Exponential Reserves of Insurance Contracts under Jump-Diffusion Process Model

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Abstract

We consider a collective risk model for the liability process that generates claims in a portfolio of insurance policies. Also we consider a financial market for investments and assume that the risky asset price process follows a geometric Brownian motion. The model for asset and liability follows jump-diffusion process.

We calculate the indifference premiums and exponential reserves that maximize the final expectation of the portfolio values. We show that the indifference premiums are calculated under the certainty equivalence principle.

It is interesting to note that that the exponential reserves under the prospective method are the difference between the current value of the future certainty equivalent of benefits and the present value of future certainty equivalent of premiums discounted by risk free rate. Under the retrospective method, the exponential reserves are the difference between the accumulated value of the past indifference premiums using the risk free rate and the current value of the certainty equivalent of the past benefit payments. We also show the probability of defaults of the portfolio at the terminal time of the contract.

Keywords: Indifference premiums, Exponential Reserves, Certainty Equivalence Principle, Jump-diffusion Process, Prospective method, Retrospective method

1 Introduction

Actuaries usually calculate the premiums and reserve values of insurance contracts under static strategies. The prices, so called actuarial premiums, of insurance policies are calculated using the equivalence principle. The premium principles are well introduced in Bowers et al (1997). In this paper, we consider a different premium principal using the exponential utility function. Exponential premiums and exponential reserves are discussed early by Gerber (1976, 1979). We consider a dynamic financial market where the risky asset price process follows a geometric Brownian motion. We consider a collective risk model for the insurance liability process which generates claims for a portfolio of insurance policies. Then the surplus of the insurer follows a jump-diffusion process. More dynamic insurance risk models can be found in Browne (1995), Moore and Young (2003), and Jaimungal and Young (2005). The indifference premiums can be calculated by equating the optimal value without liabilities and the optimal value with indifference premiums for taking the liabilities. Young and Zariphopoulou (2002) shows single indifference premiums of insurance policies using the principle of equivalent utility. In this paper, we show additional indifference premiums such as fully-continuous annual indifference premiums and semi-continuous indifference premiums as well as single indifference premiums. We find out that the indifference annual premiums are calculated by equating the discounted certainty equivalent of future indifference premiums and the current value of the certainty equivalent of future benefits, and this is called the certainty equivalence principle.

We show the optimal investment strategies that maximize the expected exponential utility of terminal wealth. Hipp and Plum (2000), and Liu and Yang (2004) consider the optimal investment policies of using the surplus of insurance companies. A more general discussion on the optimal investment policies of an insurer with jump-diffusion process can be found in Yang and Zhang (2005) with three criterions: maximizing exponential utility function, maximizing the survival probability of insurance company, and a general objective function. Wang (2007) discusses the optimal investment strategies for an insurer by maximizing the exponential utility of the reserve at a future time.

After we get the optimal investment strategies and the indifference premiums we try to calculate the exponential reserve values of the insurance contracts. Traditionally the actuaries calculate the benefit reserves using the level benefit premiums and the claim payments. Gerber (1976, 1979) develops the reserve principles using utility functions. Different approaches to the reserve valuations using differential equations can be found in Hoem (1969), Norberg (1991) and Milbrodt and Stracke (1997). Marceau and Gaillardetz (1999) calculates the reserves under a stochastic mortality and interest rate environment for a portfolio of life insurance policies. In this paper, we develop the exponential reserve principle in parallel to the development of indifference premiums. We note that the exponential reserves under the prospective method are the difference between the current value of the future certainty equivalent of benefits and the present value of future certainty equivalent of premiums discounted by risk free rate. Also we can calculate the exponential reserves using the retrospective formula and the exponential reserves are the difference between the accumulated value of the past indifference premiums and the current value of the certainty equivalent of the past benefit payments.

We also show the premium difference formula and the paid-up insurance formula for the exponential reserves.

2 A Continuous-Time Surplus Model

We employ an utility function, u(ω), for a seller (an insurer), where ω denotes the current wealth of the insurer measured in monetary terms. Then the acceptable premium (price) H for assuming random loss (liability) L is calculated by

$$u(\omega) = E[u(\omega + H - L)]. \tag{2.1}$$

The premium H is a price such that the insurer is indifferent between the current position and providing insurance for L at premium H.

A few elementary functions, such as exponential, fractional power, quadratic and logarithmic functions, can be used as the utility function based on the decision maker's preferences for various distributions of L.

In this paper, we consider the exponential utility function,

$$u(\omega) = - e^{-\gamma\omega} \text{ for all } \omega \text{ and for a fixed } \gamma>0. \tag{2.2}$$

By simple calculation, the indifference premium is

$$H = \frac{\log M_L(\gamma)}{\gamma}, \tag{2.3}$$

where $M_L(t) = E[e^{tL}]$ is the moment generating function of L.

We consider a financial market given by a probability space (Ω, *F*, P), with the actual measure P, and a finite time horizon T. Let S = $\{S_t, 0 \le t \le T\}$ be the price process of tradable risky assets that are (P, *F*)-semi-martingale. We assume that the risky asset price process follows a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{2.4}$$

where W_t is a standard Brownian motion in a probability space (Ω, *F*, P), $\mu>0$ is the mean rate of return and $\sigma>0$ is the volatility of S.

We consider a collective risk model for L, i.e. we assume that the liability process L generates claims for a portfolio of insurance policies. Let us denote N = $\{N_t, 0 \le t \le T\}$ to be the counting process and N_t is the number of claims produced until time t. Let Y_j denote the amount of j-th claim and we assume the individual claim amounts are independent and identically distributed. The model we want to analyze is the compound Poisson claim process,

$$L_t = \sum_{j=1}^{N_t} Y_j, \tag{2.5}$$

where N = $\{N_t, 0 \le t \le T\}$ is the Poisson process with rate λ.

For the surplus (net wealth) X_t at time t, we assume that the insurer is willing to invest the amount of π_t to risky assets and the amount of $X_t - \pi_t$ to risk free assets. Then the surplus (net wealth) process X = $\{X_t, 0 \le t \le T\}$ of the insurer is expressed by

$$dX_t = (\mu - r)\pi_t dt + \sigma \pi_t dW_t + rX_t dt - dL_t, \tag{2.6}$$

where r is the risk free rate.

3 Single Indifference Premium via Exponential Utility Maximization

We consider an optimal investment problem of an insurer with an initial wealth ω and a liability L = $\{L_t,\ 0 \le t \le T\}$. The insurer wants to maximize the expected utility of the surplus (net wealth) X_T at time T. For a fixed $\gamma > 0$, we define a value function $V(X_t, t)$ with a liability process L by the maximum expected utility of the surplus at expiration time T,

$$V(X_t, t) = V(x, t; L) = \sup_{\pi} E\left[-\exp\{-\gamma X_T\} | X_t = x\right], \tag{3.1}$$

where x is the surplus at current time t.

We wish to find the single indifference premium h at time 0 such that

$$V(X_0, 0; 0) = V(X_0 + h, 0; L), \tag{3.2}$$

where $X_0 = \omega$. The single indifference premium h is a price such that the insurer is indifferent between the current position without any liability and providing insurance for L at premium h.

For simplicity we let r = 0. Let $V(X_t, t)$ be a $C^{2,1}$-function. We consider the following Itô's formula,

$$V(X_t, t) = V(X_0, 0) + \int_0^t V_t(X_s, s)ds + \int_0^t V_x(X_s, s)dX_s^c + \frac{1}{2}\int_0^t V_{xx}(X_s, s)d < X^c, X^c >_s$$
$$+ \sum_{0<s\le t}\left[V(X_s, s) - V(X_{s-}, s-)\right]. \tag{3.3}$$

If $V(X_t, t) = -e^{-\gamma X_t} f(t)$ then the Itô formula becomes

$$V(X_t, t) = V(X_0, 0) + \int_0^t V_t(X_s, s)ds + \int_0^t V_x(X_s, s)dX_s^c + \frac{1}{2}\int_0^t V_{xx}(X_s, s)d < X^c, X^c >_s$$
$$+ \sum_{0<s\le t}\left[-\exp(-\gamma(X_{s-} - Y_s)) + \exp(-\gamma X_{s-})\right] f(s)\Delta N_s$$
$$= V(X_0, 0) + \int_0^t V_t(X_s, s)ds + \int_0^t V_x(X_s, s)dX_s^c + \frac{1}{2}\int_0^t V_{xx}(X_s, s)d < X^c, X^c >_s$$
$$+ \int_0^t \exp(-\gamma X_{s-}) f(s) dQ_s , \tag{3.4}$$

where Y_i is the i-th claim amount, Y_s is the claim amount at time s if a claim occurs at times, $\Delta N_s = N_s - N_{s-}$, and $Q_s = \sum_{i=1}^{N_s}\{1 - \exp(\gamma Y_i)\} \le 0$.

Let us define a martingale process M = $\{M_s,\ 0 \le s \le T\}$,

$$M_s = Q_s - \lambda\left(1 - E[\exp \gamma Y_i]\right)s . \tag{3.5}$$

We show that the process M is martingale in Appendix A.

We can rewrite the Itô formula using the martingale process M,

$$V(X_t, t) = V(X_0, 0) + \int_0^t V_t(X_s, s)ds + \int_0^t V_x(X_s, s)dX_s^c + \frac{1}{2}\int_0^t V_{xx}(X_s, s)d < X^c, X^c >_s$$
$$+ \int_0^t \exp(-\gamma X_{s-}) f(s) dM_s + \int_0^t \exp(-\gamma X_{s-}) f(s)\, \lambda\left(1 - E[\exp(\gamma Y_i)]\right)ds . \tag{3.6}$$

This Itô formula will be used to set up the Hamilton-Jacobi-Bellman (HJB) equation and to find the optimal investment strategy and the optimal value of the expected utility of terminal surplus.

Under the dynamic programming principle the value function $V(X_t, t)$ satisfies the Hamilton-Jacobi-Bellman (HJB) equation,

$$\frac{\partial V}{\partial t}+\max_{\{\pi\}}\left\{\pi\mu V_x+\frac{1}{2}\pi^2\sigma^2V_{xx}\right\}-V(X_t,t)\lambda\left[1-E[e^{\gamma Y}]\right]=0, \tag{3.7}$$

with the boundary condition at time T,

$$V(x, T) = -e^{-\gamma x}, \quad \forall x\in R. \tag{3.8}$$

We consider $V(x,t) = -e^{-\gamma x} f(t)$ then HJB equation equals

$$-e^{-\gamma x}f'(t) + \max_{\{\pi\}}\left\{\gamma\pi\mu e^{-\gamma x}f(t)-\frac{1}{2}\gamma^2\pi^2\sigma^2e^{-\gamma x}f(t)\right\} + e^{-\gamma x}f(t)\lambda\beta = 0, \tag{3.9}$$

where $\beta = 1-E[e^{\gamma Y}] < 0$.

Canceling $e^{-\gamma x}$, we can rewrite HJB,

$$-f'(t) + \max_{\{\pi\}}\left\{\gamma\pi\mu-\frac{1}{2}\gamma^2\pi^2\sigma^2\right\}f(t) + f(t)\lambda\beta = 0. \tag{3.10}$$

The first order condition in HJB is

$$\gamma\mu-\pi\gamma^2\sigma^2 = 0,$$

and the optimal investment strategy is

$$\pi^* = \frac{\mu}{\gamma\sigma^2}. \tag{3.11}$$

We can find the optimal value $V^*(x, t)$ by plugging π^* in (3.10),

$$-f'(t) + \frac{1}{2}\frac{\mu^2}{\sigma^2}f(t) + f(t)\lambda\beta = 0.$$

Solving the equation, we have

$$f(t) = \exp\left\{-\left(\frac{1}{2}\frac{\mu^2}{\sigma^2}+\lambda\beta\right)(T-t)\right\},$$

and the optimal value (when r = 0) is

$$V^*(x, t) = V^*(x, t; L) = -e^{-\gamma x}\exp\left\{-\left(\frac{1}{2}\frac{\mu^2}{\sigma^2}+\lambda\beta\right)(T-t)\right\}, \tag{3.12}$$

where $\beta = 1-E[e^{\gamma Y}] = 1-M_Y(\gamma)$.

When r > 0, the surplus (net wealth) process X = $\{X_s, 0 \le s \le T\}$ of the insurer is expressed by

$$dX_s = (\mu - r)\pi_s\,ds + \sigma\,\pi_s\,dW_s + rX_s ds - dL_s.$$

We can simplify the surplus process by multiplying $e^{r(T-s)}$,

$$\begin{aligned} d(e^{r(T-s)}X_s) &= e^{r(T-s)}dX_s - r\,e^{r(T-s)}X_s ds \\ &= e^{r(T-s)}[(\mu - r)\pi_s\,ds + \sigma\,\pi_s\,dW_s - dL_s]. \end{aligned}$$

Let $\widetilde{X}_s = e^{r(T-s)}X_s$ and $\widetilde{\pi}_s = e^{r(T-s)}\pi_s$, then we have

$$d\widetilde{X}_s = (\mu - r)\,\widetilde{\pi}_s\,ds + \sigma\,\widetilde{\pi}_s\,dW_s - e^{r(T-s)}dL_s. \tag{3.13}$$

We can follow the same steps as if the risk free rate equals 0.

For a fixed γ>0, we define a value function $V(\widetilde{X}_t, t)$ with a liability process L by the maximum expected utility of the surplus at expiration time T,

$$V(\widetilde{X}_t, t) = V(\widetilde{x}, t; L) = \sup_{\pi} E\left[-\exp\left\{-\gamma\widetilde{X}_T\right\}|\widetilde{X}_t=\widetilde{x}\right]. \tag{3.14}$$

For a $C^{2,1}$-function V($\widetilde{X}_t$, t), we consider the following Itô's formula,

$$V(\widetilde{X}_t, t) = V(\widetilde{X}_0, 0) + \int_0^t V_t(\widetilde{X}_s, s)ds + \int_0^t \widetilde{V}_{\widetilde{x}}(\widetilde{X}_s, s)d\widetilde{X}_s^c + \frac{1}{2}\int_0^t V_{\widetilde{x}\widetilde{x}}(\widetilde{X}_s, s)d < \widetilde{X}^c, \widetilde{X}^c >_s$$
$$+ \sum_{0<s\le t}\left[V(\widetilde{X}_s, s) - V(\widetilde{X}_{s-}, s-)\right]. \tag{3.15}$$

If V($\widetilde{X}_t$, t) = $-\, e^{-\gamma \widetilde{X}_t}\widetilde{f}(t)$ then the Itô formula becomes

$$V(\widetilde{X}_t, t) = V(\widetilde{X}_0, 0) + \int_0^t V_t(\widetilde{X}_s, s)ds + \int_0^t V_{\widetilde{x}}(\widetilde{X}_s, s)d\widetilde{X}_s^c + \frac{1}{2}\int_0^t V_{\widetilde{x}\widetilde{x}}(\widetilde{X}_s, s)d < \widetilde{X}^c, \widetilde{X}^c >_s$$
$$+ \sum_{0<s\le t}\left[-\exp(-\gamma(\widetilde{X}_{s-} - e^{r(T-s)}Y_s)) + \exp(-\gamma \widetilde{X}_{s-})\right]\widetilde{f}(s)\Delta N_s$$
$$= V(\widetilde{X}_0, 0) + \int_0^t V_t(\widetilde{X}_s, s)ds + \int_0^t V_{\widetilde{x}}(\widetilde{X}_s, s)d\widetilde{X}_s^c + \frac{1}{2}\int_0^t V_{\widetilde{x}\widetilde{x}}(\widetilde{X}_s, s)d < \widetilde{X}^c, \widetilde{X}^c >_s$$
$$+ \int_0^t \exp(-\gamma \widetilde{X}_{s-})\widetilde{f}(s)d\widetilde{Q}_s, \tag{3.16}$$

where $\widetilde{Q}_s = \sum_{i=1}^{N_s}\{1 - \exp(\gamma e^{r(T-S_i)}Y_i)\} \le 0$, where Y_i is the i-th claim amount, Y_s is the claim amount at time s if a claim occurs at time s, and S_i is the i-th event time.

Let us define a martingale process $\widetilde{M} = \{\widetilde{M}_s, 0 \le s \le T\}$,

$$\widetilde{M}_s = \widetilde{Q}_s - \lambda s\left(1 - \frac{1}{s}\int_0^s E\left[\exp\left(\gamma e^{r(T-u)}Y\right)\right]du\right). \tag{3.17}$$

We show that the process $\widetilde{M}$ is martingale in Appendix B.

Using the martingale process $\widetilde{M}$ we have the following Itô formula

$$V(\widetilde{X}_t, t) = V(\widetilde{X}_0, 0) + \int_0^t V_t(\widetilde{X}_s, s)ds + \int_0^t V_{\widetilde{x}}(\widetilde{X}_s, s)d\widetilde{X}_s^c + \frac{1}{2}\int_0^t V_{\widetilde{x}\widetilde{x}}(\widetilde{X}_s, s)d < \widetilde{X}^c, \widetilde{X}^c >_s$$
$$+ \int_0^t \exp(-\gamma \widetilde{X}_{s-})\widetilde{f}(s)d\widetilde{M}_s + \int_0^t \exp(-\gamma \widetilde{X}_{s-})\widetilde{f}(s)\lambda\left(1 - E[\exp(\gamma e^{r(T-s)}Y_i)]\right)ds. \tag{3.18}$$

The value function V($\widetilde{X}_t$, t) satisfies the Hamilton-Jacobi-Bellman (HJB) equation,

$$\frac{\partial \widetilde{V}}{\partial t} + \max_{\{\pi\}}\left\{\widetilde{\pi}(\mu - r)\widetilde{V}_{\widetilde{x}} + \frac{1}{2}\widetilde{\pi}^2\sigma^2 V_{\widetilde{x}\widetilde{x}}\right\} - V(\widetilde{X}_t, t)\lambda\left[1 - E[\exp\left(\gamma e^{r(T-t)}Y\right)]\right] = 0, \tag{3.19}$$

If we try V($\widetilde{x}$, t) = $-e^{-\gamma\widetilde{x}}\,\widetilde{f}(t)$ then we can get the optimal investment strategy by solving HJB equation,

$$\widetilde{\pi}^* = \frac{\mu - r}{\gamma\sigma^2}. \tag{3.20}$$

Using $\widetilde{\pi}^*$ we can find the optimal value $V^*(\widetilde{x}, t)$. Note that

$$-\widetilde{f}'(t) + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\widetilde{f}(t) + \widetilde{f}(t)\lambda\widetilde{\beta}_t = 0,$$

where

$$\widetilde{\beta}_t = 1 - E[\exp\left(\gamma e^{r(T-t)}Y\right)]. \tag{3.21}$$

Solving the equation, we have

$$\widetilde{f}(t) = \exp\left[-\int_t^T \left(\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} + \lambda\widetilde{\beta}_s\right) ds\right],$$

and the optimal value is

$$V^*(\widetilde{x}, t) = V^*(\widetilde{x}, t; L) = -e^{-\gamma\widetilde{x}} \exp\left[-\int_t^T \left(\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} + \lambda\widetilde{\beta}_s\right) ds\right]. \quad (3.22)$$

Now we have the following theorem.

Theorem 1 *For an insurer with the current time t surplus x and the liability process L, the optimal investment strategy is*

$$\pi_t^{\,*} = e^{-r(T-t)} \frac{\mu - r}{\gamma\sigma^2}, \quad (3.23)$$

and the optimal value is

$$V^*(x, t) = V^*(x, t; L)$$

$$= -\exp\{-\gamma e^{r(T-t)} x\} \exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}(T-t)\right] \exp\left[-\int_t^T \left(\lambda\widetilde{\beta}_s\right) ds\right], \quad (3.24)$$

where $\widetilde{\beta}_s = 1 - E[\exp\{\gamma e^{r(T-s)} Y] = 1 - M_Y(\gamma e^{r(T-s)})$.

The single indifference premium h at time 0 is such that

$$V^*(\omega, 0; 0) = V^*(\omega + h, 0; L), \quad (3.25)$$

where $\omega = X_0$ is the initial wealth of the insurer.
The optimal value at time 0 with the initial wealth ω and without liability L is

$$V^*(\omega, 0; 0) = -\exp\{-\gamma e^{rT} \omega\} \exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} T\right]. \quad (3.26)$$

The optimal value at time 0 with the initial wealth ω and with additional single premium h for taking the liability L is

$$V^*(\omega + h, 0; L) = -\exp\{-\gamma e^{rT}(\omega + h)\} \exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2} T\right] \exp\left[-\int_0^T \left\{\lambda\widetilde{\beta}_s\right\} ds\right]. \quad (3.27)$$

Equating the equations (3.26) and (3.27) we have the single indifference premium h.

Theorem 2 *For an insurer with the initial surplus* ω*, the single indifference premium h for assuming the liability process L is*

$$h = e^{-rT}\,{}_TC_0 = e^{-rT} \frac{\lambda\int_0^T \left[M_Y(\gamma e^{r(T-s)}) - 1\right] ds}{\gamma}, \quad (3.28)$$

where

${}_TC_0 = \dfrac{\lambda\int_0^T \left[M_Y(\gamma e^{r(T-s)}) - 1\right] ds}{\gamma}$ *is the certainty equivalent of liability benefits from time 0 to time T.*

We will also use the notation $\bar{A}_{\overline{T}|}$ = h for the single indifference premium,

$$\bar{A}_{\overline{T}|} = e^{-rT} \; {}_{T}C_0 = e^{-rT} \frac{\lambda \int_0^T \left[M_Y(\gamma e^{r(T-s)}) - 1 \right] ds}{\gamma}.$$

The certainty equivalent of benefits from time 0 to time T, ${}_{T}C_0$, is treated as a certain amount, and then the discount factor at time 0, e^{-rT}, is applied to obtain the single indifference premium $\bar{A}_{\overline{T}|}$.

It is interesting to compare the indifference premium H without investing the surplus and the indifference premium h under the optimal investment strategy.

$$\text{H} = \frac{\log M_L(\gamma)}{\gamma} = \frac{\lambda [M_Y(\gamma) - 1] T}{\gamma}. \tag{3.29}$$

If we let r = 0 then h = H. So the indifference premium h under the optimal investment strategy is a generalized version of H considering a discount rate $r \geq 0$. Note that the indifference premium h is independent of the insurer's initial wealth ω, μ and σ^2.

4 Annual Indifference Premium

We want to find the fully continuous annual indifference premium p for a liability process L = $\{L_t, 0 \leq t \leq T\}$ under an optimal investment strategy of an insurer with an initial wealth ω. The financial market is given by a probability space (Ω, *F*, P), with the actual measure P, and a finite time horizon T. Let S = $\{S_t, 0 \leq t \leq T\}$ be the price process of tradable risky assets that are (P, *F*)-semi-martingale. We assume that the risky asset price process S follows a geometric Brownian motion (2.4) and the liability process L follows the compound Poisson claim process (2.5).

We assume that the insurer's investment strategy is to invest the amount of π_t to risky assets and the amount of $X_t - \pi_t$ to risk free assets. Then the surplus process X = $\{X_t, 0 \leq t \leq T\}$ of the insurer is expressed by

$$dX_t = (\mu - r)\pi_t \, dt + \sigma \, \pi_t \, dW_t + rX_t dt + p_t \, dt - dL_t, \tag{4.1}$$

where r is the risk free rate and p_t is the fully continuous annual indifference premium at time t.

For a fixed $\gamma > 0$, the value function $V(X_t, t)$ with the liability process L is defined by the maximum expected utility of the surplus at expiration time T,

$$V(X_t, t) = V(x, t; L, p_t) = \sup_{\pi} E\left[-\exp\{-\gamma X_T\} | X_t = x\right], \tag{4.2}$$

where x is the surplus at current time t.

The continuous annual indifferent premium p at time 0 is a price that satisfies

$$V(X_0, 0; 0, 0) = V(X_0, 0; L, p), \tag{4.3}$$

where $X_0 = \omega$. The annual indifference premium p is a price such that the insurer is indifferent between the current position without any liability and providing insurance for L at the premium rate c paid continuously.

For a moment, we assume r = 0. We assume that the value function, $V(X_t, t)$, is a $C^{2,1}$-function and satisfies the Hamilton-Jacobi-Bellman (HJB) equation,

$$\frac{\partial V}{\partial t}+\max_{\{\pi\}}\left\{(\pi\mu+p_t)V_x+\frac{1}{2}\pi^2\sigma^2V_{xx}\right\}-V(X_t,t)\lambda\left[1-E[e^{\gamma Y}]\right]=0, \tag{4.4}$$

with the boundary condition at time T,

$$V(x, T) = -e^{-\gamma x} \quad , \qquad \forall x\in R. \tag{4.5}$$

We try $V(x,t) = -e^{-\gamma x} f(t)$ then HJB equation equals

$$-f'(t) + \max_{\{\pi\}}\left\{\gamma(\pi\mu+p_t)-\frac{1}{2}\gamma^2\pi^2\sigma^2\right\}f(t) + f(t)\lambda\beta = 0, \tag{4.6}$$

where $\beta = 1-E[e^{\gamma Y}] < 0$.

The first order condition in HJB is

$$\gamma\mu-\pi\gamma^2\sigma^2 = 0,$$

and the optimal investment strategy is

$$\pi^* = \frac{\mu}{\gamma\sigma^2}. \tag{4.7}$$

By plugging π^* in HJB equation (4.6), we have

$$-f'(t) + \left(\frac{1}{2}\frac{\mu^2}{\sigma^2}+\gamma p_t\right)f(t) + f(t)\lambda\beta = 0.$$

Solving the equation, we find f(t),

$$f(t) = \exp\left\{-\left(\frac{1}{2}\frac{\mu^2}{\sigma^2}+\lambda\beta+\gamma p_t\right)(T-t)\right\},$$

and the optimal value (when r = 0), $V^*(x, t)$,is

$$V^*(x, t) = V^*(x, t; L, p_t) = -e^{-\gamma x}\exp\left\{-\left(\frac{1}{2}\frac{\mu^2}{\sigma^2}+\lambda\beta+\gamma c\right)(T-t)\right\}, \tag{4.8}$$

where $\beta = 1-E[e^{\gamma Y}] = 1-M_Y(\gamma)$.

When r > 0, the surplus (net wealth) process X = $\{X_s,\ 0 \le s \le T\}$ of the insurer is expressed by

$$dX_s = (\mu - r)\pi_s\, ds + \sigma\,\pi_s\, dW_s + rX_s ds + p_s\, ds - dL_s.$$

We can simplify the surplus process by multiplying $e^{r(T-s)}$,

$$\begin{aligned} d(e^{r(T-s)} X_s) &= e^{r(T-s)}\, dX_s - r\, e^{r(T-s)} X_s ds \\ &= e^{r(T-s)} [(\mu - r)\pi_s\, ds + \sigma\,\pi_s\, dW_s + p_s\, ds - dL_s]. \end{aligned}$$

Let $\widetilde{X}_s = e^{r(T-s)} X_s$, $\widetilde{p}_s = e^{r(T-s)} p_s$, and $\widetilde{\pi}_s = e^{r(T-s)}\pi_s$, then we have

$$d\widetilde{X}_s = (\mu - r)\,\widetilde{\pi}_s\, ds + \sigma\,\widetilde{\pi}_s\, dW_s + \widetilde{p}_s\, ds - e^{r(T-s)}\, dL_s. \tag{4.9}$$

We derive the optimal investment strategy and the optimal value in the following theorem

Theorem 3 *Let the risk free rate r > 0. We assume that the current time t surplus of an insurer is x, $X_t = x$, and the insurer assumes the liability process L with continuously paid premium p_t. Then the optimal investment strategy at time t is*

$$\pi_t^{\ *} = e^{-r(T-t)} \frac{\mu - r}{\gamma\sigma^2}, \tag{4.10}$$

and the optimal value is

$V^*(x, t) = V^*(x, t; L, \ p_t)$

$$= -\exp\{-\gamma e^{r(T-t)} x\} \ \exp\left[-\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}(T-t)\right] \exp\left[-\gamma \frac{p_t}{r}\left\{e^{r(T-t)} - 1\right\}\right] \exp\left[-\int_t^T \lambda \tilde{\beta}_s ds\right], \tag{4.11}$$

where $\tilde{\beta}_s = 1 - E[\exp\{\gamma e^{r(T-s)} Y] = 1 - M_Y(\gamma e^{r(T-s)})$.

Proof For a $C^{2,1}$-function $V(\tilde{X}_t, t)$, we consider the Hamilton-Jacobi-Bellman (HJB) equation,

$$\frac{\partial V}{\partial t} + \max_{\{\tilde{\pi}\}}\left\{(\tilde{\pi}(\mu - r) + \tilde{p}_t)V_{\tilde{x}} + \frac{1}{2}\tilde{\pi}^2\sigma^2 V_{\tilde{x}\tilde{x}}\right\} - V(\tilde{X}_t, t)\lambda\left[1 - E[e^{\gamma\tilde{Y}_t}]\right] = 0, \tag{4.12}$$

where $\tilde{Y}_t = Y\, e^{r(T-t)}$.

The boundary condition at time T is,

$$V(x, T) = -e^{-\gamma x} \quad , \qquad \forall x \in R. \tag{4.13}$$

We try $V(\tilde{x}, t) = -\exp(-\gamma\tilde{x})\, f(t)$ then HJB equation becomes

$$-f'(t) + \max_{\{\tilde{\pi}\}}\left\{\gamma\{\tilde{\pi}(\mu - r) + \tilde{p}_t\} - \frac{1}{2}\gamma^2\tilde{\pi}^2\sigma^2\right\} f(t) + f(t)\lambda\tilde{\beta}_t = 0, \tag{4.14}$$

where $\tilde{\beta}_t = 1 - E\left[\exp(\gamma\tilde{Y}_t)\right] < 0$.

The first order condition in HJB is

$$\gamma(\mu - r) - \tilde{\pi}\gamma^2\sigma^2 = 0,$$

and the optimal investment strategy is

$$\tilde{\pi}_t^{\ *} = e^{r(T-t)}\ \pi^*_{\ t} = \frac{\mu - r}{\gamma\sigma^2},$$

so

$$\pi^*_{\ t} = e^{-r(T-t)}\ \frac{\mu - r}{\gamma\sigma^2}.$$

By plugging $\tilde{\pi}_t^{\ *}$ in HJB equation (4.14), we have

$$-f'(t) + \left(\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} + \gamma\tilde{p}_t\right) f(t) + f(t)\lambda\tilde{\beta}_t = 0.$$

Solving the equation, we find f(t),

$$f(t) = \exp\left[-\int_t^T \left\{\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} + \gamma e^{r(T-s)} p_t + \lambda\tilde{\beta}_s\right\} ds\right],$$

and we have the optimal value $V^*(x, t)$. □

At current time t, we can calculate the fully continuous annual indifference premium p_t.

Theorem 4 (Certainty Equivalence Principle) *For an insurer, the fully continuous annual indifference premium, p_t = $\bar{P}(\bar{A}_{\overline{T-t|}})$, at time t for assuming the liability process L is calculated by*

$$\bar{P}(\bar{A}_{\overline{T-t|}}) = \frac{{}_T C_t}{\bar{s}_{\overline{T-t|}}} = \frac{e^{-r(T-t)}\,{}_T C_t}{\bar{a}_{\overline{T-t|}}} = \frac{\bar{A}_{\overline{T-t|}}}{\bar{a}_{\overline{T-t|}}}, \tag{4.15}$$

where

$$\bar{s}_{\overline{T-t|}} = \frac{e^{r(T-t)}-1}{r}, \tag{4.16}$$

$${}_T C_t = \frac{\lambda}{\gamma}\int_t^T \left[M_Y(\gamma e^{r(T-s)})-1\right]ds, \tag{4.17}$$

$$\bar{a}_{\overline{T-t|}} = \frac{1-e^{-r(T-t)}}{r}, \tag{4.18}$$

and

$$\bar{A}_{\overline{T-t|}} = e^{-r(T-t)}\,{}_T C_t. \tag{4.19}$$

Proof At time t, the indifference annual premium is the price such that

$$V^*(x, t; 0, 0) = V^*(x, t; L, p_t), \tag{4.20}$$

where x = X_t is the t time surplus of the insurer.
The optimal value at time t without assuming the liability L is

$$V^*(x, t; 0, 0) = -\exp\{-\gamma e^{r(T-t)}x\}\exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}(T-t)\right]. \tag{4.21}$$

The optimal value at time t with additional premium p_t for taking the liability L is
$V^*(x, t; L, p_t)$ =

$$-\exp\{-\gamma e^{r(T-t)}x\}\exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}(T-t)\right]\exp\left[-\gamma\frac{p_t}{r}\left\{e^{r(T-t)}-1\right\}\right]\exp\left[-\int_t^T \lambda\tilde{\beta}_s ds\right], \tag{4.22}$$

where $\tilde{\beta}_s = 1 - E[\exp\{\gamma e^{r(T-s)}Y\}] = 1 - M_Y(\gamma e^{r(T-s)})$.
Equating the equations (4.21) and (4.22) we have the fully continuous annual indifference premium. □

Note that ${}_T C_t = \frac{\lambda}{\gamma}\int_t^T \left[M_Y(\gamma e^{r(T-s)})-1\right]ds$ is the certainty equivalent of benefits from time t to time T. And then we apply a discount factor $e^{-r(T-t)}$ to obtain the single indifference premium at time t, $\bar{A}_{\overline{T-t|}} = e^{-r(T-t)}\,{}_T C_t$, called the current value of the certainty equivalent of future benefits. Also note that $\bar{P}(\bar{A}_{\overline{T-t|}})\,\bar{a}_{\overline{T-t|}}$ is the discounted certainty equivalent of future annual indifference premiums. So the annual indifference premium, $\overline{P}(\overline{A}_{\overline{T-t|}})$, is calculated by

The discounted certainty equivalent of future indifference premiums
= The current value of the certainty equivalent of future benefits,

$$\overline{P}(\overline{A}_{\overline{T-t|}})\ \overline{a}_{\overline{T-t|}} = \overline{A}_{\overline{T-t|}}. \tag{4.23}$$

This premium calculation method is called *the certainty equivalence principle*.

At time 0, the fully continuous annual indifference premium, $p_0 = \overline{P}(\overline{A}_{\overline{T|}})$ is

$$\overline{P}(\overline{A}_{\overline{T|}}) = \frac{{}_TC_0}{\overline{s}_{\overline{T|}}} = \frac{r}{e^{rT}-1}\frac{\lambda}{\gamma}\int_0^T\left[M_Y(\gamma e^{r(T-s)})-1\right]ds = \frac{\overline{A}_{\overline{T|}}}{\overline{a}_{\overline{T|}}}. \tag{4.24}$$

Note that the single indifference premium $\overline{A}_{\overline{T|}}$ is

$$\overline{A}_{\overline{T|}} = e^{-rT}\ {}_TC_0 = \frac{e^{-rT}\ \lambda\int_0^T\left[M_Y(\gamma e^{r(T-s)})-1\right]ds}{\gamma}.$$

It is easy to check that the total of the annual indifference premiums is bigger than the single indifference premium, i.e., $T\,\overline{P}(\overline{A}_{\overline{T|}}) > \overline{A}_{\overline{T|}}$ when $r > 0$. This is because of the discount effects. Also the relationship between the single indifference premium $\overline{A}_{\overline{T|}}$ and the annual indifference premium $\overline{P}(\overline{A}_{\overline{T|}})$ is

$$\overline{A}_{\overline{T|}} = \overline{P}(\overline{A}_{\overline{T|}})\ \overline{a}_{\overline{T|}}. \tag{4.25}$$

When r = 0, the annual premium p is

$$p = \frac{\lambda\left[M_Y(\gamma)-1\right]}{\gamma} = \frac{\log M_{L_1}(\gamma)}{\gamma},$$

where L_1 is the liability process with one year of maturity, T = 1.
The single indifference premium when r = 0 is

$$H = \frac{\log M_L(\gamma)}{\gamma} = \frac{\lambda\left[M_Y(\gamma)-1\right]}{\gamma}T = pT.$$

So the traditional indifference premium H does not consider the discount effects. But the indifference premiums $\overline{A}_{\overline{T|}}$ and $\overline{P}(\overline{A}_{\overline{T|}})$ under the optimal investment strategies consider the discount effects on the liability payments and the premium incomes with risk free rate $r > 0$. This is an improvement since the time value of the cash flows should be considered in pricing.

We consider the insurance contracts with a premium paying period h that is shorter than the period over which benefits are paid, h < T. We can show that the indifference premium is calculated by

$${}_h\overline{P}(\overline{A}_{\overline{T|}}) = \frac{\overline{A}_{\overline{T|}}}{\overline{a}_{\overline{h|}}}. \tag{4.26}$$

We can also derive the discrete type annual indifference premium, $P(\bar{A}_{\overline{T|}})$. The first premium is payable when the insurance is issued and the subsequent premiums are payable on anniversaries of the policy issue, so the set of annual premiums form an annuity due,

$$P(\bar{A}_{\overline{T|}}) = \frac{\bar{A}_{\overline{T|}}}{\ddot{a}_{\overline{T|}}}, \tag{4.27}$$

where $\ddot{a}_{\overline{T|}} = \frac{1-v^T}{d}$, d =1 – v, v = $\frac{1}{1+i}$, and 1 + i = e^r.

We consider that premiums are payable m times a year with no adjustment in the benefits. The level annual indifference premium payable in m-thly installments at the beginning of each m-thly period is denoted by $P^{(m)}(\bar{A}_{\overline{T|}})$. Using the certainty equivalence principle we can calculate $P^{(m)}(\bar{A}_{\overline{T|}})$ by

$$P^{(m)}(\bar{A}_{\overline{T|}}) = \frac{\bar{A}_{\overline{T|}}}{\ddot{a}^{(m)}_{\overline{T|}}}, \tag{4.28}$$

where $\ddot{a}^{(m)}_{\overline{T|}} = \frac{1-v^T}{d^{(m)}}$, and $d^{(m)} = m\left[1-(1-d)^{1/m}\right]$.

The formulas for the indifference premiums look similar to those for the benefit premiums under equivalence principle. But there are a few differences between them. First, note that i is not the expected interest rate of the insurer used for the benefit premiums under equivalence principle. It is the annual risk free rate. The present value of future benefits is considered under the equivalence principle, but the current value of the certainty equivalent of future benefits is considered under the certainty equivalence principle.

5 Exponential Reserves

At time t, 0 ≤ t ≤ T, we want to find, so called, the exponential reserves. The insurer, under the utility maximization principle of terminal surplus, will be indifference between continuing the risk while receiving premiums and paying the amount of ${}_t\bar{V}$ to a reinsurer to assume the risk.

Definition 1 *The exponential reserve, ${}_t\bar{V}$, is the amount such that*

$$V^*(x - {}_t\bar{V}, t; 0) = V^*(x, t; L), \tag{5.1}$$

where x = X_t is the surplus of the insurer at time t.

Let us first consider the insurance contract with the single indifference premium h = $\bar{A}_{\overline{T|}}$.

Theorem 5 *The exponential reserves,* ${}_t\overline{V}$ *, of the insurance contract with the single indifference premium* $\overline{A}_{\overline{T}|}$ *is*

$$ {}_t\overline{V} \;=\; \overline{A}_{\overline{T-t}|}. \tag{5.2}$$

Proof Applying Theorem 1 to (5.1), we have

$$-\exp\{-\gamma e^{r(T-t)} \;\; (x \;-\; {}_t\overline{V})\} \exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}(T-t)\right]$$

$$= -\exp\{-\gamma e^{r(T-t)}\, x\} \exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}(T-t)\right] \exp\left[\lambda\int_t^T \left[M_Y(\gamma e^{r(T-s)})-1\right]ds\right].$$

Simplifying the above equation, we have the exponential reserves,

$$ {}_t\overline{V} \;=\; e^{-r(T-t)}\,\frac{\lambda}{\gamma}\int_t^T \left[M_Y(\gamma e^{r(T-s)})-1\right]ds$$

$$ = e^{-r(T-t)}\;{}_TC_t \;=\; \overline{A}_{\overline{T-t}|}. \qquad \square$$

So the exponential reserve is the current value of the certainty equivalent of future benefits, $\overline{A}_{\overline{T-t}|} \;=\; e^{-r(T-t)}\;{}_TC_t$, for the insurance contracts with a single indifference premium.

For the insurance contract with the fully continuous annual indifference premium $\overline{P}(\overline{A}_{\overline{T}|})$, we can find the exponential reserve, ${}_t\overline{V}\,(\overline{A}_{\overline{T}|})$, from the following formula,

$$V^*(x - {}_t\overline{V}(\overline{A}_{\overline{T}|}),\, t;\, 0,\, 0) \;=\; V^*(x,\, t;\, L,\, \overline{P}(\overline{A}_{\overline{T}|}))\,, \tag{5.3}$$

where $x = X_t$ is the surplus of the insurer at time t.

Theorem 6 (Prospective method) *The exponential reserves,* ${}_t\overline{V}(\overline{A}_{\overline{T}|})$*, of the insurance contract with the fully continuous annual indifference premium* $\overline{P}(\overline{A}_{\overline{T}|})$ *is*

$$ {}_t\overline{V}(\overline{A}_{\overline{T}|}) \;=\; \overline{A}_{\overline{T-t}|} \;-\overline{P}(\overline{A}_{\overline{T}|})\;\overline{a}_{\overline{T-t}|}, \tag{5.4}$$

where $\overline{a}_{\overline{T-t}|} \;=\; \dfrac{1-e^{-r(T-t)}}{r}$.

Proof Applying Theorem 3 to (5.3), we have

$$V^*(x - {}_t\overline{V}(\overline{A}_{\overline{T}|}),\, t;\, 0,\, 0) = -\exp\{-\gamma e^{r(T-t)} \;\; (x \;-\; {}_t\overline{V}(\overline{A}_{\overline{T}|})\} \exp\left[-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}(T-t)\right],$$

and we find the exponential reserve,

$$ {}_t\overline{V}(\overline{A}_{\overline{T}|}) \;=\; e^{-r(T-t)}\,\frac{\lambda}{\gamma}\int_t^T \left[M_Y(\gamma e^{r(T-s)})-1\right]ds \;-\; \overline{P}(\overline{A}_{\overline{T}|})\;\frac{1-e^{-r(T-t)}}{r}$$

$$ = \overline{A}_{\overline{T-t}|} \;-\; \overline{P}(\overline{A}_{\overline{T}|})\;\overline{a}_{\overline{T-t}|} \qquad \square$$

It is interesting to note that that the exponential reserves ${}_t\overline{V}(\overline{A}_{\overline{T|}}) = \overline{A}_{\overline{T-t|}} - \overline{P}(\overline{A}_{\overline{T|}})\ \overline{a}_{\overline{T-t|}}$ are the difference between the current value of the future certainty equivalent of benefits and the present value of future certainty equivalent of premiums discounted by risk free rate, based on the indifference premiums determined by the certainty equivalence principle;

Exponential Reserves =
Current value of future certainty equivalent of benefits
– Present value of future certainty equivalent of premiums.

This calculation method of exponential reserves is called the prospective method.

For the insurance contracts with a premium paying period h that is shorter than the period over which benefits are paid, h < T, the exponential reserve at time t is

$$ {}_t^h\overline{V}(\overline{A}_{\overline{T|}}) = \begin{cases} \overline{A}_{\overline{T-t|}} - {}_h\overline{P}(\overline{A}_{\overline{T|}})\overline{a}_{\overline{h-t|}}, & t \le h \le T \\ \overline{A}_{\overline{T-t|}}, & h < t \le T \end{cases} . \tag{5.5} $$

From Thoerem 4, the fully continuous annual indifference premium $\overline{P}(\overline{A}_{\overline{T-t|}})$ at time t for assuming the liability process L is calculated by

$$ \overline{P}(\overline{A}_{\overline{T-t|}})\ \overline{a}_{\overline{T-t|}} = \overline{A}_{\overline{T-t|}} . \tag{5.6} $$

We can derive the premium-difference formula for ${}_t\overline{V}(\overline{A}_{\overline{T|}})$ by factoring $\overline{a}_{\overline{T-t|}}$ out of the reserve formula,

$$ \begin{aligned} {}_t\overline{V}(\overline{A}_{\overline{T|}}) &= \overline{A}_{\overline{T-t|}} - \overline{P}(\overline{A}_{\overline{T|}})\ \overline{a}_{\overline{T-t|}} \\ &= \overline{P}(\overline{A}_{\overline{T-t|}})\ \overline{a}_{\overline{T-t|}} - \overline{P}(\overline{A}_{\overline{T|}})\ \overline{a}_{\overline{T-t|}} \\ &= (\overline{P}(\overline{A}_{\overline{T-t|}}) - \overline{P}(\overline{A}_{\overline{T|}}))\ \overline{a}_{\overline{T-t|}} . \end{aligned} \tag{5.7} $$

The above formula shows the reserve as the present value of the premium difference payable over the remaining premium-payment term using the risk free discount rate r. The premium difference is obtained by subtracting the time 0 annual indifference premium $\overline{P}(\overline{A}_{\overline{T|}})$ from the fully continuous annual indifference premium $\overline{P}(\overline{A}_{\overline{T-t|}})$ for an insurance issued at time t for the remaining term.

We can also derive a formula, called paid-up insurance formula, for ${}_t\overline{V}(\overline{A}_{\overline{T|}})$ by factoring the current value of the equivalent of future benefits, $\overline{A}_{\overline{T-t|}}$, out of the reserve formula,

$$ {}_t\overline{V}(\overline{A}_{\overline{T|}}) = \overline{A}_{\overline{T-t|}} - \overline{P}(\overline{A}_{\overline{T|}})\ \overline{a}_{\overline{T-t|}} $$

$$= \left[1 \ - \ \bar{P}(\bar{A}_{\overline{T|}}) \frac{\bar{a}_{\overline{T-t|}}}{\bar{A}_{\overline{T-t|}}} \right] \bar{A}_{\overline{T-t|}}$$

$$= \left[1 \ - \ \frac{\bar{P}(\bar{A}_{\overline{T|}})}{\bar{P}(\bar{A}_{\overline{T-t|}})} \right] \bar{A}_{\overline{T-t|}} . \tag{5.8}$$

Theorem 7 (Retrospective method) *The exponential reserves,* ${}_t\bar{V}(\bar{A}_{\overline{T|}})$, *of the insurance contract with the fully continuous annual indifference premium* $\bar{P}(\bar{A}_{\overline{T|}})$ *is calculated by*

$$ {}_t\bar{V}(\bar{A}_{\overline{T|}}) \ = \ \bar{P}(\bar{A}_{\overline{T|}}) \ \bar{s}_{\overline{t|}} - {}_tK_0 \ , \tag{5.9}$$

where ${}_tK_0 \ = {}_tC_0 \ e^{-r(T-t)}$ *and* $\bar{s}_{\overline{t|}} \ = \ \dfrac{e^{rt}-1}{r}$.

Proof From the prospective formula of reserves,

$$\begin{aligned}
{}_t\bar{V}(\bar{A}_{\overline{T|}}) \ &= \ \bar{A}_{\overline{T-t|}} \ - \bar{P}(\bar{A}_{\overline{T|}}) \ \bar{a}_{\overline{T-t|}} \\
&= e^{-r(T-t)} \ \frac{\lambda}{\gamma} \int_t^T \left[M_Y(\gamma e^{r(T-s)}) - 1 \right] ds \ - \ \bar{P}(\bar{A}_{\overline{T|}}) \ \frac{1-e^{-r(T-t)}}{r} \\
&= e^{-r(T-t)} \ \frac{\lambda}{\gamma} \left[\int_0^T \left[M_Y(\gamma e^{r(T-s)}) \ - \ 1 \right] ds - \int_0^t \left[M_Y(\gamma e^{r(T-s)}) \ - \ 1 \right] ds \right] \\
&\quad - \ \bar{P}(\bar{A}_{\overline{T|}}) \ \frac{1-e^{-r(T-t)}}{r} \\
&= e^{-r(T-t)} \ \frac{\lambda}{\gamma} \int_0^T \left[M_Y(\gamma e^{r(T-s)}) \ - \ 1 \right] ds \ - e^{-r(T-t)} \ \frac{\lambda}{\gamma} \int_0^t \left[M_Y(\gamma e^{r(T-s)}) \ - \ 1 \right] ds \\
&\quad - \ \bar{P}(\bar{A}_{\overline{T|}}) \ \frac{1-e^{-r(T-t)}}{r} \\
&= e^{rt} \left[e^{-rT} \ \frac{\lambda}{\gamma} \int_0^T \left[M_Y(\gamma e^{r(T-s)}) \ - \ 1 \right] ds - \bar{P}(\bar{A}_{\overline{T|}}) \frac{e^{-rt} - e^{-rT}}{r} \right] - e^{-r(T-t)} \ {}_tC_0 \\
&= e^{rt} \left[\bar{P}(\bar{A}_{\overline{T|}}) \frac{1-e^{-rT}}{r} - \bar{P}(\bar{A}_{\overline{T|}}) \frac{e^{-rt} - e^{-rT}}{r} \right] - {}_tK_0 \\
&= \bar{P}(\bar{A}_{\overline{T|}}) \ e^{rt} \left[\frac{1-e^{-rt}}{r} \right] - {}_tK_0 \\
&= \bar{P}(\bar{A}_{\overline{T|}}) \ \bar{s}_{\overline{t|}} \ - {}_tK_0 \ .
\end{aligned}$$

□

The retrospective formula for the exponential reserves, ${}_t\bar{V}(\bar{A}_{\overline{T|}}) = \bar{P}(\bar{A}_{\overline{T|}}) \ \bar{s}_{\overline{t|}} \ - {}_tK_0$, is the difference between the accumulated value of the past indifference premiums using the risk free rate and the current value of the certainty equivalent of the past benefit payments from time 0 to current time t,

Exponential Reserves =
Accumulation of past premiums
– Current value of the past certainty equivalent of benefit payments.

For the insurance contracts with the discrete type annual indifference premium $P(\bar{A}_{\overline{T|}}) = \dfrac{\bar{A}_{\overline{T|}}}{\ddot{a}_{\overline{T|}}}$, the exponential reserve at time t is calculated by

$$
\begin{aligned}
{}_tV(\bar{A}_{\overline{T|}}) &= \bar{A}_{\overline{T-t|}} - P(\bar{A}_{\overline{T|}})\, \ddot{a}_{\overline{T-t|}} \\
&= P(\bar{A}_{\overline{T|}})\, \ddot{s}_{\overline{t|}} - {}_tK_0 , \qquad (5.10)
\end{aligned}
$$

where $\ddot{a}_{\overline{T-t|}} = \dfrac{1-v^{T-t}}{d}$, and $\ddot{s}_{\overline{t|}} = \dfrac{e^{rt}-1}{d}$.

For the insurance contracts with the level annual indifference premium payable in m-thly installments at the beginning of each m-thly period, $P^{(m)}(\bar{A}_{\overline{T|}}) = \dfrac{\bar{A}_{\overline{T|}}}{\ddot{a}^{(m)}_{\overline{T|}}}$ the exponential reserve at time t is calculated by

$$
\begin{aligned}
{}_tV^{(m)}(\bar{A}_{\overline{T|}}) &= \bar{A}_{\overline{T-t|}} - P^{(m)}(\bar{A}_{\overline{T|}})\, \ddot{a}^{(m)}_{\overline{T-t|}} \\
&= P^{(m)}(\bar{A}_{\overline{T|}})\, \ddot{s}^{(m)}_{\overline{t|}} - {}_tK_0 , \qquad (5.11)
\end{aligned}
$$

where $\ddot{a}^{(m)}_{\overline{T-t|}} = \dfrac{1-v^{T-t}}{d^{(m)}}$, $\ddot{s}^{(m)}_{\overline{t|}} = \dfrac{e^{rt}-1}{d^{(m)}}$, and $d^{(m)} = m\left[1-(1-d)^{1/m}\right]$.

6 Probability of Default at Terminal Time

At terminal time T the surplus X_T can be less than 0. We say that default occurs when the surplus goes below 0. The probability of default can be positive at any time even though we have the optimal investment strategy. In this section we calculate the probability of default at the maturity time T.

For the insurance contracts with the single indifference premium h = $\bar{A}_{\overline{T|}}$ and with r > 0, we have the following formula by plugging the optimal investment strategy $\tilde{\pi}^* = \dfrac{\mu - r}{\gamma\sigma^2}$ to (3.13),

$$
d\tilde{X}_s = \frac{(\mu-r)^2}{\gamma\sigma^2}\, ds + \frac{\mu-r}{\gamma\sigma}\, dW_s - e^{r(T-s)}\, dL_s, \qquad (6.1)
$$

where $\tilde{X}_s = e^{r(T-s)} X_s$.

The final surplus X_T is expressed by solving the above formula,

$$
X_T = e^{rT}(\omega + h) + \frac{(\mu-r)^2}{\gamma\sigma^2}\, T + \frac{\mu-r}{\gamma\sigma}\, W_T - L, \qquad (6.2)
$$

where ω is the initial surplus

and

$$L = \sum_{0<s\le T} e^{r(T-s)} Y_s \Delta N(s)\,. \tag{6.3}$$

The probability of default at terminal time T is

$$\Pr\{X_T \le 0\} = E[1_{\{X_T\le 0\}}] = E[E[1_{\{X_T\le 0\}} \mid L]]\,. \tag{6.4}$$

We consider the conditional expectation,

$$\begin{aligned}
E[1_{\{X_T\le 0\}} \mid L] &= \Pr\{X_T \le 0 \mid L\} \\
&= \Pr\left\{ e^{rT}(\omega+h) + \frac{(\mu-r)^2}{\gamma\sigma^2}T + \frac{\mu-r}{\gamma\sigma}W_T - L \le 0 \right\} \\
&= \Pr\left\{ W_T \le \frac{\gamma\sigma}{\mu-r}L - e^{rT}(\omega+h)\frac{\gamma\sigma}{\mu-r} - \frac{\mu-r}{\sigma}T \right\}, \text{ we assume } \mu-r>0, \\
&= \Pr\left\{ \frac{W_T}{\sqrt{T}} \le \frac{1}{\sqrt{T}}\frac{\gamma\sigma}{\mu-r}L - \frac{e^{rT}}{\sqrt{T}}(\omega+h)\frac{\gamma\sigma}{\mu-r} - \frac{\mu-r}{\sigma}\sqrt{T} \right\} \\
&= \Phi\left\{ \frac{1}{\sqrt{T}}\frac{\gamma\sigma}{\mu-r}L - \frac{e^{rT}}{\sqrt{T}}(\omega+h)\frac{\gamma\sigma}{\mu-r} - \frac{\mu-r}{\sigma}\sqrt{T} \right\},
\end{aligned} \tag{6.5}$$

where Φ is the cumulative density function of the standard normal distribution.
The probability of default at terminal time T is

$$\begin{aligned}
\Pr\{X_T \le 0\} &= E[1_{\{X_T\le 0\}}] = E[E[1_{\{X_T\le 0\}} \mid L]] \\
&= E\left[\Phi\left\{ \frac{1}{\sqrt{T}}\frac{\gamma\sigma}{\mu-r}L - \frac{e^{rT}}{\sqrt{T}}(\omega+h)\frac{\gamma\sigma}{\mu-r} - \frac{\mu-r}{\sigma}\sqrt{T} \right\} \right] \qquad (6.6) \\
&= E\left[\Phi\{aK+b\}\right],
\end{aligned}$$

where

$$a = \frac{e^{rT}}{\sqrt{T}}\frac{\gamma\sigma}{\mu-r}\,, \tag{6.7}$$

$$b = -\frac{e^{rT}}{\sqrt{T}}\frac{\gamma\sigma}{\mu-r}\,\omega - \frac{\gamma\sigma}{\mu-r}\frac{1}{\sqrt{T}}\frac{\lambda}{\gamma}\int_0^T \left[M_Y(\gamma e^{r(T-s)}) - 1\right]ds - \frac{\mu-r}{\sigma}\sqrt{T}\,, \tag{6.8}$$

$$K = \sum_{i=1}^{N(T)} e^{-rS_i} Y_i\,, \tag{6.9}$$

and S_i is the time of i-th claim, and Y_i is the amount of i-th claim.
We can simplify the probability of default at terminal time T,

$$\begin{aligned}
\Pr\{X_T \le 0\} &= E\left[\Phi\ (aK\ +\ b)\right], \\
&= E\left[E\left[\Phi\ (aK\ +\ b)\ \mid\ N(T)\right]\right] \\
&= \sum_{n=0}^{\infty} E\left[\Phi(aK_n + b)\right]\Pr(N(T)=n) \\
&= \sum_{n=0}^{\infty}\int_0^{\infty}\Phi(ak+b)\,dF_{K_n}(k)\Pr(N(T)=n)\,,
\end{aligned} \tag{6.10}$$

where

$$\mathrm{K_n} = \sum_{i=1}^{n} e^{-rS_i} Y_i \,, \tag{6.11}$$

$$\begin{aligned} F_{K_n}(\mathrm{k}) &= \Pr(\mathrm{K_n} \le \mathrm{k}) \\ &= \Pr\left[\sum_{i=1}^{n} e^{-rS_i} Y_i \le k\right] \\ &= \int_0^T \Pr\left[\sum_{i=1}^{n} e^{-rs} Y_i \le k\right] \Pr\left[S_i = s\right] ds \\ &= \frac{1}{T}\int_0^T F_Y^{*n}(ke^{rs})\,ds \,, \end{aligned} \tag{6.12}$$

and F_Y^{*n} is the n-th convolution of F_Y ,

$$F_Y^{*n}(\mathrm{y}) = \Pr(\mathrm{Y_1 + Y_2 + Y_3 + \ldots\ldots + Y_n \le y}). \tag{6.13}$$

We want to know the expectation of the final surplus under the optimal investment strategies. The expected value of the final surplus X_T is expressed by

$$\begin{aligned} \mathrm{E}[\mathrm{X_T}] &= e^{rT}(\omega + \mathrm{h}) + \frac{(\mu - r)^2}{\gamma\sigma^2}\mathrm{T} - E\left[\sum_{i=1}^{N(T)} e^{r(T-S_i)} Y_i\right] \\ &= e^{rT}\omega + \frac{\lambda\int_0^T \left[M_Y(\gamma e^{r(T-s)}) - 1\right] ds}{\gamma} + \frac{(\mu - r)^2}{\gamma\sigma^2}\mathrm{T} - E\left[N(T)\, E\left[e^{r(T-S_i)}\right] E\left[Y_i\right]\right] \\ &= e^{rT}\omega + {}_T C_0 + \frac{(\mu - r)^2}{\gamma\sigma^2}\mathrm{T} - E\left[N(T)\, E\left[e^{r(T-S_i)}\right] E\left[Y_i\right]\right] \\ &= e^{rT}\omega + {}_T C_0 + \frac{(\mu - r)^2}{\gamma\sigma^2}\mathrm{T} - \lambda\, E[Y]\, \frac{e^{rT} - 1}{r} \\ &= e^{rT}\omega + {}_T C_0 + \frac{(\mu - r)^2}{\gamma\sigma^2}\mathrm{T} - \lambda\, E[Y]\, \bar{s}_{\overline{T}|}. \end{aligned} \tag{6.14}$$

We also calculate the variance of the final surplus,

$$\begin{aligned} \mathrm{Var}(\mathrm{X_T}) &= \frac{(\mu - r)^2}{\gamma^2\sigma^2}\mathrm{T} + Var\left(\sum_{i=1}^{N(T)} e^{r(T-S_i)} Y_i\right) \\ &= \frac{(\mu - r)^2}{\gamma^2\sigma^2}\mathrm{T} + E\left[Var\left(\sum_{i=1}^{N(T)} e^{r(T-S_i)} Y_i \mid N(T)\right)\right] + Var\left(E\left[\sum_{i=1}^{N(T)} e^{r(T-S_i)} Y_i \mid N(T)\right]\right) \\ &= \frac{(\mu - r)^2}{\gamma^2\sigma^2}\mathrm{T} + E\left[N(T) Var\left(e^{r(T-U)} Y\right)\right] + Var\left(N(T)\, E[Y] E\left[e^{r(T-U)}\right]\right) \\ &= \frac{(\mu - r)^2}{\gamma^2\sigma^2}\mathrm{T} + E\left[N(T) Var\left(e^{r(T-U)} Y\right)\right] + Var\left(N(T) E[Y] \frac{e^{rT-1}}{rT}\right) \\ &= \frac{(\mu - r)^2}{\gamma^2\sigma^2}\mathrm{T} + \lambda T\left\{E\left[e^{2r(T-U)}\right] E\left[Y^2\right] - E\left[e^{r(T-U)}\right]^2 E\left[Y\right]^2\right\} + \lambda T E\left[Y\right]^2 \frac{(e^{rT} - 1)^2}{r^2 T^2} \\ &= \frac{(\mu - r)^2}{\gamma^2\sigma^2}\mathrm{T} + \lambda E\left[Y^2\right] \frac{(e^{2rT} - 1)}{2r} \end{aligned}$$

$$= \frac{(\mu - r)^2}{\gamma^2 \sigma^2} \text{T} + \lambda E\left[Y^2\right] \bar{s}_{\overline{T|}2r}, \tag{6.15}$$

where $\bar{s}_{\overline{T|}2r}$ is the accumulated value of continuous annuity of 1 per time period with the force of interest 2r and maturity T.

For the insurance contracts with the fully continuous annual indifference premium p = $\bar{P}(\bar{A}_{\overline{T|}})$ and with r > 0, we have the following formula by plugging the optimal investment strategy $\tilde{\pi}^* = \frac{\mu - r}{\gamma \sigma^2}$,

$$\text{d}\,\tilde{X}_s = \frac{(\mu - r)^2}{\gamma \sigma^2} \text{ds} + \frac{\mu - r}{\gamma \sigma} \text{dW}_s + \tilde{p}_s \text{ds} - \text{e}^{\text{r(T-s)}} \text{dL}_s. \tag{6.16}$$

The final surplus X_T for the annual premium p is expressed by,

$$\text{X}_\text{T} = \omega\, \text{e}^{\text{rT}} + \frac{(\mu - r)^2}{\gamma \sigma^2} \text{T} + \frac{\mu - r}{\gamma \sigma} \text{W}_\text{T} + \bar{P}(\bar{A}_{\overline{T|}}) \int_0^T e^{r(T-s)} ds - \text{L}. \tag{6.17}$$

By noting that

$$\bar{A}_{\overline{T|}}\, \text{e}^{\text{rT}} = {}_T C_0 = \bar{P}(\bar{A}_{\overline{T|}})\, \bar{s}_{\overline{T|}}, \tag{6.18}$$

we can prove that the final surplus X_T for the insurance contracts with a single premium is same as that for the insurance contracts with an annual premium. So we have an interesting fact that the probability of default at terminal time T, the expected value and the variance of the final surplus for the insurance contracts with a single premium are equal to those for the insurance contracts with an annual premium.

7 Numerical Example

As an example, we assume that the claim amount Y follows an exponential distribution with one parameter θ,

$$f_Y(y) = \frac{e^{-y/\theta}}{\theta}. \tag{7.1}$$

The moment generating function of Y is

$$M_Y(t) = (1 - \theta t)^{-1}, \quad \text{t} < 1/\theta. \tag{7.2}$$

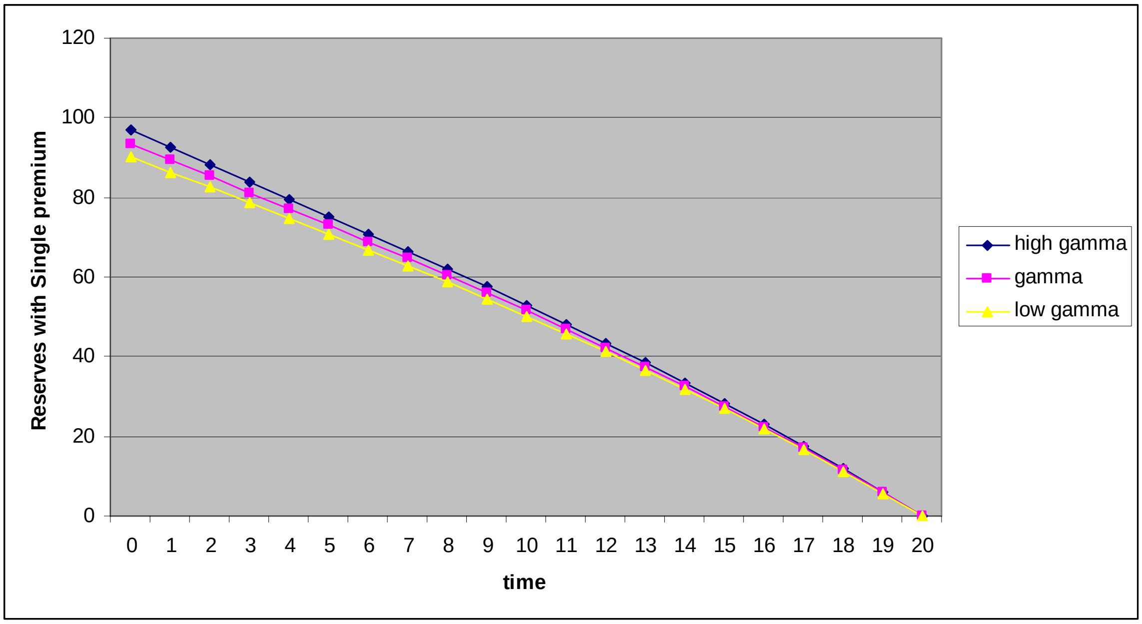
The certainty equivalent of benefits from time t to time T is

$$\begin{aligned} {}_T C_t &= \frac{\lambda}{\gamma} \int_t^T \left[M_Y(\gamma e^{r(T-s)}) - 1\right] ds \\ &= \frac{\lambda}{\gamma r} \ln\left[\frac{1 - \theta\gamma}{1 - \theta\gamma e^{r(T-t)}}\right]. \end{aligned} \tag{7.3}$$

For illustration purposes, we consider a 20-year insurance contract, T = 20, with 10 year premium payment periods, h = 10. We assume that the parameters are λ = 0.00005, γ = 0.0000016, and θ = 100,000. The risk free rate r is assumed to be 4%. The single indifference premium is 93.31916 and the 10-year annual indifference premium is 11.32239.

We show the graph of the exponential reserves, $\overline{A}_{\overline{T-t}|}$, for the insurance contracts with a single indifference premium. The reserves are decreasing to 0 as time approaches to the maturity year 20.

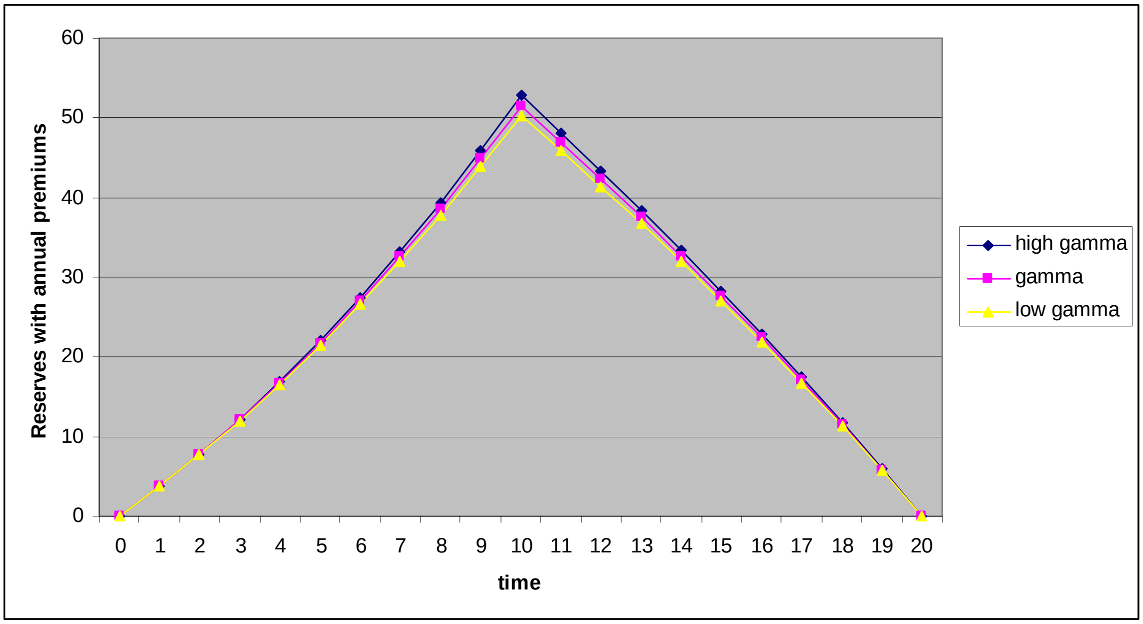
Figure 1. Reserves with Single Indifference Premiums



We use 1% higher and lower risk aversion coefficients to calculate the reserve values. We can check that if the risk aversion coefficient γ is increased then the single indifference is increased to 96.85288 and the reserves are also increased. when the risk aversion coefficient is decreased by 1% then the single indifference is decreased to 90.05325 and the reserves are also decreased.

We also show the graph of the exponential reserves, ${}_{t}^{h}\overline{V}(\overline{A}_{\overline{T}|})$, for the insurance contracts with the annual premiums for 10 years. For higher risk aversion the 10-year annual indifference premium is increased to 11.75114 and the reserves are also increased. For lower risk aversion the 10-year annual indifference premium is decreased to 10.92614 and the reserves are also decreased. The initial reserves are zero. The reserve values are increasing during the premium payment periods and then decreasing to zero. We may notice that the reserve values can be negative for some insurance contracts. This should be a consideration when we develop insurance policies because of the regulatory issues.

Figure 2. Reserves with 10-Year Annual Indifference Premiums



8 Conclusions

We consider a collective risk model for the liability process of insurance claims from a portfolio of insurance policies. We assume that the insurer is willing to invest to risky assets and risk free assets and that the surplus process follows a jump-diffusion process. We calculate the single indifference premium such that the optimal value without liability is the same as the optimal value with additional single premium for taking the liability. We notice that the annual indifference premium is calculated under the certainty equivalence principle: the discounted certainty equivalent of future indifference premiums equals the current value of the certainty equivalent of future benefits. This is similar to the equivalence principle used to calculate the benefit premiums in life insurance policies.

We find out that the exponential reserves are the difference between the current value of the future certainty equivalent of benefits and the present value of future certainty equivalent of premiums discounted by risk free rate under the prospective method. This is similar to the benefit reserve which is the conditional expectation of the difference between the present value of future benefits and the present value of future benefit premiums. The exponential reserves under the retrospective method are the difference between the accumulated value of the past indifference premiums using the risk free rate and the current value of the certainty equivalent of the past benefit payments from initial time to current time. For some insurance contracts, the exponential reserves can be negative when the premium payment period approaches the insurance period. This can be an issue of regulatory.

Appendix A

In this section we show the process M in (3.5) is martingale. Let us consider a process Q_s,

$$Q_s = \sum_{i=1}^{N_s} \{1 - \exp(\gamma Y_i)\} \le 0. \tag{A.1}$$

We define a process M = {M_s, 0 ≤ s ≤ T},

$$M_s = Q_s - \lambda\left(1 - E[\exp \gamma Y_i]\right)s\ . \tag{A.2}$$

For 0 ≤ s ≤ t,

$$E\left[M_t \mid F_s\right] = E\left[\sum_{i=1}^{N_t}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t \mid F_s\right]$$

$$= E\left[\sum_{i=N_s+1}^{N_t}(1 - e^{\gamma Y_i}) \mid F_s\right] + \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t$$

$$= E\left[\sum_{i=N_s+1}^{N_t}(1 - e^{\gamma Y_i})\right] + \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t$$

$$= E\left[\sum_{i=1}^{N_{t-s}}(1 - e^{\gamma Y_i})\right] + \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t$$

$$= E\left[E\left[\sum_{i=1}^{N_{t-s}}(1 - e^{\gamma Y_i}) \mid N_{t-s}\right]\right] + \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t$$

$$= \sum_{k=0}^{\infty} E\left[\sum_{i=1}^{k}(1 - e^{\gamma Y_i}) \mid N_{t-s} = k\right] \frac{e^{-\lambda(t-s)}\{\lambda(t-s)\}^k}{k!} + \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t$$

$$= \sum_{k=0}^{\infty} k\left(1 - E\left[e^{\gamma Y_i}\right]\right) \frac{e^{-\lambda(t-s)}\{\lambda(t-s)\}^k}{k!} + \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t$$

$$= \lambda(t-s)\left(1 - E[e^{\gamma Y_i}]\right) + \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)t$$

$$= \sum_{i=1}^{N_s}(1 - e^{\gamma Y_i}) - \lambda\left(1 - E[e^{\gamma Y_i}]\right)s$$

$= M_s$.

Appendix B

We want to show that the process $\widetilde{M} = \{\widetilde{M}_s$, 0 ≤ s ≤ T} in (3.17) is martingale. Let us define a process

$$\widetilde{Q}_s = \sum_{i=1}^{N_s} \{1 - \exp(\gamma e^{r(T-S_i)} Y_i)\} \le 0, \tag{B.1}$$

where S_i is the i-th event time. We consider a process $\widetilde{M} = \{\widetilde{M}_s$, 0 ≤ s ≤ T},

$$\widetilde{M}_s = \widetilde{Q}_s - \lambda s \left(1 - \frac{1}{s} \int_0^s E\left[\exp\left(\gamma e^{r(T-u)} Y \right) \right] du \right). \tag{B.2}$$

For 0 ≤ s ≤ t, we calculate the conditional expectation of $\widetilde{Q}_t$

$$E\left[\widetilde{Q}_t \mid F_s \right] = E\left[\sum_{i=1}^{N_t} (1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \mid F_s \right]$$

$$= E\left[\sum_{i=N_s+1}^{N_t} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\} \mid F_s \right] + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$= E\left[\sum_{i=N_s+1}^{N_t} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\} \right] + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$= E\left[E\left[\sum_{i=N_s+1}^{N_t} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\} \mid N_t - N_s \right] \right] + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$= \sum_{k=0}^{\infty} E\left[\sum_{i=1}^{k} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\} \mid N_{t-s} = k \right] \frac{e^{-\lambda(t-s)} \{\lambda(t-s)\}^k}{k!} + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$= \sum_{k=0}^{\infty} k \left(1 - E\left[\exp\left(\gamma e^{r(T-U)} \right) Y \right] \right) \frac{e^{-\lambda(t-s)} \{\lambda(t-s)\}^k}{k!} + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\},$$

here U is the uniform random variable on the interval (s, t)[1],

$$= \lambda(t-s) \left(1 - E\left[\exp\left(\gamma e^{r(T-U)} \right) Y \right] \right) + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$= \lambda(t-s) \left(1 - E\left[E\left[\exp\left(\gamma e^{r(T-U)} \right) Y \right] \mid Y \right] \right) + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$= \lambda(t-s) \left(1 - E\left[\frac{1}{t-s} \int_s^t \exp\left(\gamma e^{r(T-u)} Y \right) du \right] \right) + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$= \lambda(t-s) \left(1 - \frac{1}{t-s} \int_s^t E\left[\exp\left(\gamma e^{r(T-u)} Y \right) du \right] \right) + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}.$$

Now let us consider the conditional expectation of $\widetilde{M}_t$,

$$E\left[\widetilde{M}_t \mid F_s \right] = E\left[\widetilde{Q}_t - \lambda t \left(1 - \frac{1}{t} \int_0^t E\left[\exp\left(\gamma e^{r(T-u)} Y \right) \right] du \right) \mid F_s \right]$$

$$= \lambda(t-s) \left(1 - \frac{1}{t-s} \int_s^t E\left[\exp\left(\gamma e^{r(T-u)} Y \right) du \right] \right) + \sum_{i=1}^{N_s} \left\{ 1 - \exp\left(\gamma e^{r(T-S_i)} Y_i \right) \right\}$$

$$- \lambda t \left(1 - \frac{1}{t} \int_0^t E\left[\exp\left(\gamma e^{r(T-u)} Y \right) \right] du \right)$$

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[1] S_i is the i-th order statistic of the uniform distribution U, but we can use U instead of S_i in the summation.

$$= \sum_{i=1}^{N_s} \left\{1 - \exp\left(\gamma e^{r(T-S_i)} Y_i\right)\right\} - \lambda s \quad - \lambda \int_s^t E\left[\exp\left(\gamma e^{r(T-u)} Y\right) du\right] \quad + \lambda \int_0^t E\left[\exp\left(\gamma e^{r(T-u)} Y\right) du\right]$$

$$= \sum_{i=1}^{N_s} \left\{1 - \exp\left(\gamma e^{r(T-S_i)} Y_i\right)\right\} - \lambda s \left(1 - \frac{1}{s} \int_0^s E\left[\exp\left(\gamma e^{r(T-u)} Y\right)\right] du\right)$$

$$= \widetilde{M}_s .$$

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