Bivariate Analysis of Survivorship and Persistency

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Abstract. Voluntary non-payment of premiums leads to policy termination. When policies are terminated, it is costly to several parties of insurance contracts. These costs include the inability of the insurance company to recover acquisition expenses, loss of income from renewal premiums by the insurance agent, and the loss of premiums paid and insurance coverage by the contractholder. While most of these costs can be directly accounted for, there is the additional hidden cost resulting from mortality selection. This refers to the tendency of contractholders who are generally healthier to select against the insurance company by voluntarily terminating their policies. This paper explores a methodology to quantify the cost of mortality selection and to examine for the presence of such selection. While standard actuarial models of survivorship and persistency consist of specifying distributions for times until death and withdrawal, the typical assumption is that these random times are independent. We will use a more general approach of specifying the bivariate distributions using copulas for these random times without having to assume independence. We demonstrate procedures to estimate parameters in the model, and we show how one can use these estimates to predict more accurate future cash‡ows.

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1. Introduction. As pointed out in Lian, Yuan and Loi (1998), it is costly to all parties of insurance contracts when policies are terminated because of voluntary non-payment of premiums. Some of these costs include: inability to recover acquisition expenses by the insurance company, loss of income from renewal premiums by the agent, and the loss of premiums paid and insurance coverage by the contractholder. Most of these expenses can be directly accounted for. However, there is an additional cost which is hidden and typically difficult to measure or quantify: the cost of mortality selection. Mortality selection refers to the tendency of contractholders who are generally healthier to select against the insurer by voluntarily terminating their policies. It is therefore hypothesized that those policyholders who tend to lapse their policies are generally healthier than those who stay or remain. This is to say that the mortality experience of policyholders who lapse will generally be more favorable. Thus, when insurance companies experience high rates of policy lapsation or termination, not only do they lose the collection of premiums from these terminated policies, but they can expect to experience a worse mortality.

In this paper, we demonstrate how to empirically assess the cost of mortality selection, if any. First, we formally define what is meant by antiselection. We derive a condition for assessing the presence of antiselection, whether the rate of mortality for those who have lapsed is generally more favorable than those who keep their policies. Using a dataset drawn from a textbook on survival models, we show how to estimate the parameters in a model where there are two competing risks, mortality and withdrawal. However, there is the issue of non-identifiability or estimability. This is because the insurer typically cannot observe the survivorship pattern of terminated policies. However, we may be able to avoid this by parametrically specifying the joint distribution of times for survival and persistency of which they may not necessarily be independent. As such, this joint distribution must contain parameter(s) describing this possible dependence between persistency and survival. This joint distribution can be specified using a statistical tool called copulas.

The problem of the relationship between persistency and survivorship can be of

considerable importance in pricing contracts and valuation of insurance liabilities. Yet there is very little research done about the true nature of this relationship. The common practice is to select average mortality and lapse rates, on the "aggregate" as Jones (1998) pointed out, appropriate for a class of contractholders. In subsequent periods, to re‡ect the possible mortality selection, there must be excess lapse rates for renewed policies. See also Atkinson (1990), Becker (1984), and Dukes and MacDonald (1980). One of the main objective of this paper is to o¤er a statistical methodology for analyzing the relationship between survivorship and persistency. In light of this objective, we analyze a dataset hypothetically developed in London (1988) that possibly describe the mortality and lapse experience of an insurance company. Such empirical analysis will further our understanding of the true nature of the relationship.

The rest of this paper has been organized as follows. In Section 2, we review some of the early works that examine the relationship between mortality and lapsation. In Section 3, we review fundamental concepts of the competing risk model and introduce notations used throughout the paper. In Section 4, we formally describe conditions for analyzing the presence of antiselection. Using copulas to specify the joint distribution of the times until death and withdrawal, we are able to reduce the condition for antiselection into a single ratio of which we de…ne as the ratio of antielection.. Section 5 provides a discussion about model estimation and the results using the London's dataset. Section 6 demonstrates the possible discrepancy of the emerging costs resulting from a portfolio of life insurance policies if the dependence assumption is ignored. We conclude in section 7.

2. Early Models of Mortality and Lapsation.

Although the problem of the relationship between mortality and lapse rates is of considerable importance in pricing and reserving, there is very little research made about the true nature of this relationship. The common practice is to select average mortality and lapse rates, on the "aggregate" as Jones (1998) points out, appropriate for a class of contracts and policyholders. Then in subsequent periods, to re‡ect the possible mortality selection,

there will be excess lapse rates for renewed policies. See also Atkinson (1990), Becker (1984), and Dukes and MacDonald (1980).

Jones (1998) analyzed the relationship between mortality and lapsation based on the model that uses the concept of "frailty." Frailty models appeared in Vaupel, Manton, and Stallard (1979) and Hougaard (1984) as a method to model mortality with heterogeneity. Jones demonstrated how the heterogeneity concept can be adapted to analyze the relationship of mortality and withdrawal in conjunction with a multiple state model. In Renshaw and Haberman (1986), a similar concept of heterogeneity was applied to examine the e¤ect of policy duration on withdrawals.

Sigalotti (1988) suggested the approach using Bayesian inference to derive mortality estimates based on data from insurance company records. He developed a model that could possibly account for the "correlation between mortality and withdrawal." Berger (1976) earlier suggested the use of di¤usion models of mortality to examine the e¤ect of antiselection on the overall mortality pattern. By following each risk element periodically and suggesting how each move to the various risk classes, Berger applied simulation procedures to examine evidence of "physical health and lapse behavior." No actual experience data was used from this study.

None of these early papers on the relationship between mortality and withdrawal validates the suggested models through empirical evidence. There are several reasons for this. One is the di¢culty to obtain actual mortality and withdrawal experience data. In a recent study by Albert and Bragg (1996), it was found that there is "little evidence of a direct relationship between lapses and the changes in mortality results by duration." The authors further explained that lapses do not occur because of the policyholder's state of health, but instead because of economic reasons.

By partitioning the classes of policyholders into "low" and "high" risk and assuming withdrawal patterns are di¤erent for the two classes because of the presence of antiselection, Valdez (2000a) recently demonstrated how misestimation of the cost of insurance can result out of ignoring mortality/lapsation relationship.

3. A Competing Risk Model of Mortality and Lapsation. In this section, we provide a brief introduction to the subject of competing risks with time-until-death and time-until-withdrawal as competing risks. This is meant to develop preliminaries, de…nitions, and notations useful in later sections. Several textbooks provide good foundations of the theory; for example, see Bowers et al. (1997), Birnbaum (1979), David and Moeschberger (1978), and Elandt-Johnson and Johnson (1980).

An individual has a lifetime random variable T and is exposed to two various causes of failure: for our purposes, they will be death and withdrawal. We shall denote the cause of failure random variable by J. Thus, $J = d$ indicates that failure is due to death and $J = w$ indicates that failure is due to withdrawal. It is often convenient to introduce the theoretical "net" lifetime random variables: T_d and T_w. T_d denotes the time-until-death and T_w denotes the time-until-withdrawal random variables. We shall assume that their marginal distribution, survival, and density functions exist and are denoted by $F_j; S_j;$ and f_j, respectively, where $j = d; w$.

The net lifetimes T_d and T_w will never be observed simultaneously. Instead, when an individual dies, we observe the cause of death J and the lifetime $T = \min(T_d; T_w)$. Let the net lifetime random vector be denoted by $\vec{T} = (T_d; T_w)$ whose joint density we assume exists and is denoted by $h(t_d; t_w)$. Its distribution and survival functions are to be denoted by $H(t_d; t_w)$ and $S(t_d; t_w)$, respectively. In e¤ect, we have

$$H(t_d; t_w) = \text{Prob}(T_d \cdot t_d; T_w \cdot t_w) \tag{1}$$

and

$$S(t_d; t_w) = \text{Prob}(T_d > t_d; T_w > t_w) \tag{2}$$

The marginal distribution functions can be derived using

$$F_d(t_d) = H(t_d; 1) \text{ and } F_w(t_w) = H(1; t_w)$$

and survival functions using

$$S_d(t_d) = S(t_d; 0) \text{ and } S_w(t_w) = S(0; t_w):$$

The probability density functions are clearly

$$f_d(t_d) = \frac{@F_d(t_d)}{@t_d} \text{ and } f_w(t_w) = \frac{@F_w(t_w)}{@t_w}:$$

To simplify illustrations, we additionally assume that a person cannot die and withdraw at the same time i.e., the causes of failure are mutually exclusive events. Thus, the value of J is uniquely determined with probability one. We can write this as follows $Prob(T_d = T_w) = 0$. This assumption appears realistic.

The joint distribution of $(T; J)$ can be derived as follows. For $J = d$, we have

$$\begin{aligned} F_{T;J}(t; d) &= Prob(T \cdot t; J = d) = Prob(\min(T_d; T_w) \cdot t; J = d) \\ &= Prob(T_d \cdot t; T_d < T_w) = \int_0^t \int_{z_1}^{\infty} h(z_1; z_2)\, dz_2 dz_1: \end{aligned} \tag{3}$$

Similarly, we have for $J = w$;

$$F_{T;J}(t; w) = \int_0^t \int_{z_2}^{\infty} h(z_1; z_2)\, dz_1 dz_2: \tag{4}$$

The corresponding joint density is determined by

$$f_{T;J}(t; j) = \frac{@F_{T;J}(t; j)}{@t} \text{ for } j = d; w: \tag{5}$$

To calculate the marginal distribution of T, we can sum the distribution functions in equations (3) and (4) as

$$F_T(t) = F_{T;J}(t; d) + F_{T;J}(t; w) = 1 - \int_t^{\infty} \int_t^{\infty} h(z_1; z_2)\, dz_1 dz_2: \tag{6}$$

or directly compute it based on the survival function of T by noting that

$$F_T(t) = 1 - S(t; t): \tag{7}$$

The survival functions of the net lifetimes T_j as calculated are often called the net survival functions. The survival function of the lifetime T, denoted by $S_T(t) = S(t; t)$, is called the total or overall survival function. Finally, the survival function

that corresponds to $(T;J)$ and de…ned by $S^{(j)}(t) = \text{Prob}(T > t; J = j)$ for $j = d;w$, are called the crude survival functions. For this reason, we shall call probabilities associated with $(T;J)$ to be crude probabilities and they are useful for developing the likelihood function as discussed in our section on empirical analysis. Corresponding to the net, overall, and crude functions, we can de…ne life table notations and we suggest the reader to consult Bowers et al. (1997) and Carriere (1994) for life table notations.

Equation (3) is an extremely useful probability statement but can be particularly di¢cult to evaluate because of the number of times integration has to be performed. An alternative may be developed as follows. Note that we can write the overall survival function as

$$S(t_d;t_w) = \int_{t_d}^{\infty}\int_{t_w}^{\infty} h(z_1;z_2)\,dz_2 dz_1$$

from which it follows that

$$\frac{\partial S(t_d;t_w)}{\partial t_d}\Big|_{t_w=t_d} = -\int_{t_d}^{\infty} h(t_d;z_2)\,dz_2.$$

Thus, we have the following formula to calculate the crude joint distribution when $j = d$:

$$F_{T;J}(t;d) = \int_0^t \left[-\frac{\partial S(t_d;t_w)}{\partial t_d}\Big|_{t_w=t_d}\right] dt_d. \tag{8}$$

Similarly, when $j = w$, we have

$$F_{T;J}(t;w) = \int_0^t \left[-\frac{\partial S(t_d;t_w)}{\partial t_w}\Big|_{t_d=t_w}\right] dt_w. \tag{9}$$

Equations (8) and (9) involve only a single integration and is therefore useful for numerically evaluating the joint distribution. A similar formula appears in Tsiatis (1975) to prove the identi…ability in the case of independence and in Carriere (1994) as a simple representation of the crude survival function. Their formulas extend to several competing causes.

4. Analysis of the Presence of Antiselection. Carriere (1998) o¤ered a definition of antiselection in a dependent double decrement model and showed that the withdrawal bene…t is smaller in the presence of antiselection. We now de…ne what is meant by antiselection in the case of life insurance and we follow a similar de…nition imposed by Carriere. We preserve the usual notations for time-until-death and time-until-withdrawal and their corresponding survival and distribution functions. In addition, we de…ne the force of mortality as

$$\mu_d(t) = f_d(t)/S_d(t)$$

and the force of withdrawal as

$$\mu_w(t) = f_w(t)/S_w(t):$$

Now, consider the conditional density of T_d given that $T_w = t$: We shall denote this by $f_{djw}(t_d|t)$ with its corresponding survival function as

$$S_{djw}(t_d|t) = \int_{t_d}^{1} f_{djw}(z|t)\,dz$$

and force of mortality as

$$\mu_{djw}(t_d|t) = f_{djw}(t_d|t) \cdot S_{djw}(t_d|t);$$

assuming it exists. It is clear that

$$S_{djw}(t_d|t) = \exp\left[-\int_0^{t_d} \mu_{djw}(z|t)\,dz\right]:$$

Note that in the case of independence between T_d and T_w, i.e. $h(t_d; t_w) = f_d(t_d) \cdot f_w(t_w)$, we have

$$f_{djw}(t_d|t) = f_d(t_d);$$

$$S_{djw}(t_d|t) = S_d(t_d);$$

and

$$\mu_{djw}(t_d|t) = \mu_d(t_d):$$

We are now ready to de…ne antiselection.

Definition 1. (Antiselection) There is presence of antiselection at withdrawal in life insurance if

$$S_{djw}(t_d|t_w) > S_d(t_d) \tag{10}$$

for every t_d , t_w.

In other words, there is greater survival rate after withdrawal of the insurance policy. Those lives who withdrew their insurance policies tend to be generally healthier than those who kept their policies. Antiselection for life annuities can be similarly defined by simply reversing the inequality in the definition.

Definition 1 is a consequence of the definition offered by Carriere (1998) for which the condition for antiselection is

$$\mu_{djw}(t_d|t_w) < \mu_d(t_d) \text{ for every } t_d \text{ , } t_w.$$

It is straightforward to see that this implies

$$\begin{aligned} S_{djw}(t_d|t_w) &= \exp\left\{-\int_0^{t_d} \mu_{djw}(z|t_w)\,dz\right\} \\ &> \exp\left\{-\int_0^{t_d} \mu_d(z)\,dz\right\} = S_d(t_d); \end{aligned}$$

our condition for antiselection. Carriere (1998) further proved that the withdrawal benefit in the presence of antiselection should be smaller than that assuming death and withdrawal are stochastically independent. For purposes of this paper, we are interested in testing for the presence of antiselection given a set of observed data. To do this, we may wish to test the null hypothesis

$$H_0 : S_{djw}(t_d|t_w) = S_d(t_d), \text{ for every } t_d \text{ , } t_w$$

against the alternative

$$H_a : S_{djw}(t_d|t_w) > S_d(t_d), \text{ for every } t_d \text{ , } t_w.$$

We next consider a condition for the presence of antiselection in terms of the joint distribution function in equation (1) expressed in terms of a copula function.

Brie‡y, a two-dimensional copula, denoted by $C(u;v)$, is a two-dimensional probability distribution function de…ned on the unit square $[0;1]^2$ and whose univariate marginals are uniform on $[0;1]$. Thus, it follows that for all $u;v \in [0;1]$, we have $C(u;0) = C(0;v) = 0$, $C(u;1) = u$, and $C(1;v) = v$: Furthermore, it is true that $C(u_1;v_1) - C(u_1;v_2) - C(u_2;v_1) + C(u_2;v_2) \geq 0$ whenever $u_1 \leq u_2; v_1 \leq v_2$ for all $u_1;v_1;u_2;v_2 \in [0;1]$: The existence of a copula function for any multivariate distribution was established by Sklar (1959). In the context of our competing risk model described in the previous section, Sklar showed that for the random vector $(T_d;T_w)$ whose bivariate distribution function is expressed as in equation (1), there will always be a copula function C that will satisfy

$$H(t_d;t_w) = C(F_d(t_d);F_w(t_w)): \tag{11}$$

Because of (11), copulas are often referred to as functions that link or join or couple multivariate distribution functions to their marginal distribution functions. In the case of independence, the associated copula is clearly $C(u;v) = uv$. A family of copula called the Frank (1979) copula has the form

$$C(u;v) = \frac{1}{\theta}\log\left[1 + \frac{(e^{\theta u}-1)(e^{\theta v}-1)}{(e^{\theta}-1)}\right] \tag{12}$$

where θ, the dependence parameter, can take any values between -1 and 1. Because of its ‡exibility to accommodate a wide range of dependence, the Frank copula has had many empirical applications. See, for example, Frees, Carriere, and Valdez (1996) and Frees and Valdez (1998). A good introduction of copulas with potential actuarial applications is provided in Frees and Valdez (1998). For further understanding of copulas, see also Hutchinson and Lai (1990) and Nelsen (1999). We now state and prove our …rst theorem which gives a condition for the presence of antiselection.

Theorem 2. In a competing risk model of deaths and withdrawals with

$$H(t_d;t_w) = \text{Prob}(T_d \leq t_d; T_w \leq t_w) = C(F_d(t_d);F_w(t_w))$$

where $C(\cdot;\cdot)$ is the copula function that links the univariate margins F_d and F_w to their bivariate distribution, denote by

$$C_2(u;v) = \partial C / \partial v;$$

then there is presence of antiselection in life insurance at withdrawal if

$$C_2(F_d(t_d); F_w(t_w)) < F_d(t_d) \tag{13}$$

for every t_d , t_w:

Proof: The proof is straightforward and begins with

$$\begin{aligned} S_{d|w}(t_d|t_w) &= \int_{t_d}^{\infty} f_{d|w}(z|t_w)\,dz \\ &= \int_{t_d}^{\infty} \frac{h(z;t_w)}{f_w(t_w)}dz \\ &= \int_{t_d}^{\infty} \frac{f_d(z)\, f_w(t_w)\, C_{12}(F_d(t_d); F_w(t_w))}{f_w(t_w)}dz \\ &= \int_{t_d}^{\infty} f_d(z)\, C_{12}(F_d(t_d); F_w(t_w))\,dz. \end{aligned}$$

Applying change of variable with $u = F_d(z)$, we have

$$S_{d|w}(t_d|t_w) = \int_{F_d(t_d)}^{\infty} C_{12}(u; F_w(t_w))\,du.$$

Applying another change of variable with $z = C_2(u; F_w(t_w))$, we have

$$\begin{aligned} S_{d|w}(t_d|t_w) &= \int_{C_2(F_d(t_d);F_w(t_w))}^{C_2(1;F_w(t_w))} dz \\ &= C_2(1; F_w(t_w)) - C_2(F_d(t_d); F_w(t_w)) \\ &= 1 - C_2(F_d(t_d); F_w(t_w)). \end{aligned}$$

Note that $S_{d|w}(t_d|t_w) > S_d(t_d)$ if and only if

$$1 - C_2(F_d(t_d); F_w(t_w)) > 1 - F_d(t_d)$$

and for which gives the result in (13) follows.

It is interesting to note that in the case of independence, i.e. the copula is $C(u;v) = uv$, then $C_2(u;v) = u = F_d(t_d)$. Thus, it will never satisfy the result in the theorem. One implication is that when independence is assumed, there will always be no presence of antiselection. This is why the assumption of independence is considered unrealistic.

We shall call the ratio

$$R(t_d|t_w) = \frac{C_2(F_d(t_d);F_w(t_w))}{F_d(t_d)} \tag{14}$$

the ratio of antiselection and thus

$$R(t_d|t_w) < 1 \text{ for every } t_d \text{ , } t_w$$

is the condition for the presence of antiselection in life insurance at withdrawal.

Example 1: As an illustration, using the Frank copula in (12), it can be shown that

$$C_2(u;v) = @C/@v = \frac{e^{®v}(e^{®u} ¡ 1)}{e^{®C}(e^{®} ¡ 1)}:$$

Therefore with $u = F_d(t_d)$ and $v = F_w(t_w)$, the ratio of antiselection in (14) becomes

$$R(t_d|t_w) = \frac{1}{F_d(t_d)}\left(\frac{e^{®F_w(t_w)}}{e^{®H(t_d;t_w)}}\right)\left(\frac{e^{®F_d(t_d)} ¡ 1}{e^{®} ¡ 1}\right); \tag{15}$$

in the case of Frank copula.

Example 2: The Gumbel-Hougaard copula has the form

$$C(u;v) = \exp\left\{¡[(¡\log u)^{®} + (¡\log v)^{®}]^{1=®}\right\} \tag{16}$$

where the dependence parameter has the constraint $® \text{ , } 1$. Since

$$C_2(u;v) = \left(\frac{\log v}{\log C}\right)^{®¡1}\frac{C}{v};$$

then the ratio of antiselection can be expressed as

$$R(t_d|t_w) = \frac{C}{uv}\left(\frac{\log v}{\log C}\right)^{®¡1}: \tag{17}$$

Example 3: Another important copula is the Cook-Johnson which has the form

$$C(u;v) = \left[u^{-\alpha} + v^{-\alpha} - 1\right]^{-1/\alpha} \tag{18}$$

where $\alpha \geq 0$. Here, it can be shown that

$$C_2(u;v) = \left(\frac{C}{v}\right)^{\alpha+1};$$

so that the ratio of antiselection can be expressed as

$$R(t_d|t_w) = \frac{C}{uv}\left(\frac{C}{v}\right)^{\alpha}: \tag{19}$$

5. Empirical Estimation. In this section, we discuss the procedures used to estimate the parameters in a competing risk model given a set of observed data. We apply these procedures using a basic death-withdrawal data taken from pages 262-263 of London (1988). For this data, we assume age-at-death X_d follows a Gompertz (1825) distribution of the form

$$F_{X_d}(x) = 1 - \exp\left[e^{-(x-\mu)/\sigma}\left(1 - e^{x/\sigma}\right)\right] \tag{20}$$

and age-at-withdrawal X_w follows an exponential distirbution of the form

$$F_{X_w}(x) = 1 - \exp(-\lambda x): \tag{21}$$

From equation (20), we can clearly describe the distribution function for $T_d = X_d - x$, given the individual is currently age x. Its expression is given by

$$\begin{aligned} F_d(t) &= \text{Prob}(T_d \leq t) \\ &= 1 - \frac{1 - F_{X_d}(x+t)}{1 - F_{X_d}(x+t)} \\ &= 1 - \exp\left[e^{-(\mu-x)/\sigma}\left(1 - e^{t/\sigma}\right)\right]: \end{aligned} \tag{22}$$

By the memoryless property of the exponential, it is easy to see that the distribution for $T_w = X_w - x$, given that the individual is currently age x, is also exponential with the same parameter

$$F_w(t) = 1 - \exp(-\lambda t): \tag{23}$$

We will further assume that the random vector $(T_d; T_w)$ has a joint distribution function whose form can be expressed as in equation (11). We are therefore realistically specifying the dependence structure only starting from the beginning of the period for which the observation is made. In this case, it is at age x. We never observe both T_d and T_w, but only $T = \min(T_d; T_w)$. We will also need the joint survival function as de…ned in equation (2) and it can be expressed in terms of the copula function as

$$\begin{aligned} S(t_d; t_w) &= 1 - F_d(t_d) - F_w(t_w) + H(t_d; t_w) \\ &= 1 - C(F_d(t_d); 1) - C(1; F_w(t_w)) + C(F_d(t_d); F_w(t_w)): \end{aligned} \quad (24)$$

We now suppose we have a set of n observations each of which consists of a triplet in the form

$$(x_i; t_i; (\pm_{i;d}; \pm_{i;w}))$$

recorded where x_i denotes the individual's age at the beginning of the observation period, or entry age (if later),

$$t_i = \min(t_{d_i}; t_{w_i}; t_{c_i})$$

is the smallest of the observed death, or withdrawal, or right-censored time, and $(\pm_{i;1}; \pm_{i;2})$ is a pair of indicator variables de…ned as

$$\pm_{i;d} = \begin{cases} 1 & \text{if the ith person died before the end of the observation period} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pm_{i;w} = \begin{cases} 1 & \text{if the ith person withdrew before the end of the observation period} \\ 0 & \text{otherwise.} \end{cases}$$

The pair (1; 0) indicates death during the observation period, (0; 1) indicates withdrawal during the observation period, and (0; 0) indicates a right-censored observation. The pair (1; 1) indicating both death and withdrawal is not possible, as we

have said once withdrawal occurs, the observation is lost to follow-up. Follow-up of withdrawn policies is not possible.

We use maximum likelihood procedures to estimate the parameters. Maximum likelihood estimation results in estimators with properties desireable for further statistical inference and the procedure handles well with censored observation as obviously most survival data possess. To develop the likelihood function to maximize, we must distinguish the contributions made by those who were observed to die, withdraw, or survived to the end of the observation period. Without loss of generality, we assume that the period of observation is …xed so that the right-censored time $t_c = t_{c_i}$ for all observations.

If an individual dies during the observation period, i.e. $(\pm_{i;d}; \pm_{i;w}) = (1; 0)$, then his or her contribution to the likelihood function is given by

$$\left[\frac{@}{@t_i}\text{Prob}(T \cdot t_i; J = d)\right]^{\pm_{i;d}}; \tag{25}$$

where

$$\text{Prob}(T \cdot t_i; J = d) = F_{T;J}(t_i; d):$$

From equations (8) and (24), we can show that

$$\begin{aligned}\frac{@}{@t_i}F_{T;J}(t_i; d) &= \frac{@}{@t_i}\left\{\int_0^{t_i}\left[\frac{@S(t_d; t_w)}{@t_d}\right]_{t_w = t_d} dt_d\right\}; \\ &= \left[\frac{@S(t_d; t_w)}{@t_d}\right]_{t_w = t_d = t_i} \\ &= f_d(t_i)[1 - C_1(F_d(t_i); F_w(t_i))]\end{aligned}$$

where

$$C_1(u; v) = @C/@u:$$

Similarly, if an individual withdraws during the observation period, i.e. $(\pm_{i;d}; \pm_{i;w}) = (0; 1)$, then his or her contribution to the likelihood function is given by

$$\left[\frac{@}{@t_i}\text{Prob}(T \cdot t_i; J = w)\right]^{\pm_{i;w}}; \tag{26}$$

where

$$\text{Prob}(T \cdot t_i; J = w) = F_{T;J}(t_i; w):$$

From equations (9) and (24), it can similarly be shown that

$$\begin{aligned}\frac{@}{@t_i}F_{T;J}(t_i; w) &= \frac{@}{@t_i}\left\{\int_0^{t_i}\left[\frac{¡\,@S(t_d; t_w)}{@t_w}\right]_{t_d=t_w} dt_w\right\};\\ &= \left[\frac{¡\,@S(t_d; t_w)}{@t_w}\right]_{t_d=t_w=t_i}\\ &= f_w(t_i)[1 ¡ C_2(F_d(t_i); F_w(t_i))];\end{aligned}$$

where

$$C_2(u; v) = @C/@v:$$

On the other hand, for an observation who survived to attain $x_i + t_i$, where $t_i = t_c$ obviously, his or her contribution to the likelihood function is given by

$$[\text{Prob}(T > t_i)]^{1 ¡ ±_{i;d} ¡ ±_{i;w}}; \tag{27}$$

where using equation (7), we have

$$\text{Prob}(T > t_i) = 1 ¡ F_T(t_i) = S(t_i; t_i):$$

From equations (25) - (27), the full likelihood can be aggregated in the following manner:

$$\begin{aligned}L(-; x_i; t_i; (±_{i;d}; ±_{i;w})) &= \prod_{i=1}^{n} f f_d(t_i)[1 ¡ C_1(F_d(t_i); F_w(t_i))]g^{±_{i;d}}\\ &\quad £ f f_w(t_i)[1 ¡ C_2(F_d(t_i); F_w(t_i))]g^{±_{i;w}}\\ &\quad £ [S(t_i; t_i)]^{1 ¡ ±_{i;d} ¡ ±_{i;w}};\end{aligned} \tag{28}$$

where - denotes the vector of parameters to estimate. Thus, the total full log-likelihood function to maximize simpli…es to

$$\begin{aligned}\log L(-; x_i; t_i; (±_{i;d}; ±_{i;w})) &= \sum_{i=1}^{n} ±_{i;d} \log f f_d(t_i)[1 ¡ C_1(F_d(t_i); F_w(t_i))]g\\ &\quad + \sum_{i=1}^{n} ±_{i;w} \log f f_w(t_i)[1 ¡ C_2(F_d(t_i); F_w(t_i))]g\\ &\quad + \sum_{i=1}^{n} (1 ¡ ±_{i;d} ¡ ±_{i;w}) \log [S(t_i; t_i)]:\end{aligned} \tag{29}$$

A more detailed derivation and discussion of a similar likelihood function, but in the context of dependent causes of death, in (29) can be found in Valdez (2000b).

5.1. London's Data. We apply the previously described estimation procedures to a basic death-withdrawal data taken from London's survival models textbook. As it was not clearly stated in the textbook, we had to assume the period of observation is between January 1, 1978 until December 31, 1990 for this dataset. The data consists of 70 observations, each of which carries a triplet of observed data of the form

$$(x_i; t_i; (\pm_{i;d}; \pm_{i;w}))$$

where x_i, t_i, and $(\pm_{i;d}; \pm_{i;w})$ are similarly de…ned in the early part of this section and $i = 1; 2; :::; 70$: The information in Table 1 summarizes a death-withdrawal pattern of the observations over the 12 calendar years beginning with 1978. The new entrants for each calendar year are not provided in this table, but crude death and withdrawal rates are estimated here for each calendar year with the exposure for the year roughly estimated. The entry age is de…ned to be the age at the beginning of the observation period which is 1-1-78 if entry was prior to this date, otherwise, it is the exact age at entry. The terminal age is the age at death or withdrawal for deaths/withdrawals during the observation period, or the age at the end of the observation which is 12-31-90 if the individual survived to this date. Figure 1 provides a scatter plot of entry age versus terminal age for all 70 observations, coded according to reason for termination: 0 (if censored), 1 (if withdrawal), or 2 (if death). As somewhat depicted from the …gure, the ages at entry vary between age 27 to approximately age 37, with most observations between ages 29 to 33, a very short range of ages for …tting a mortality pattern for all ages. This short age-range may be re‡ected in the large standard errors in the estimate. It is therefore di¢cult to justify large sample properties of the parameter estimators. However, this data is meant only for illustrative purposes and the estimation results here may not be fully justi…ed. In practice, however, for insurance companies with generally large amount of experience, this should not be a

major problem.

Table 1

Summary of Death-Withdrawal Pattern

with London's Data[1]

Calendar Year	(a)	(b)	(c)	(d)	(e)
1978	-	0	0	0.0000	0.0000
1979	5	0	2	0.0000	0.3077
1980	8	1	3	0.1111	0.3333
1981	10	1	1	0.0870	0.087
1982	13	0	0	0.0000	0.0000
1983	20	0	1	0.0000	0.0435
1984	26	1	2	0.0377	0.0755
1985	27	0	1	0.0000	0.0345
1986	31	2	3	0.0625	0.0938
1987	33	4	4	0.1194	0.1194
1988	34	5	5	0.1587	0.1587
1989	29	1	5	0.0357	0.1786

[Insert Figure 1 here]

The marginal distributions assume that time-until-death random variable has a Gompertz form as expressed in equation (22), time-until-withdrawal has an exponential form as expressed in (23), and that their joint distribution is expressed in terms of a Frank's copula as expressed in equation (12). The vector of parameters to be estimated is therefore given by

$$(m; ¾; ¸; ®):$$

[1](a) = number alive at the start of year; (b) = number of deaths; (c) = number of withdrawals; (d) = death rate; and (e) = withdrawal rate.

The resulting parameter estimates and corresponding standard errors are summarized in Table 2. The table provides the estimates when using the standard assumption of independent time-until-death and time-until-withdrawal. Please note that for the dependent case, we are de…ning the dependence only starting from the age-at-issue, and not from birth. This explains the fact that the joint distribution is expressed for $(T_d; T_w)$ and not for $(X_d; X_w)$. This is well-justi…ed as the insurance company only observe the possible dependence of survivorship and persistency from the time the policy has been issued.

Table 2

Parameter Estimates: Dependence versus Independence

	Frank Copula		Independence	
Parameter	Estimates	Standard Errors	Estimates	Standard Errors
m	69.951	12.536	77.471	24.306
¾	12.214	1.785	9.483	5.614
¸	0.252	0.029	0.251	0.030
®	-16.230	1.859	not applicable	not applicable
Log-likelihood	-166.7075		-170.6261	

For the Frank's copula in (12), if we take the limit as ® goes to zero, we have the independent copula. This suggests that to test for the signi…cance of the dependence parameter, we perform the null hypothesis test $H_0 : ® = 0$ against the alternative hypothesis $H_a : ® \neq 0$. From the table, assuming large sample properties are well justi…ed, the test statistic

$$\left| \frac{\hat{®}}{se(\hat{®})} \right| = \frac{16:230}{0:859} = 8:73$$

suggests a rejection of the null hypothesis. Thus, the dependence of time-until-death and time-until-withdrawal is fairly signi…cant and must therefore be accounted for when assessing mortality and lapses.

5.2. Alternative Copula Models. One difficulty with competing risk models is that the marginals are never observed. Hence, it becomes a challenging task to evaluate the quality of fit of the marginals. Coupled with this problem is an equally difficult problem of finding the copula to fit the joint distribution. Our data is censored which contributes to this difficulty. As a first step, we investigated the effect of fitting alternative copula models. In the result as outlined in the previous section whereby we fitted the Frank copula, the dependence parameter has a point estimate of -16.230. This translates to a Kendall's tau coefficient of approximately 77%, suggesting a positive correlation between death and withdrawal. Alternative copulas would therefore be those that can accommodate such positive dependencies. Specifically, the Gumbel-Hougaard copula in equation (16) and the Cook-Johnson copula in equation (18) were prime candidates. The results of fitting these copula models assuming similar marginals are summarized in Table 3.

All the copula models produced larger log-likelihood values than the model assuming independence. This suggests that the copula models may well be better models than the case of independence. However, there is a fairly high degree of differences among the estimates for the marginals for death. For example, in the Frank copula model, the estimate for m, a measure of location for the Gompertz distribution, is 69.951 years; they are 58.748 and 74.632 years, respectively, for the Gumbel-Hougaard and Cook-Johnson models. The estimate for σ, a measure of dispersion for the Gompertz, varies in the range of 9.025 years to 18.691 years. However, with respect to the marginal for withdrawals, the parameter estimate for γ is about 0.25 for all models. With respect to the parameter for dependence, the estimate of 5.781 for θ in the Gumbel-Hougaard translates to a Kendall's tau coefficient of

$$1 - \frac{1}{\theta} = 0.827.$$

On the other hand, the estimate of 32.818 for θ in the Cook-Johnson translates to a Kendall's tau coefficient of

$$\frac{\theta}{\theta + 2} = 0.943.$$

See Frees and Valdez (1998) for a further discussion on how the dependence parameter of some of the copula models relate to the well-known measures of dependence such as the Kendall's tau coe¢cient. In the Gumbel-Hougaard model, the case where ® = 1 leads to the independence copula. Therefore, testing the null hypothesis $H_0 : ® = 1$ will lead to acceptance of this hypothesis with the p-value slightly above 5%. Similarly, in the Cook-Johnson model, the case where ® = 0 leads to the independence copula. It is easy to see that because of the large standard error corresponding to the estimate for this parameter, we conclude that the estimate of ® is not statistically signi…cantly di¤erent from zero. In conclusion, because of these results, the Frank copula model produces the best among all the alternative models examined.

Table 3

Parameter Estimates of Fitting Alternative Copula Models

	Gumbel-Hougaard		Cook-Johnson	
Parameter	Estimates	Standard Errors	Estimates	Standard Errors
m	58.748	16.779	74.632	9.401
¾	9.025	2.005	18.691	8.283
¸	0.251	0.030	0.251	0.030
®	5.781	2.420	32.818	34.398
Log-likelihood	-166.7156		-169.1559	

5.3. Evidence of Antiselection. In this subsection, we focus on the results obtained by …tting the Frank copula model. Note that with a Kendall's tau coe¢cient of 77%, the dependence parameter suggests a fairly strong positive relationship between the time-until-death T_d and the time-until-withdrawal T_w random variables. This relationship can already suggest evidence of possible antiselection. This positive relationship indicates that high T_d leads to high T_w, and vice-versa, that low T_d leads to low T_w. Intuitively, this means that individuals who may be exposed to low mortality (i.e. high T_d) tend to have very low withdrawal (i.e. high T_w). Furthermore,

individuals with possible high mortality (i.e. low T_d) may have more pressures to withdraw (i.e. low T_w).

In section 4, we suggested a more formal way of examining any possible evidence of antiselection. This requires an examination of the ratio of antiselection de…ned in equation (14) of which equation (15) is the special form when the Frank copula is used to model the bivariate distribution of T_d and T_w. This ratio varies for a given value of T_w. Thus, we …rst …x the value of t_w and then compute the ratio for various t_d , t_w. Assuming the parameter estimates for the Frank copula as outlined in Table 2, we provide Figure 2 which graphically displays the ratio of antiselection for a low value of t_w (2 years) and a moderate value of t_w (7 years). The darker shades give the values in the case where the withdrawal is on the seventh year. According to the …gure, it shows that there is no evidence of antiselection for individuals who are withdrawing in their second policy year because the ratio is larger than one for years 2 through 6. On the other hand, the ratio is always less than one for the case where withdrawals are occurring on the seventh year. There is therefore evidence of antiselection in this case.

[Insert Figure 2 here]

6. Measuring Financial Implications Using the Method of Emerging Costs.

In this section, we describe cash ‡ow procedures for analyzing the impact of not accounting for the possible presence of antiselection. To make the illustrations clear and simple, we consider a portfolio of discrete whole life insurance policies issued to a homogeneous group of policyholders who are age x. We analyze the projected cash‡ows emerging from this portfolio for many years into the future, making a comparison of these cash‡ows with assumption of independent and dependent withdrawals and death. Thus, we can examine the pro…ts that emerges from the portfolio with this comparison.

For a policy that is in force at the beginning of year t, denote by G_t the premium to be paid at the start of the year, E_t the expenses associated with each policy for

the year, i_t will be interest rate earned for the year, D_t the expected death bene…t payable at the end of the year, and W_t the expected withdrawal bene…t payable at the end of the year. Therefore, the expected cash‡ows for year t can be expressed as

$$CF_t = (G_t - E_t)(1 + i_t) - D_t \cdot (aq)^d_{x+t} - W_t \cdot (aq)^w_{x+t}; \tag{30}$$

where

$$(aq)^d_{x+t} = \text{Prob}(t < T \cdot t + 1; J = d)$$

and

$$(aq)^w_{x+t} = \text{Prob}(t < T \cdot t + 1; J = w)$$

are the probabilities of death and withdrawal, respectively, for the year. For evaluating these probabilities, see Valdez (1998). The expected emerging cost for year t can be derived then using

$$EC_t = {}_t(ap)_x \cdot CF_t$$

where

$${}_t(ap)_x = \text{Prob}(T > t)$$

is the probability that the individual has not died and not withdrawn at age x + t, conditional that the individual has not died and not withdrawn at age x. We shall denote the resulting emerging cost assuming independent death and withdrawals by EC^{IND}_t and assuming dependent death and withdrawals by EC^{COP}_t. Then we compare these two emerging costs for thirty years into the future, i.e. for years $1; 2; :::; 30$:

Assume that death bene…t is …xed at $10; 000$ and that withdrawal bene…t will be equal to the return of premium without interest. Thus, $W_t = (t + 1) G_t$ for $t = 0; 1; 2; ::::$ Set interest rate to be constant at $i_t = 5\%$ and expenses at 10% of gross premiums, i.e. $E_t = 0{:}1G_t$: Setting the present value of gross premiums equal to the present values of the death and withdrawal bene…ts as well as the expenses,

we have an equation for the gross premium level per policy at

$$G = \frac{10{,}000 \sum_{t=0}^{\omega} v^{t+1}\, {}_tp_x\, q^{d}_{x+t}}{\sum_{t=0}^{\omega} v^{t}\, {}_tp_x \left[0.90 - v(t+1)\, q^{w}_{x+t}\right]}. \tag{31}$$

Using the parameter estimates summarized in Table 2 from section 5 and assuming $x = 35$, Figure 3 provides a graphical display of the difference in the emerging costs that is defined as $EC_t^{COP} - EC_t^{IND}$. A positive difference therefore indicates an underestimation when dependence is not properly accounted for and a negative difference is the opposite. From these results, we therefore conclude that there can be substantial difference in the resulting annual emerging costs if the portfolio valuation fails to recognize the possible significance of the dependence between the time-until-death and the time-until-withdrawal randm variable. It appears that there would be an overestimation of emerging costs in the early years and an underestimation in later years. This pattern in the emerging costs is to be expected in the presence of anti-selection. This is because a significant discrepancy in the cash‡ows will be derived from death benefits resulting from the deterioration of the mortality of remaining policyholders.

[Insert Figure 3 here]

7. Conclusion. There is very little empirical evidence as to the true nature of the relationship between mortality and lapses or withdrawal in life insurance. One problem is applying the proper methodology so that the assumption of independence can be relaxed and therefore the effect of antiselective lapsation can be properly evaluated. In this paper, we offer the "competing risk" methodology as a procedure to analyze the presence or absence of antiselection. It is important to account for the possibility of antiselection as the paper further demonstrates that potential miscalculation of emerging cash‡ows can result. We also showed, using the London's dataset, how one can go about empirically estimating the parameters in the "competing risk"

model. We also de…ned conditions for the presence of antiselection in life insurance. These conditions can therefore be used to assess whether a particular portfolio of insurance contracts is faced with the problem of antiselective lapsation.

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Figure 1: SCATTERPLOT OF ENTRY AGE VS TERMINAL AGE FROM LONDON'S DATA

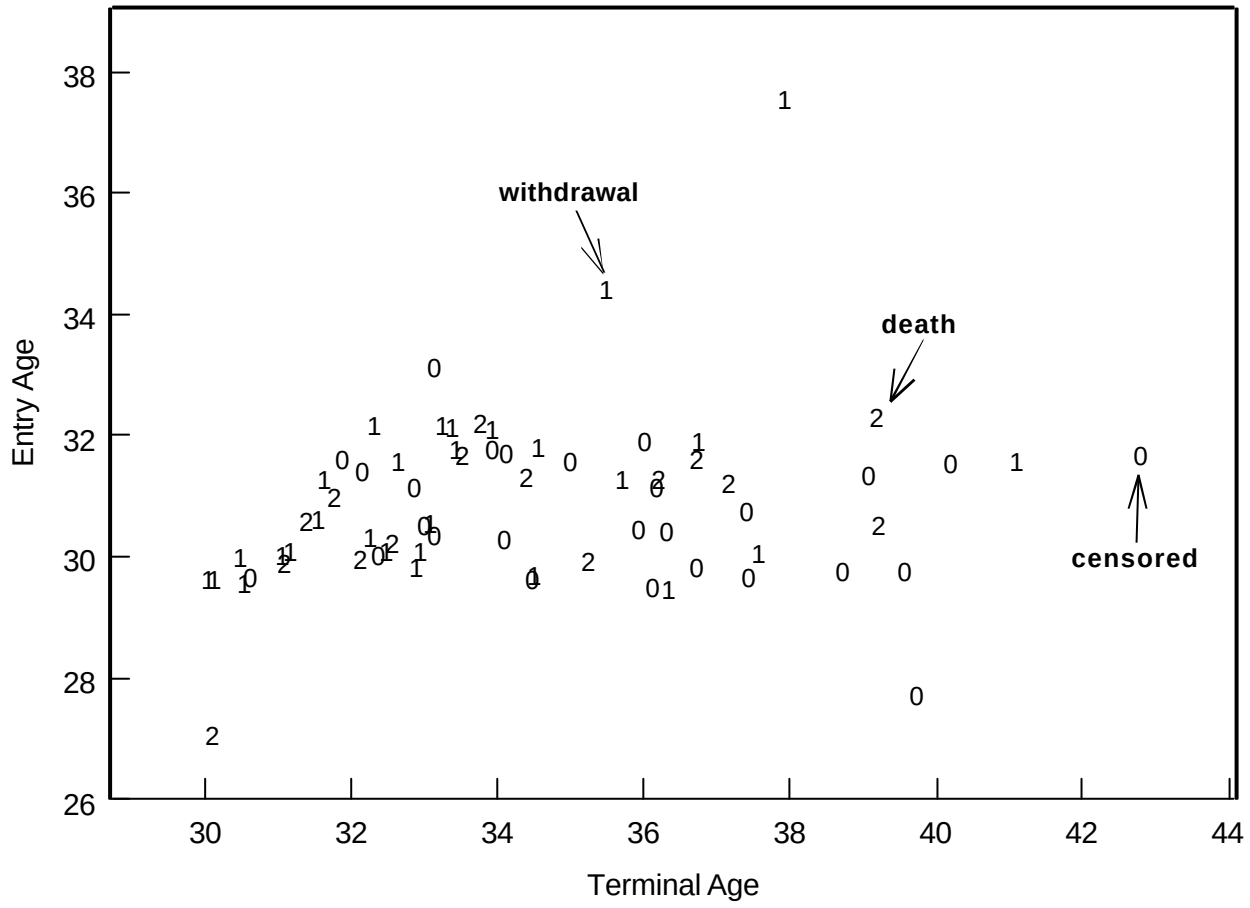

Figure 2: THE RATIO OF ANTISELECTION FOR t_w = *2 years* AND FOR t_w = *7 years*

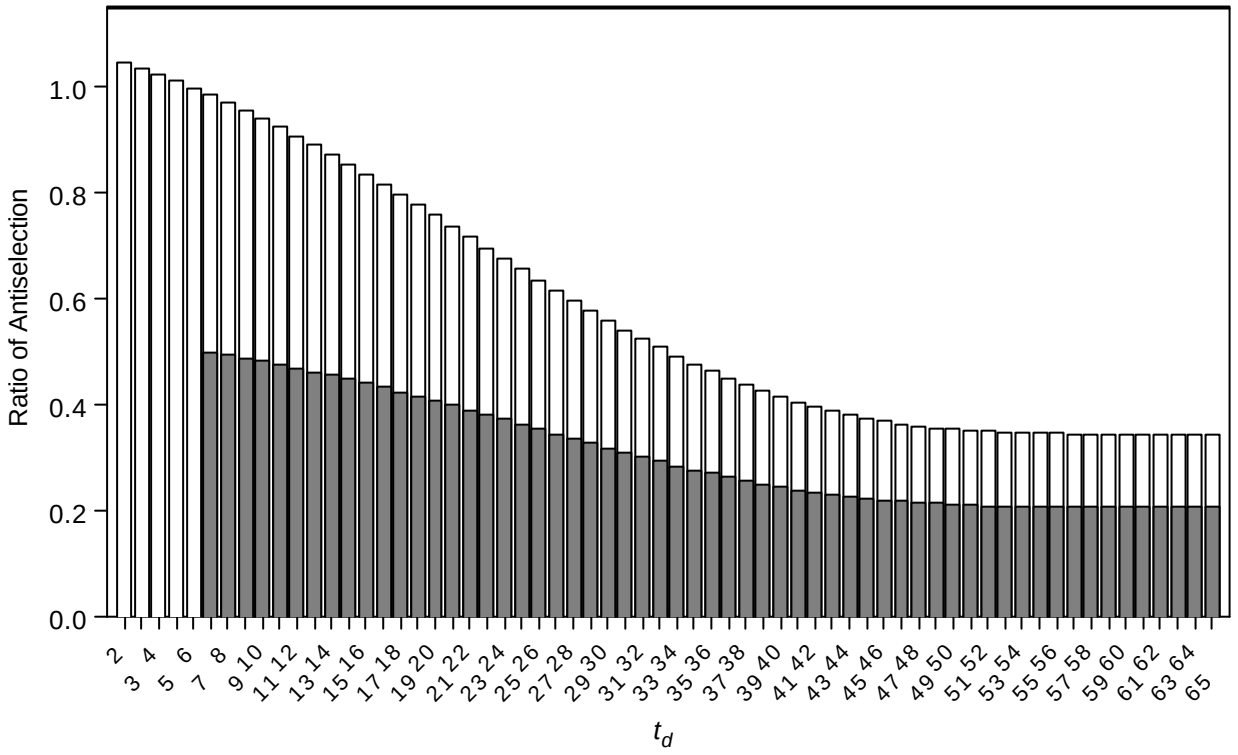

Figure 3: DIFFERENCE IN ANNUAL EMERGING COSTS: FRANK COPULA VS INDEPENDENCE