends of mortality improvement, we refer to the two reports "The RP-2000 Mortality Tables" from the Society of Actuaries (2000) and "Report of the Society of Actuaries Mortality Improvement Survey Subcommittee" from the Society of Actuaries (2003). The Society of Actuaries conducted a study of uninsured pension plan mortality in response to the Retirement Protection Act of 1994 and presented the RP-2000 Mortality Tables.

Table 1. Annualized Mortality Improvement for Male

Male			
Age	**Study Data 1990-1994**	**Social Security 1990-1994**	**Federal Civil Service 1988-1996**
20-24		0.31%	
25-29	-1.07%	0.99%	
30-34	4.83%	-1.58%	
35-39	2.15%	-1.41%	
40-44	-1.87%	-2.85%	
45-49	2.01%	0.06%	
50-54	3.63%	0.47%	
55-59	4.48%	1.83%	1.13%
60-64	2.45%	1.26%	1.72%
65-69	1.50%	0.96%	0.93%
70-74	0.75%	1.06%	1.22%
75-79	1.10%	1.08%	1.59%
80-84	0.32%	0.47%	1.43%
85-89	0.18%	-0.49%	0.78%
90-94	-0.81%	-0.82%	0.41%

Source : *The RP-2000 Mortality Tables*. Society of Actuaries.

Table 2. Annualized Mortality Improvement for Female

Female			
Age	**Study Data 1990-1994**	**Social Security 1990-1994**	**Federal Civil Service 1988-1996**
20-24		0.21%	
25-29	13.88%	-0.59%	
30-34	-15.60%	-1.24%	
35-39	-7.51%	-2.19%	
40-44	-1.66%	-1.42%	
45-49	-4.61%	0.56%	
50-54	-5.72%	0.94%	
55-59	5.27%	1.09%	0.92%
60-64	-3.23%	0.49%	0.10%
65-69	0.38%	-0.07%	0.44%
70-74	-1.00%	0.06%	1.07%
75-79	-0.93%	-0.13%	1.10%
80-84	-0.24%	-0.30%	0.64%
85-89	-1.25%	-0.49%	0.30%
90-94	0.15%	-0.47%	0.08%

Source : *The RP-2000 Mortality Tables*. Society of Actuaries.

The tables were developed from experience on mortality for uninsured pension plans. The rates of 1992 base year were projected to year 2000 based on a review of three sets of data, Social Security dada, federal retiree data, and the study data. To better observe the trends of mortality rate changes, they calculated least-squares regression lines through the logarithms of the raw mortality rates by year for each quinquennial age group for each gender for each data set. The best-fit log-linear mortality improvement trends were calculated using the slopes of these regression lines. For each regression line, the best-fit log-linear mortality improvement trend equals one minus the antilog of the slope. Table 1 presents the annualized best-fit log-linear mortality improvement trends for male by age and data source. Table 2 shows the annualized best-fit log-linear mortality improvement trends for female by age and data source. These tables compare recent mortality improvement from the data collected for the study on employees and healthy annuitants combined (1990-1994), from Social Security data (1990-1994), and from Federal Civil Service data (1988-1996).

From Table 1 and Table 2, we can observe that mortality improvement trends for male from age 55 through age 80 for Social Security and Federal Civil Service are all remarkably positive. But the trends for females at all ages exhibit mixed results including many negative and insignificant trends. Based on these observations they projected annual mortality improvement rates to RP 2000 mortality table only for male. It is not easy to infer a specific relationship between attained age and mortality improvement from the observations. We do not know any reasons of the recent mortality improvement. We cannot conjecture a quantitative mortality improvement for the future either. We just make an observation that mortality improvement has been experienced in recent years. And we have noticed the following trends on mortality improvement.

(1) The trend of mortality improvement is apparent for male mortality rates rather than female rates. Especially mortality improvement has been higher for males than for females.
(2) There is no clear indication of mortality improvement at attained ages under 45 and attained ages above 85. It is an interesting observation that mortality improvement has been experienced especially for the *middle-aged males* for the last decades.
(3) Almost 1.0% of annual mortality improvement has been realized for males ages 55 – 80 for Social Security and Federal Civil Service data.
(4) For females, almost 0.5% - 1.0% of annual mortality improvement has been realized ages 45 – 64 for Social Security data, and ages 55 – 84 for Federal Civil Service data.
(5) For the study data, female mortality has tended to decrease.

2 Sensitivity Test and Hedge Ratio

From the previous observation, we notice that mortality improvement is a trend in recent decade especially for the middle-aged male and should be investigated by actuaries for the mortality rate risk management. We define the mortality rate risks.

Definition 1 The *mortality rate risk* is the risk that the actual claims associated to death are more frequent than the anticipated resulting in unexpected losses.

In this section, we assume that the force of mortality may increase for specific ages unexpectedly and that there may exist mortality rate risks. If the life table constructed from a force of mortality model predicts lower mortality rates than actually experienced by the life insurance policy holders then the company will face losses in the future. We consider the changes in mortality rates, called the mortality rate shocks, and want to check the premium differences when there exist mortality rate shocks.

For illustration purpose, we use Gompertz's model,

$$\mu(x) = \mu(0)\, c^x . \tag{2.1}$$

The survival function, $s(x)$, is calculated from this force of mortality function,

$$\begin{aligned} s(x) &= \exp\left[-\int_0^x (\mu(0)c^s)ds\right] \\ &= \exp[-m(c^x - 1)], \end{aligned} \tag{2.2}$$

where $m = \mu(0)/\log c$.

For illustration purpose, we assume that the parameter c in the force of mortality function is increased by 1%, i.e. c is changed to c*1.01. Under this assumption, we will construct the changed life table, calculate the premiums of 10-year term insurance and 10-year pure endowment, check the gains or losses of premiums, and compute the hedge ratios.

The premiums of 10-year term insurance $A^{1}_{x:\overline{10|}}$ are increasing after the mortality rate shock, and the premiums of 10-year pure endowment $A_{x:\overline{10|}}^{\;\;\,1}$ are decreasing after the mortality rate shock.[1] Table 3 shows the changes of premiums.

To find the hedge ratio, let us denote the premium of 10-year term insurance after mortality rate shock by $\tilde{A}^{1}_{x:\overline{10|}}$, and denote the premium of 10-year pure endowment after mortality rate shock by $\tilde{A}_{x:\overline{10|}}^{\;\;\,1}$. The premium loss amount from 10-year term insurance is

$$\text{Loss} = \tilde{A}^{1}_{x:\overline{10|}} - A^{1}_{x:\overline{10|}} > 0,$$

and the premium gain from 10-year pure endowment is

$$\text{Gain} = A_{x:\overline{10|}}^{\;\;\,1} - \tilde{A}_{x:\overline{10|}}^{\;\;\,1} > 0.$$

To offset the losses from the 10-year term insurance sales, we have to sell 10-year pure endowment at the same time, that is it is better for insurance companies to sell 10-year endowment insurance policies than to sell only 10-year term insurance policies. Then they can hedge the mortality rate risk from the increasing mortality rate shock. The hedge ratio, R_x, is defined to be the face amount of 10-year pure endowment to be sold to offset the losses from the 10-year term insurance of face amount 1 payable at the end of the year when (x) dies. The hedge ratio R_x is such that

$$\tilde{A}^{1}_{x:\overline{10|}} - A^{1}_{x:\overline{10|}} = R_x\,(A_{x:\overline{10|}}^{\;\;\,1} - \tilde{A}_{x:\overline{10|}}^{\;\;\,1}),$$

and we have

$$R_x = (\tilde{A}^{1}_{x:\overline{10|}} - A^{1}_{x:\overline{10|}})/(A_{x:\overline{10|}}^{\;\;\,1} - \tilde{A}_{x:\overline{10|}}^{\;\;\,1}). \tag{2.3}$$

Table 4 shows the gains and losses from mortality rate shock and the hedge ratio R_x. We assume $\mu(0) = 0.0001$, c = 0.40987, and annual interest rate i = 5%.

Table 3. Change of Premiums with Mortality Rate Shock

Change of Premiums					
Before Shock			**After Shock**		
Age(x)	**Term Life** $A^{1}_{x:\overline{10|}}$	**Pure Endowment** $A_{x:\overline{10|}}^{\;\;\,1}$	**Age(x)**	**Term Life** $\tilde{A}^{1}_{x:\overline{10|}}$	**Pure Endowment** $\tilde{A}_{x:\overline{10|}}^{\;\;\,1}$
35	0.02183	0.59598	35	0.03239	0.58722

[1] Thorough out this paper, we try to follow the general rules for symbols of actuarial functions. See the Appendix 4 in Bowers et al (1997).

40	0.03299	0.58684	40	0.05119	0.57178
45	0.04967	0.57318	45	0.08031	0.54791
50	0.07438	0.55301	50	0.12460	0.51180
55	0.11047	0.52366	55	0.18995	0.45891
60	0.16204	0.48195	60	0.28192	0.38544

Table 4. Premium Gains and Losses from Mortality Rate Shock

Age(x)	**Losses from Term Life**	**Gains from Pure Endowment**	R_x **(Hedge Ratio)**
35	0.01056	0.00876	1.20548
40	0.01820	0.01506	1.20845
45	0.03064	0.02527	1.21250
50	0.05022	0.04121	1.21864
55	0.07948	0.06475	1.22749
60	0.11988	0.09651	1.24215

3 Hedging Strategy for Mortality Rate Risks in Developing Endowment Policies

We made an assumption that the mortality rate shock follows a particular movement and the parameter c in the force of mortality function is increased by 1%. This is just for illustration purpose and not realistic. We cannot predict the exact amount of change in the mortality rate shock, even though we try to observe the trends of mortality improvement as precisely as possible. We generalize this assumption.

In this section, we do not assume the amount of mortality rate shock. We just assume that there exist mortality rate shocks. For a given life table, the premium of n-year term life insurance with face amount 1 payable at the end of the year when (x) dies is

$$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \frac{d_{x+k}}{l_x}, \tag{3.1}$$

where $v = 1/(1+i)$, i = annual effective interest rate, d_{x+k} is the expected number of deaths between age x+k and x+k+1, and l_x is the number of policy holders at age x.

Let us assume that the force of mortality μ_{x+t} at age x+t is increased to μ^{ε}_{x+t}, by $\varepsilon(x,t) > 0$,

$$\mu_{x+t} \;\rightarrow\; \mu^{\varepsilon}_{x+t} = \mu_{x+t} + \varepsilon(x,t),\; t \ge 0. \tag{3.2}$$

Then the survival probability ${}_t p_x$ will be changed to ${}_t p^{\varepsilon}_x$,

$$\begin{aligned} {}_t p^{\varepsilon}_x &= \exp\left(-\int_0^t (\mu_{x+s} + \varepsilon(x,s))ds\right) \\ &= \exp\left(-\int_0^t \mu_{x+s} ds\right) \exp\left(-\int_0^t \varepsilon(x,s) ds\right) \\ &= {}_t p_x \exp\left(-\int_0^t \varepsilon(x,s) ds\right) < {}_t p_x . \end{aligned} \tag{3.3}$$

The premium of term insurance after mortality rate shock $\varepsilon(x,t)$ is

$$\widetilde{A}^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} \frac{\widetilde{d}_{x+k}}{l_x} > \sum_{k=0}^{n-1} v^{k+1} \frac{d_{x+k}}{l_x} = A^{1}_{x:\overline{n}|}. \tag{3.4}$$

The amount of loss from term insurance after the mortality rate shock is

$$Loss_x = \widetilde{A}^{1}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|} > 0. \tag{3.5}$$

As a hedging strategy, we consider n-year pure endowment to offset the losses from n-year term insurance**.** The net single premium of n-year pure endowment issued to (x) before mortality rate shock is

$$A_{x:\overline{n}|}^{\;\;1} = {}_nE_x = v^n\,{}_np_x. \tag{3.6}$$

After the force of mortality μ_{x+t} is increased to μ^{ε}_{x+t}, by $\varepsilon(x,t) > 0$, the net single premium of n-year pure endowment will be decreased to $\widetilde{A}_{x:\overline{n}|}^{\;\;1}$. The amount of gain from n-year pure endowment after mortality rate shock is

$$Gain_x = A_{x:\overline{n}|}^{\;\;1} - \widetilde{A}_{x:\overline{n}|}^{\;\;1} = {}_nE_x - {}_n\widetilde{E}_x > 0. \tag{3.7}$$

We want to offset the losses from n-year term insurance with the gains from n-year pure endowment. The hedge ratio, R_x, is the ratio of n-year pure endowment to offset the losses from n-year term insurance in n-year endowment policies.

Note that the ratio between term insurance and pure endowment is 1 in traditional endowments. Here we want to develop new kind of endowments, called modified endowments, that pay 1 when the insured (x) dies in n years or pay R_x when the insured survives at age x+n.

Definition 2 The *hedge ratio* R_x for age x is the ratio of n-year pure endowment to offset the losses from n-year term insurance in a modified endowment, i.e. R_x is the number such that

$$Loss_x = \widetilde{A}^{1}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|} = R_x({}_nE_x - {}_n\widetilde{E}_x) = Gain_x. \tag{3.8}$$

We want to calculate R_x for each age x. From the definition of R_x, we have

$$\widetilde{A}^{1}_{x:\overline{n}|} + R_x\,{}_n\widetilde{E}_x = A^{1}_{x:\overline{n}|} + R_x\,{}_nE_x. \tag{3.9}$$

For convenience, let us denote the liability of a modified endowment before mortality rate shock to be L_x and the liability after mortality rate shock to be $\widetilde{L}_x$,

$$L_x = A^{1}_{x:\overline{n}|} + R_x\,{}_nE_x, \tag{3.10}$$

and

$$\widetilde{L}_x = \widetilde{A}^{1}_{x:\overline{n}|} + R_x\,{}_n\widetilde{E}_x. \tag{3.11}$$

We want to find the hedge ratio R_x such that

$$L_x = A^{1}_{x:\overline{n}|} + R_x\,{}_nE_x = \widetilde{L}_x = \widetilde{A}^{1}_{x:\overline{n}|} + R_x\,{}_n\widetilde{E}_x. \tag{3.12}$$

Note that the premium of traditional n-year endowment is

$$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x$$

$$= 1 - d\, \ddot{a}_{x:\overline{n|}},$$

where $\ddot{a}_{x:\overline{n|}}$ is the premium of n-year temporary annuity-due.

The liability L_x of a modified endowment before the mortality rate shock is

$$\begin{aligned} L_x &= A^{1}_{x:\overline{n|}} + R_x\, {}_nE_x \\ &= A_{x:\overline{n|}} - {}_nE_x + R_x\, {}_nE_x \\ &= A_{x:\overline{n|}} + (R_x - 1)\, {}_nE_x \\ &= 1 - d\, \ddot{a}_{x:\overline{n|}} + (R_x - 1)\, {}_nE_x \\ &= 1 - d \sum_{k=0}^{n-1} v^k\, {}_kp_x + (R_x - 1)\, v^n\, {}_np_x . \end{aligned} \tag{3.13}$$

By the same way, the liability $\widetilde{L}_x$ after the mortality rate shock is

$$\widetilde{L}_x = 1 - d \sum_{k=0}^{n-1} v^k\, {}_k\widetilde{p}_x + (R_x - 1)\, v^n\, {}_n\widetilde{p}_x . \tag{3.14}$$

The difference ΔL_x between the liabilities $\widetilde{L}_x$ and L_x is

$$\begin{aligned} \Delta L_x &= \widetilde{L}_x - L_x \\ &= - d \sum_{k=0}^{n-1} v^k ({}_k\widetilde{p}_x - {}_kp_x) + (R_x - 1)\, v^n ({}_n\widetilde{p}_x - {}_np_x) . \end{aligned} \tag{3.15}$$

We want to find R_x such that the difference between the liabilities, ΔL_x, is as small as possible,

$$\begin{aligned} \Delta L_x &= \widetilde{L}_x - L_x \\ &= - d \sum_{k=0}^{n-1} v^k ({}_k\widetilde{p}_x - {}_kp_x) + (R_x - 1)\, v^n ({}_n\widetilde{p}_x - {}_np_x) = 0. \end{aligned}$$

Let us analyze the difference $\Delta L_x = \widetilde{L}_x - L_x$ between the liabilities.

$$\begin{aligned} \Delta L_x = \widetilde{L}_x - L_x &= - d \sum_{k=0}^{n-1} v^k ({}_k\widetilde{p}_x - {}_kp_x) + (R_x - 1)\, v^n ({}_n\widetilde{p}_x - {}_np_x) \\ &= - d \sum_{k=0}^{n-1} v^k \frac{({}_k\widetilde{p}_x - {}_kp_x)}{{}_kp_x}\, {}_kp_x + (R_x - 1)\, v^n \frac{({}_n\widetilde{p}_x - {}_np_x)}{{}_np_x}\, {}_np_x . \end{aligned}$$

Define the function

$$f(k) = \frac{{}_k\widetilde{p}_x - {}_kp_x}{{}_kp_x} . \tag{3.16}$$

Note that $f(0) = (1-1)/1 = 0$. The difference between the liabilities is

$$\Delta L_x = - d \sum_{k=0}^{n-1} v^k f(k)\, {}_kp_x + (R_x - 1)\, v^n f(n)\, {}_np_x . \tag{3.17}$$

If we assume that the function $f(k)$ is twice differentiable then, by Taylor's formula with integral remainder, the function $f(k)$ can be expressed as

$$f(k) = f(0) + f'(0)k + \int_0^k (k-w) f''(w)\, dw$$

$$= f'(0)k + \int_0^k (k-w) f''(w)dw.$$

Now the difference between the liabilities is

$$\Delta L_x = - d \sum_{k=0}^{n-1} v^k f'(0) k\, {}_k p_x - d \sum_{k=0}^{n-1} [v^k\, {}_k p_x \int_0^k (k-w) f''(w)dw]$$

$$+ (R_x - 1)\ v^n n f'(0)\, {}_n p_x + (R_x - 1)\ v^n\, {}_n p_x \int_0^n (n-w) f''(w)dw$$

$$= f'(0)[- d \sum_{k=0}^{n-1} v^k k\, {}_k p_x + (R_x - 1)\ v^n n\, {}_n p_x]$$

$$- d \sum_{k=0}^{n-1} [v^k\, {}_k p_x \int_0^k (k-w) f''(w)dw] + (R_x - 1)\ v^n\, {}_n p_x \int_0^n (n-w) f''(w)dw$$

$$= \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = f'(0)[- d \sum_{k=0}^{n-1} v^k k\, {}_k p_x + (R_x - 1)\ v^n n\, {}_n p_x],$$

and

$$\mathrm{II} = - d \sum_{k=0}^{n-1} [v^k\, {}_k p_x \int_0^k (k-w) f''(w)dw] + (R_x - 1)\ v^n\, {}_n p_x \int_0^n (n-w) f''(w)dw.$$

Now let us find R_x such that the difference ΔL_x between the liabilities is as small as possible,

$$\Delta L_x = \mathrm{I} + \mathrm{II} \approx 0.$$

One strategy is to find R_x such that the first term I of ΔL_x equals 0,

$$\mathrm{I} = f'(0)[- d \sum_{k=0}^{n-1} v^k k\, {}_k p_x + (R_x - 1)\ v^n n\, {}_n p_x] = 0, \tag{3.18}$$

and the second term II of ΔL_x becomes nearly 0,

$$\mathrm{II} = - d \sum_{k=0}^{n-1} [v^k\, {}_k p_x \int_0^k (k-w) f''(w)dw] + (R_x - 1)\ v^n\, {}_n p_x \int_0^n (n-w) f''(w)dw \approx 0.$$

From the equation (3.18) we obtain

$$- d \sum_{k=0}^{n-1} v^k k\, {}_k p_x + (R_x - 1)\ v^n n\, {}_n p_x = 0 \tag{3.19}$$

$$\Leftrightarrow - d\ [(I\ddot{a})_{x:\overline{n|}} - \ddot{a}_{x:\overline{n|}}] + (R_x - 1)\ n\, {}_n E_x = 0 \tag{3.20}$$

$$\Leftrightarrow - d\ (Ia)_{x:\overline{n-1|}} + (R_x - 1)\ n\, {}_n E_x = 0 \tag{3.21}$$

$$\Leftrightarrow (R_x - 1)\ n\, {}_n E_x = d\ (Ia)_{x:\overline{n-1|}}, \tag{3.22}$$

where

$$(Ia)_{x:\overline{n-1|}} = \sum_{k=0}^{n-1} v^k k\, {}_k p_x.$$

Now we have the following theorem.

Theorem 1 *The hedge ratio R_x of n-year pure endowment for age x to offset the losses from the sale of 1 unit of n-year term insurance (with face amount 1) is approximately*

$$R_x = 1 + \frac{d}{n} \frac{(Ia)_{x:\overline{n-1}|}}{{}_nE_x}. \tag{3.23}$$

Remark 1 The hedge ratio R_x in theorem 1 is independent of the amount of mortality rate shock $\varepsilon(x,t)$. And it is greater than 1 for any age x, this may not be true for some age and we will solve this problem using a stochastic mortality rate model and Malliavin calculus later.

We can interpret theorem 1 using the sensitivity of the liabilities L_x with respect to the change of mortality rates with the following assumption. For the survival probability, ${}_k p_x = \exp\left(-\int_0^k \mu_{x+s} ds\right)$, we assume that the sensitivity of ${}_k p_x$ with respect to the change of mortality rates is approximately

$$\frac{\partial_k p_x}{\partial \mu} \approx -k \exp\left(-\int_0^k \mu_{x+s} ds\right) = -k \; {}_k p_x. \tag{3.24}$$

Theorem 2 *With the assumption above, the hedge ratio R_x of n-year pure endowment for age x to offset the losses from n-year term insurance is the number which satisfies*

$$\frac{\partial L_x}{\partial \mu} = 0. \tag{3.25}$$

Proof From (3.13) we have

$$\begin{aligned} L_x &= A^1_{x:\overline{n}|} + R_x \, {}_nE_x \\ &= 1 - d \sum_{k=0}^{n-1} v^k \, {}_k p_x + (R_x - 1)\, v^n \, {}_n p_x. \end{aligned}$$

We have the following equivalences

$$\begin{aligned} &\frac{\partial L_x}{\partial \mu} = 0 \\ \Leftrightarrow\ & - d \sum_{k=0}^{n-1} v^k \frac{\Delta_k p_x}{\Delta \mu} + (R_x - 1)\, v^n \frac{\Delta_n p_x}{\Delta \mu} = 0 \\ \Leftrightarrow\ & d \sum_{k=0}^{n-1} v^k k \, {}_k p_x - (R_x - 1)\, v^n n \, {}_n p_x = 0 \\ \Leftrightarrow\ & (R_x - 1)\, n \, {}_nE_x = d\, (Ia)_{x:\overline{n-1}|} \\ \Leftrightarrow\ & R_x = 1 + \frac{d}{n} \frac{(Ia)_{x:\overline{n-1}|}}{{}_nE_x}. \end{aligned}$$

Remark 2 If the force of mortality $\mu_x = \mu$, a constant, then we have

$$ {}_k p_x = e^{-\int_0^k \mu_{x+s} ds} = e^{-\mu k}. \tag{3.26} $$

From (3.13) we have

$$ \begin{aligned} L_x &= A^1_{x:\overline{n|}} + R_x \, {}_nE_x \\ &= 1 - d \sum_{k=0}^{n-1} v^k \, {}_k p_x + (R_x - 1) \, v^n \, {}_n p_x . \end{aligned} $$

Now we consider L_x as a function of both mortality rate μ and interest rate i. Then we have the following equivalences on the change of the liabilities L_x with respect to the change of mortality rate μ,

$$ \begin{aligned} & \frac{\partial L_x}{\partial \mu} = 0 \qquad (3.27) \\ \Leftrightarrow \; & \frac{\partial \left\{ 1 - d\sum_{k=0}^{n-1} v^k \, {}_k p_x + (R_x - 1) v^n \, {}_n p_x \right\}}{\partial \mu} = 0 \\ \Leftrightarrow \; & -d\sum_k v^k \frac{\partial \, {}_k p_x}{\partial \mu} + (R_x - 1) v^n \frac{\partial \, {}_n p_x}{\partial \mu} = 0 \\ \Leftrightarrow \; & d \sum_{k=0}^{n-1} v^k k \, {}_k p_x - (R_x - 1) \, v^n n \, {}_n p_x = 0 \\ \Leftrightarrow \; & (R_x - 1) \, n \, {}_n E_x = d \, (Ia)_{x:\overline{n-1|}} \\ \Leftrightarrow \; & R_x = 1 + \frac{d}{n} \frac{(Ia)_{x:\overline{n-1|}}}{{}_n E_x} . \end{aligned} $$

Recall that the difference ΔL_x between the liabilities is

$$ \Delta L_x = \mathrm{I} + \mathrm{II}, $$

where

$$ \mathrm{I} = f'(0) [- \, d \sum_{k=0}^{n-1} v^k k \, {}_k p_x + (R_x - 1) \, v^n n \, {}_n p_x], $$

and

$$ \mathrm{II} = - \, d \sum_{k=0}^{n-1} [v^k \, {}_k p_x \int_0^k (k-w) f''(w) dw] + (R_x - 1) \, v^n \, {}_n p_x \int_0^n (n-w) f''(w) dw . $$

If we set R_x such that

$$ R_x = 1 + \frac{d}{n} \frac{(Ia)_{x:\overline{n-1|}}}{{}_n E_x}, $$

then the first term I of ΔL_x equals 0,

$$ \mathrm{I} = f'(0) [- \, d \sum_{k=0}^{n-1} v^k k \, {}_k p_x + (R_x - 1) \, v^n n \, {}_n p_x] = 0, $$

and

$$ \Delta L_x = \mathrm{II}. $$

So the hedging strategy with $R_x = 1 + \dfrac{d}{n}\dfrac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}$ does not guarantee a perfect hedging. There still remain mortality rate risks since ΔL_x = II may not equal to 0.

Definition 3 The *residual risk* RR_x for age x is the amount ΔL_x = II when we have the hedging strategy with $R_x = 1 + \dfrac{d}{n}\dfrac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}$ (i.e. I = 0).

Now let us analyze the residual risk

$$
\begin{aligned}
RR_x &= \Delta L_x = \text{II} \\
&= -\,d\sum_{k=0}^{n-1}[v^k\,{}_kp_x\int_0^k (k-w)f''(w)dw] + (R_x-1)\;v^n\,{}_np_x\int_0^n (n-w)f''(w)dw. \qquad (3.28)\\
&= -\,d\sum_{k=0}^{n-1}[v^k\,{}_kp_x(f(k)-kf'(0))] + (R_x-1)\;v^n\,{}_np_x(f(n)-nf'(0)) \\
&= -\,d\sum_{k=0}^{n-1}[v^k\,{}_kp_x(f(k)-kf'(0))] + \frac{d}{n}\,(Ia)_{x:\overline{n-1|}}(f(n)-nf'(0)) \\
&= -\,d\sum_{k=0}^{n-1}v^k\,{}_kp_x\,f(k) + d\;f'(0)\sum_{k=0}^{n-1}kv^k\,{}_kp_x + \frac{d}{n}\,(Ia)_{x:\overline{n-1|}}(f(n)-nf'(0)) \\
&= -\,d\sum_{k=0}^{n-1}v^k\,{}_kp_x\,f(k) + d\;f'(0)\,(Ia)_{x:\overline{n-1|}} + \frac{d}{n}\,(Ia)_{x:\overline{n-1|}}(f(n)-nf'(0)) \\
&= -\,d\sum_{k=0}^{n-1}v^k\,{}_kp_x\,f(k) + d\;(Ia)_{x:\overline{n-1|}}\frac{f(n)}{n} \\
&= d\,[(Ia)_{x:\overline{n-1|}}\frac{f(n)}{n} - \sum_{k=0}^{n-1}v^k\,{}_kp_x\,f(k)]. \qquad (3.29)
\end{aligned}
$$

Since we cannot determine the function $f(k)$ exactly, (3.29) does not give us the residual risk RR_x. But we have another way to estimate RR_x if we assume that RR_x is a measure of the difference between a non-perfect hedging strategy and a perfect hedging strategy. If we set up the liability L_x as follows

$$L_x = A^1_{x:\overline{n|}} + {}_nE_x + d\;\ddot{a}_{x:\overline{n|}}, \qquad (3.30)$$

then ΔL_x = 0, i.e. a perfect hedging strategy. For the perfect hedging strategy, the liability $\widetilde{L}_x$ after the mortality rate shock is

$$
\begin{aligned}
\widetilde{L}_x &= L_x + \Delta L_x \\
&= A^1_{x:\overline{n|}} + {}_nE_x + d\;\ddot{a}_{x:\overline{n|}}.
\end{aligned}
$$

For the non-perfect hedging strategy, the liability $\widetilde{L}_x$ after the mortality rate shock is

$$\widetilde{L}_x = L_x + \Delta L_x$$

$$= L_x + \mathrm{I} + \mathrm{II}$$
$$= A^1_{x:\overline{n|}} + R_x\,{}_nE_x + RR_x .$$

If we set

$$R_x\,{}_nE_x + RR_x = {}_nE_x + d\,\ddot{a}_{x:\overline{n|}},$$

then the residual risk is

$$\begin{aligned} RR_x &= {}_nE_x + d\,\ddot{a}_{x:\overline{n|}} - R_x\,{}_nE_x \\ &= (1 - R_x)\,{}_nE_x + d\,\ddot{a}_{x:\overline{n|}} \\ &= -\frac{d}{n}\frac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}\,{}_nE_x + d\,\ddot{a}_{x:\overline{n|}}, \text{ by (3.23)} \\ &= -\frac{d}{n}(Ia)_{x:\overline{n-1|}} + d\,\ddot{a}_{x:\overline{n|}} \\ &= d\,[\ddot{a}_{x:\overline{n|}} - \frac{1}{n}(Ia)_{x:\overline{n-1|}}] \\ &= \frac{d}{n}(D\ddot{a})_{x:\overline{n|}}, \end{aligned}$$

where

$$(D\ddot{a})_{x:\overline{n|}} = \sum_{k=0}^{n-1}(n-k)v^k\,{}_kp_x .$$

Now we have the following theorem.

Theorem 3 *Let us assume that RR_x is a measure of the difference between the non-perfect hedging strategy with the hedge ratio $R_x = 1 + \frac{d}{n}\frac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}$ and the perfect hedging strategy (3.30). We have that the residual risk of the non-perfect hedging strategy is*

$$RR_x = \frac{d}{n}(D\ddot{a})_{x:\overline{n|}}. \tag{3.31}$$

Remark 3 Let us find a perfect hedge ratio, $R_x^{perfect}$, of n-year pure endowment. Comparing

$$L_x = A^1_{x:\overline{n|}} + {}_nE_x + d\,\ddot{a}_{x:\overline{n|}}$$

and

$$L_x = A^1_{x:\overline{n|}} + R_x^{perfect}\,{}_nE_x,$$

we have

$$R_x^{perfect}\,{}_nE_x = {}_nE_x + d\,\ddot{a}_{x:\overline{n|}}.$$

So a perfect hedge ratio of n-year pure endowment to hedge the mortality rate risks in n-year endowment[2] is

$$R_x^{perfect} = 1 + d\,\frac{\ddot{a}_{x:\overline{n|}}}{{}_nE_x}$$
$$= 1 + d\,\ddot{s}_{x:\overline{n|}}. \qquad (3.32)$$

Table 5 shows the hedge ratios R_x and residual risks RR_x when we assume that the force of mortality follows Gompertz's model. In the first column of Table 5, we assume that the parameter c in the force of mortality function is increased by 1%. In the second column of Table 5, we calculate the hedge ratios, and in the third column of Table 5, we calculate the residual risks.

Table 5. Hedge Ratios and Residual Risks

Age(x)	R_x **(1% increase in** c **)**	$R_x = 1 + \frac{d}{10}\frac{(Ia)_{x:\overline{10-1|}}}{{}_{10}E_x}$	$RR_x = \frac{d}{10}(D\ddot{a})_{x:\overline{10|}}$
35	1.20548	1.18792	0.22639
40	1.20845	1.19507	0.22564
45	1.21250	1.20386	0.22452
50	1.21864	1.21480	0.22283
55	1.22749	1.22867	0.22030
60	1.24215	1.24689	0.21656

4 Stochastic Force of Mortality using Brownian Gompertz Model

Until now we assume that there exist mortality rate shocks and find the hedge ratio R_x which is independent of the amount of mortality rate shocks. But the hedging strategy is not perfect and there remain residual risks since the approximation using Taylor's formula may have error terms. To improve the hedging strategies we can consider stochastic mortality rate models and calculate the sensitivities of liabilities directly using Malliavin calculus.

There are several families of deterministic analytical laws of mortality such as De Moivre, Gompertz, Makeham, and Weibull. The Gompertz law of mortality is given by

$$\mu(x) = \mu(0)c^x, \qquad (4.1)$$

where $\mu(0) > 0$, $c > 1$, $x \geq 0$.

---

[2] Note that a perfect hedging strategy is $L_x = A^1_{x:\overline{n|}} + {}_nE_x + d\,\ddot{a}_{x:\overline{n|}} = 1$. So RR_x and $R_x^{perfect}$ are the risk measures only when $L_x = 1$. We want to find generalized hedging strategies.

For the stochastic mortality rate models, we refer to Lee and Carter (1993), Lee (2000), Yang (2001), Dahl (2003), Biffis (2005), Milevsky and Promislow (2001), Woodbury and Manton (1997), and Yashin et al. (1985).

For illustration purpose, we consider a stochastic law of mortality, especially the Brownian Gompertz (BG) model based on Milevsky and Promislow (2001)[3]. The BG force of mortality process is expected to grow exponentially, the variance is proportional to the value of the hazard rate, and this process never hits zero. In this paper, we consider a simple stochastic force of mortality process, $\{\mu(t);\ t \geq 0\}$, as follows[4],

$$\mu(t) = \mu(0)\exp(\sigma Z(t)), \tag{4.2}$$

where $\sigma > 0$, $\mu(0) > 0$, and the dynamics of $\{Z(t);\ t \geq 0\}$ are described by the stochastic differential equation,

$$dZ(t) = -b\, Z(t)\, dt + dW(t), \tag{4.3}$$

with $Z(0) = 0$, $b \geq 0$, and $W(t)$ is the standard Brownian motion. Note that if $b = 0$, the process $\mu(t)$ is the geometric Brownian motion. The process $Z(t)$ possesses mean reversion with mean reversion coefficient $b \geq 0$. We can solve (4.3) for $Z(t)$,

$$Z(t) = \int_0^t e^{-b(t-s)} dW(s). \tag{4.4}$$

The mean value of $Z(t)$ is

$$E[Z(t)] = 0, \tag{4.5}$$

and the variance of $Z(t)$ is

$$\mathrm{Var}(Z(t)) = E[Z(t)^2] = \int_0^t \{e^{-b(t-s)}\}^2 d(s) = \frac{1-e^{-2bt}}{2b}. \tag{4.6}$$

Note that $\mathrm{Var}(Z(t)) < t$, so the process has a smaller variance than $W(t)$, and $\mathrm{Var}(Z(t))$ converges to t as b goes to 0.

The expected value of the stochastic force of mortality $\mu(t)$ is

$$E[\mu(t)] = \mu(0)\ \exp\left\{\frac{\sigma^2}{2}\left(\frac{1-e^{-2bt}}{2b}\right)\right\}. \tag{4.7}$$

Note that $E[\mu(t)]$ converges to the Gompertz law of mortality as b goes to 0,

$$E[\mu(t)] \to \mu(0)e^{\frac{\sigma^2}{2}t}, \text{ as } b \to 0. \tag{4.8}$$

Let us consider the dynamics of $\{\mu(t);\ t \geq 0\}$. Using Itô's lemma, we have

$$d\mu(t) = \sigma\, \mu(t)\, dZ(t) + \frac{1}{2}\sigma^2\, \mu(t)dt \tag{4.9}$$

$$= \sigma\, \mu(t)\, \{-b\, Z(t)\, dt + dW(t)\} + \frac{1}{2}\sigma^2\, \mu(t)dt, \quad \text{by (4.3)}, \tag{4.10}$$

---

3 The choice and justification of stochastic mortality rate models may depend on the data and insurance policies.

4 Milevsky and Promislow (2001) suggests the model, $\mu(t,x) = \zeta \exp(\xi x + \sigma Z(t))$, for the force of mortality. Here we assume that age $x = 0$ for illustration purpose.

$$= \sigma\ \mu(t)\ \{-b\frac{1}{\sigma}\ln\left(\frac{\mu(t)}{\mu(0)}\right)\ dt + dW(t)\} + \frac{1}{2}\sigma^2\ \mu(t)dt, \quad \text{by (4.2)}, \tag{4.11}$$

$$= \{\frac{1}{2}\sigma^2 + b\ln\mu(0) - b\ln\mu(t)\}\ \mu(t)dt\ +\ \sigma\ \mu(t)\ dW(t). \tag{4.12}$$

We will use this formula to find a hedging strategy using Malliavin calculus later.

5 Malliavin Calculus

We assume that the dynamics of an asset $\{X(t);\ 0 \le t \le T\}$, which is an R^n-Markov process, are described by the stochastic differential equation ,

$$dX(t) = \beta(X(t))\ dt + \sigma(X(t))\ dW(t), \tag{5.1}$$

where $\{W(t); 0\le t \le T\}$ is a Brownian motion with values in R^n.

We consider the price of a contingent claim defined by the following form

$$V(x) = E[\psi(X(t_1), \dots, X(t_m)) |\ X(0) = x], \tag{5.2}$$

where ψ is the pay off function on the times $0 < t_1 \le \dots \le t_m = T$.

Now, we want to calculate the price of the path dependent contingent claims and the sensitivity of V(x) with respect to the initial condition x. We need to compute a Monte Carlo simulation of V(x) and a Monte Carlo estimator $V(x+\varepsilon)$ for a small ε, and estimate the sensitivity of V(x) by the following value

$$\frac{\partial V}{\partial x} \approx \frac{V(x+\varepsilon)-V(x)}{\varepsilon}\ . \tag{5.3}$$

One way to calculate the sensitivity of V(x) is to use the Malliavin calculus. We briefly introduce a few resulting formulas need in this paper. For more details on Malliavin calculus and its application to finance, we may refer Bichteler et al. (1987), Fournie et al. (2001, 1999), and Nualart (1995).

It is well known that, using Malliavin calculus, the differential of V(x) can be expressed as

$$\frac{\partial V}{\partial x} = E[\pi\ \psi(X(t_1), \dots, X(t_m)) |\ X(0) = x], \tag{5.4}$$

where π is a random variable to be determined.

There are many benefits of the above formula. We only need one time Monte Carlo simulation to calculate the sensitivity of V(x). So we can save simulation and computing time. And we do not need the parallel shift of x, $x + \varepsilon$, with arbitrary small value of ε. We can notice that the weight π does not depend on the pay off function ψ, which is also an important advantage of the above formula.

Let us consider a probability space (Ω, *F,* P), and a set *C* of random variables on the Wiener-space Ω of the form

$$F = F(\omega) = f\left(\int_0^\infty h_1((t)dW(t) \quad ,\dots, \quad \int_0^\infty h_n((t)dW(t)\right), \tag{5.5}$$

where ω is a path in the Wiener-space Ω, $f \in S(R^n)$, $S(R^n)$ is the set of infinitely differentiable functions on R^n, and $h_1, \dots, h_n \in L^2(\Omega\times R_+)$.

For $F\in C$, we define the Malliavin derivative DF of F by

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(\int_0^{\infty} h_1(t)dW(t) \quad ,..., \quad \int_0^{\infty} h_n(t)dW(t)\right)h_i(t), \qquad t \geq 0 \text{ a.s.} \tag{5.6}$$

We define the norm of F by

$$\|F\|_{1,2} = (E[F^2])^{1/2} + \left(E\left[\int_0^{\infty} (D_t F)^2\, dt\right]\right)^{1/2}. \tag{5.7}$$

We denote the Banach space which is the completion of C by $D^{1,2}$, with the norm $\| \bullet \|_{1,2}$. Let U be a stochastic process, $U(t) = U(\omega, t) \in L^2(\Omega \times R_+)$. For any $\phi \in D^{1,2}$ and fixed ω, both U and $D\phi$ are in $L^2(R_+) = H$. We write

$$\langle U, D\phi \rangle = \int_0^{\infty} U(t)\, D_t\phi\, dt \;, \tag{5.8}$$

for the standard inner product in $L^2(R_+)$. Note that this expression is stochastic,

$$\langle U, D\phi \rangle = \langle U, D\phi \rangle(\omega). \tag{5.9}$$

We calculate $E[\langle U, D\phi \rangle]$ by integrating over all $\omega \in \Omega$. We define δ(U), so called the Skorohod integral, as adjoint to D by

$$E[\langle U, D\phi \rangle] = E[\delta(U)\, \phi]. \tag{5.10}$$

One of the interesting facts of Malliavin Calculus is that the divergence operator δ coincides with the standard Itô integral. Let U be and adapted stochastic process in $L^2(\Omega \times R_+)$. Then we have

$$\delta(U) = \int_0^{\infty} U(t)\, dW(t). \tag{5.11}$$

For an adapted random variable $F \in D^{1,2}$, using the chain rule for Malliavin derivatives

$$D\psi(F) = \psi'(F)\, DF. \tag{5.12}$$

By integration by parts, we have

$$\begin{aligned} E[\psi'(F)] &= E\left[\left\langle D\psi(F), \frac{1}{\langle DF, DF\rangle} DF \right\rangle\right] \\ &= E\left[\psi(F)\delta\left(\frac{1}{\langle DF, DF\rangle} DF\right)\right] \\ &= E[\psi(F)\pi], \end{aligned} \tag{5.13}$$

where $\pi = \delta\left(\frac{1}{\langle DF, DF\rangle} DF\right)$ is a random variable to be determined and ψ is a pay off function. Note that (5.13) is a way of the derivation of (5.4).

For T > 0, we assume that F = X(T) is a solution to the stochastic differential equation ,

$$dX(t) = \beta(X(t))\, dt + \sigma(X(t))\, dW(t).$$

We define the tangent process $\{Y(t) = \frac{\partial X(t)}{\partial X(0)};\ t \geq 0\}$ of X(t), as the associate first variation process defined by the stochastic differential equation,

$$dY(t) = \beta'(X(t))Y(t)\,dt + \sigma'(X(t))\,Y(t)\,dW(t), \quad Y(0) = 1, \tag{5.14}$$

where primes denote derivatives.
Then the Malliavin derivative of F = X(T) is given by

$$D_s X(T) = \sigma(X(s))\,Y(s)^{-1}Y(T), \tag{5.15}$$

for s $\le$ T and zero otherwise.
For X(0) = x and F = X(T) , we want to calculate the price sensitivity with respect to x,

$$\frac{\partial}{\partial x} E[\psi(X(T))] = E[\psi'(X(T))Y(T)]. \tag{5.16}$$

Using Malliavin calculus, we want to find the weight $\pi = \delta(u)$, for some adapted process u, such that

$$\begin{aligned}
\frac{\partial}{\partial x} E[\psi(X(T))] &= E[\psi(X(T))\,\pi] \\
&= E[\psi(X(T))\,\delta(u)] \\
&= E[\langle D\psi(X(T)), u\rangle], \quad \text{by (5.10)} \\
&= E[\psi'(X(T))\langle DX(T), u\rangle], \quad \text{by (5.12)} \\
&= E\left[\psi'(X(T))\int_0^T (D_t X(T))u(t)dt\right], \text{ by (5.8)} \\
&= E\left[\psi'(X(T))Y(T)\int_0^T Y(t)^{-1}\sigma(X(t))u(t)\,dt\right], \text{ by (5.15).}
\end{aligned} \tag{5.17}$$

Comparing (5.16) and (5.17), we need

$$\int_0^T Y(t)^{-1}\sigma(X(t))\,u(t)\,dt = 1, \tag{5.18}$$

and we have a solution

$$u(t) = \frac{Y(t)}{T\sigma(X(t))}. \tag{5.19}$$

So we can find a weight $\pi = \delta(u)$ when F = X(T),

$$\pi = \delta(u) = \int_0^T \frac{Y(t)}{T\,\sigma(X(t))} dW(t) \text{ , by (5.11).} \tag{5.20}$$

By the similar way, we can find a weight $\pi = \delta(u)$ when F is the mean value of the process {X(t); 0 $\le$ t $\le$ T},

$$F = \int_0^T X(t)\,dt\,. \tag{5.21}$$

A weight $\pi = \delta(u)$ is given by

$$\pi = \delta(u) = \delta\left(\frac{2Y^2(t)}{\sigma(X(t))}\left(\int_0^T Y(s)\,ds\right)^{-1}\right). \tag{5.22}$$

6 A Hedging Strategy using BG Stochastic Mortality Model and Malliavin Calculus

Now we begin with a sale of term insurance. Under a stochastic force of mortality model, the net single premium of n-year term life insurance with face amount 1 payable at the end of the year when (x) dies is

$$A^{1}_{x:\overline{n}|} = E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\,{}_t p_x(\omega) q_{x+t}(\omega)\right], \tag{6.1}$$

where $v = 1/(1+i)$, i = annual effective interest rate, Ω is the set of scenarios, and ω is a scenario, $\omega \in \Omega$.

For each x > 0, t ≥ 0, and $\omega \in \Omega$, the survival probability ${}_t p_x(\omega)$ is

$${}_t p_x(\omega) = \exp\left(-\int_0^t \mu_x(s,\omega)ds\right), \tag{6.2}$$

where $\mu_x(s,\omega)$ is the force of mortality at age x+s on the scenario ω, and the death probability $q_{x+t}(\omega)$ is

$$q_{x+t}(\omega) = 1 - p_{x+t}(\omega). \tag{6.3}$$

We rewrite (6.1) as

$$A^{1}_{x:\overline{n}|} = E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\,{}_t p_x(\omega) - \sum_{t=0}^{n-1} v^{t+1}\,{}_{t+1} p_x(\omega)\right]. \tag{6.4}$$

As a hedging strategy, we consider the sales of n-year pure endowment to offset any losses from n-year term insurance. The net single premium of the n-year pure endowment issued to (x) is

$$A_{x:\overline{n}|}^{\;\;1} = {}_n E_x = E^{\Omega}[v^n\,{}_n p_x(\omega)]. \tag{6.5}$$

We want to offset the losses from n-year term insurance sales with the sales of n-year pure endowment. Let us define the hedge ratio, R_x, to be the number of n-year pure endowment to be sold to offset the losses from the sales of n-year term insurance policies. Let us denote the liability to be L_x,

$$L_x = A^{1}_{x:\overline{n}|} + R_x\,{}_n E_x. \tag{6.6}$$

We want to find the hedge ratio R_x such that the sensitivity of the liability with respect to the mortality rate changes equals 0,

$$\frac{\partial L_x}{\partial \mu} = 0, \tag{6.7}$$

where $\mu = \mu_x(0,\omega)$.

The liability L_x is

$$\begin{aligned} L_x &= A^{1}_{x:\overline{n}|} + R_x\,{}_n E_x \\ &= E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\,{}_t p_x(\omega) - \sum_{t=0}^{n-1} v^{t+1}\,{}_{t+1} p_x(\omega)\right] + R_x\,E^{\Omega}[v^n\,{}_n p_x(\omega)]. \end{aligned} \tag{6.8}$$

The sensitivity of the liability with respect to the mortality rate changes is

$$\frac{\partial L_x}{\partial \mu} = \frac{\partial}{\partial \mu} E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\,{}_t p_x(\omega) - \sum_{t=0}^{n-1} v^{t+1}\,{}_{t+1} p_x(\omega)\right] + R_x \frac{\partial}{\partial \mu} E^{\Omega}[v^n\,{}_n p_x(\omega)]. \tag{6.9}$$

For a fixed age x, we assume that $\mu_x(\mathrm{t},\omega) = \mu(\mathrm{x+t},\omega)$ and the dynamics of the force of mortality are given by[5]

$$\mathrm{d}\mu_x(\mathrm{t}) = \beta(\mu_x(\mathrm{t}))\,\mathrm{dt} + \sigma(\mu_x(\mathrm{t}))\,\mathrm{dW(t)}$$

$$= \{\frac{1}{2}\sigma^2 + \mathrm{b}\ln\mu_x(0) - \mathrm{b}\ln\mu_x(\mathrm{t})\}\ \mu_x(\mathrm{t})\mathrm{dt} + \sigma\ \mu_x(\mathrm{t})\,\mathrm{dW(t)}. \qquad (6.10)$$

The tangent process $\{\mathrm{Y(t)} = \frac{\partial\mu_x(t)}{\partial\mu};\ \mathrm{t} \geq 0\}$ of $\mu_x(\mathrm{t})$ is the associate first variation process defined by the stochastic differential equation,

$$\mathrm{dY(t)} = \beta'(\mu_x(\mathrm{t}))\mathrm{Y(t)}\,\mathrm{dt} + \sigma'(\mu_x(\mathrm{t}))\,\mathrm{Y(t)}\,\mathrm{dW(t)}, \quad \mathrm{Y(0)} = 1, \qquad (6.11)$$

where primes denote derivatives.
For a given x > 0, the expected value of the survival probability is

$$E^{\Omega}\left[{}_t p_x(\omega)\right] = E^{\Omega}\left[\exp\left(-\int_0^t \mu_x(s,\omega)ds\right)\right]$$

$$= E^{\Omega}\left[\psi(F(t,\omega))\right], \qquad (6.12)$$

where

$$F(t,\omega) = \int_0^t \mu_x(s,\omega)ds\,, \qquad (6.13)$$

and

$$\psi(F(t,\omega)) = \exp(-F(t,\omega)). \qquad (6.14)$$

Using the result of Malliavin Calculus, we have

$$\frac{\partial}{\partial\mu}E^{\Omega}\left[\psi(F(t,\omega))\right] = E^{\Omega}\left[\psi(F(t,\omega))\ \pi(t,\omega)\right], \qquad (6.15)$$

where the weight $\pi = \delta(u)$ is given by

$$\pi(t,\omega) = \delta(u) = \delta\left(\frac{2Y^2(s,\omega)}{\sigma(\mu_x(s,\omega))}\left(\int_0^t Y(l,\omega)\,dl\right)^{-1}\right)$$

$$= \int_0^t \frac{2Y^2(s,\omega)}{\sigma(\mu_x(s,\omega))}\left(\int_0^t Y(l,\omega)\,dl\right)^{-1} dW(s). \qquad (6.16)$$

Now the hedge ratio R_x such that the sensitivity of the liability with respect to the mortality rate changes equals 0, $\frac{\partial L_x}{\partial\mu} = 0$, is expressed as follows,

$$R_x = -\frac{E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\left\{{}_t p_x(\omega)\pi(t,\omega) - {}_{t+1}p_x(\omega)\pi(t+1,\omega)\right\}\right]}{E^{\Omega}[v^n\ {}_n p_x(\omega)\pi(n,\omega)]}. \qquad (6.17)$$

7 Numerical Examples

[5] Here we consider the dynamics of the force of mortality under the Brownian Gompertz (BG) model. The dynamics may change according to the choice of the mortality rate models.

We have conducted a simulation to calculate the hedge ratio R_x under the BG stochastic mortality model. We use the parameters b = 0.5, i = 5% for the annual effective interest rate, and σ = 0.20 and 0.23. We use $\mu_x(0) = E^{\Omega}[\mu(x,\omega)]$ as the initial values. For the 10 year term insurance, we show the hedge ratios, R_x (6.17), of the pure endowment in the Table 6.

Table 6. Hedge Ratios R_x

σ \Age	35	40	45	50	55	60
σ = 0.20	0.81378	0.90778	1.05951	1.31239	1.76686	2.72489
σ = 0.23	0.89650	0.97027	1.08998	1.29090	1.65499	2.42923

Note that the hedge ratios in Table 5 and Table 6 are different. We do not need to use any approximations so there are not residual risks when we use a stochastic mortality model and Malliavin calculus. This is an improvement of the hedging strategies. From Table 6 we can notice that the hedge ratios grow rapidly as age increases, so we need more pure endowments to hedge the mortality rate risks in the modified endowments for old ages.

8 Conclusion

We have made an observation that mortality improvement has been experienced especially for the middle-aged males for the last decades. If there exist mortality rate shocks the insurance company may face losses from life insurance sales. As a hedging strategy, the insurance company may develop modified endowments. We present the hedge ratios of pure endowments to offset the losses from term life insurance in developing modified endowment policies.

We also show the hedging strategy under the Brownian Gompertz stochastic force of mortality model using the results from Malliavin Calculus.

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