Tail Conditional Variance for Elliptically Contoured Distributions*

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Abstract

The tail conditional expectation, TCE for short, provides a measure of the riskiness of the tail of a distribution and is an index that has gained popularity over the years. On the other hand, the tail conditional variance, TCV for short, is lesser known but provides a measure of the variability of the risk along the tail of its distribution. Landsman and Valdez (2003) derive explicit formulas for computing tail conditional expectations for elliptical distributions, a family of symmetric distributions which includes the more familiar Normal and Student-t distributions. In this paper, we are able to similarly exploit the properties of the elliptical distributions to derive similar explicit forms in computing the tail conditional variance. In particular, the *tail generator* defined in the paper plays an important role in the process of developing these explicit forms. We further investigate these results in the multivariate case especially when the addition of several risks is concerned.

*Keywords: elliptical distributions, tail conditional expectations, tail variance, tail VAR, tail covariance, and coherent risk measures.

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1 Introduction

Insurance loss is often characterized as having a long right tail risk. It is along the right tail of the loss distribution that sometimes concern insurance carriers. As pointed out by Wang (1998), the right-tail risk represents events with low frequency but large loss amounts. Several types of measures of right-tail insurance risk have been proposed in the literature and used in practice. Among them include the variance (or standard deviation), the quantile (or Value-at-Risk, VaR, as is fondly called in the finance literature), the expected policyholder deficit (EPHD), Wang's right-tail actuarial index (1998), and even the Gini's index that is commonly used in economics as a measure of wealth differences. The tail conditional expectation (TCE) which measures the "average" along the right-tail, has gained popularity over the recent years both in insurance and finance. Recently, Wei and Yatracos (2004) proposed a so-called stop-loss risk index that is based on the Dutch premium calculation principle [Goovaerts, et al., 1984] and the tail-conditional expectation.

This paper considers yet another possible right-tail index, the tail conditional variance (TCV), and will be defined to be the variability of the risk along the right tail of its distribution. Consider a loss random variable X whose distribution function we shall denote by $F_X(x)$ and the tail function by $\overline{F}_X(x) = 1 - F_X(x)$. Although the context may be in terms of insurance losses, the situations discussed in this paper may be applied in finance which, for example, may refer to the total loss of income in a portfolio of investment for an individual or corporation. The *tail conditional expectation* (TCE) is defined to be

$$TCE_q(X) = E(X \,|X > x_q) \tag{1}$$

and can be interpreted as the "average worst possible loss". See Artzner, et al. (1999). Given the loss will exceed a particular value x_q, generally referred to as the q-th quantile with

$$\overline{F}_X(x_q) = 1 - q,$$

the TCE defined in (1) gives the expected loss that can potentially be experienced. The value of q is chosen to be high, typically 95% or 99%, to ensure probability the company ruins is made small.

Artzner, et al. (1999) demonstrated that the TCE satisfies all requirements for a coherent risk measure. When compared to the traditional Value-at-Risk (VaR) measure, the tail conditional expectation provides a more conservative measure of risk for the same level of degree of confidence q. To see this, note that $VaR_q(X) = x_q$ and since we can re-write formula (1) as

$$TCE_q(X) = x_q + E(X - x_q \,|X > x_q)$$

then

$$TCE_q(X) \geq VaR_q(X)$$

because the second term is clearly non-negative. Artzner and his co-authors also showed that the Value-at-Risk does not satisfy all requirements of a coherent risk measure. In particular, it violates the sub-additivity requirement of a coherent risk measure. However, the sub-additivity requirement is satisfied only when continuous risks are concerned. In Dhaene, et al. (2003), there are situations when sub-additivity is not met in cases where the risk is either purely discrete or a mixed random variable.

Now, to measure variability on the right tail, we propose the less familiar, but equally important to the TCE, statistical index of the tail of a distribution is the *tail conditional variance* defined to be

$$TCV_q(X) = Var\left(X - \mu_X \left| X > x_q \right.\right). \tag{2}$$

From hereon, since this paper is concerned mainly with Elliptical distributions, the discussion refers always to the continuous case. For the TCV measure in (2) to exist, here we assume that the mean $\mu_X = E(X)$ and the variance $\sigma_X^2 = Var(X)$ both exist. Notice that

$$\begin{aligned} Var(X) &= E\left[(X-\mu_X)^2\right] \\ &= E\left[(X-\mu_X)^2 \cdot I(X > x_q)\right] + E\left[(X-\mu_X)^2 \cdot I(X \le x_q)\right] \end{aligned}$$

where we have

$$\begin{aligned} E\left[(X-\mu_X)^2 \cdot I(X > x_q)\right] &= \int_{x_q}^{\infty} (x-\mu_X)^2 f_X(x)\,dx \\ &= (1-q)\int_{x_q}^{\infty} (x-\mu_X)^2 \frac{f_X(x)}{\overline{F}_X(x_q)}dx \\ &= (1-q) \cdot E\left[(X-\mu_X)^2 \left| X > x_q \right.\right] \\ &= (1-q) \cdot TCV_q(X). \end{aligned}$$

If x_q is chosen to be the mean μ_X, then we have what is sometimes referred to in the finance literature as the *downside semi-variance* of risk X which accounts for the variability above the mean. That is,

$$Var_+(X) = \int_{\mu_X}^{\infty} (x-\mu_X)^2 f_X(x)\,dx. \tag{3}$$

The downside semi-variance as a measure of risk in the mean-variance context has been introduced by Markowitz (1959). See also Goovaerts, et al. (1984) for a discussion of the semi-variance premium calculation principle. It is therefore clear that if the variance of X exists (i.e. finite), then so does the tail conditional variance. This is because

$$TCV_q(X) \le \frac{1}{1-q} Var(X) < \infty.$$

Note that our risk here X refers to a loss so that we are primarily concerned with the upper tail of the distribution and is the downside risk therefore in this case. Similar definition of a tail conditional variance can be made when the conditioning is on the lower tail. This is particularly useful in investment portfolios with random rates of return where the downside risk refers to extremely low levels of return. Most of the formula development in this paper can be easily extended when switching between upper and lower tails of the distribution.

For the familiar Normal distribution $N(\mu, \sigma^2)$ with mean μ and variance σ^2, Valdez (2004) demonstrated that its tail conditional variance can be expressed in the form

$$TCV_q(X) = \left[1 + \frac{\varphi(z_q)}{1 - \Phi(z_q)} z_q\right] \times \sigma^2 \tag{4}$$

where $z_q = (x_q - \mu)/\sigma = \Phi^{-1}(q)$ is the standardized q-th quantile of the distribution, and $\varphi(\cdot)$ and $\Phi(\cdot)$ are respectively, the density and distribution functions of a standard Normal $N(0,1)$ random variable. In this paper, we extend this result to the larger class of elliptical distributions for which the Normal distribution is a member of. This family consists of symmetric distributions for which the Student-t, the Uniform, Logistic, and Exponential Power distributions are other familiar examples. This is a rich family of distributions that allow for a greater flexibility in modelling risks that exhibit tails heavier than that of a Normal distribution. This can be especially useful for capturing insurance losses for extreme events.

In the univariate case, we have X belonging to the class of Elliptical distributions if its density can be expressed as

$$f_X(x) = \frac{c}{\sigma} g\left[\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right]$$

for some density generator g. Later in the paper, the definition of Elliptical distributions is expressed in terms of its characteristic function because this function always exist for a distribution. The paper demonstrates that for univariate Elliptical distributions, tail conditional variances have the form

$$TCV_q(X) = \left[\frac{\overline{F}_{Z^*}(z_q)}{\overline{F}_Z(z_q)} + z_q \frac{T(z_q)}{\overline{F}_Z(z_q)}\right] \times \sigma^2 \tag{5}$$

where we define the *tail generator* $T(\cdot)$ by

$$T(u) = \int_{u^2/2}^{\infty} cg(x)\, dx, \tag{6}$$

Z is the spherical random variable that generates the Elliptical random variable X, and $\overline{F}_{Z^*}(\cdot)$ is the tail function of another spherical random variable Z^* (later defined in the paper) that corresponds to Z. This paper shows that $Z^* = Z$, the standard Normal random variable and the tail generator is the density of a standard Normal. Hence, it immediately follows that (5) coincides with (4).

Extending this result to the multivariate case, we next consider a random vector $\mathbf{X}^T = (X_1, X_2, ..., X_n)$ jointly distributed as multivariate Elliptical with parameters $\boldsymbol{\mu}^T = (\mu_1, \mu_2, ..., \mu_n)$ and $\boldsymbol{\Sigma} = (\sigma_{ij})$ for $i, j = 1, 2, ..., n$. The typical situation is that a firm faces a portfolio of random losses and is concerned with the sum $S = X_1 + \cdots + X_n$. Note that the sum of Elliptical random losses is again another Elliptical so that the

TCV for a sum also has the form in (5). In Valdez (2004), it was noticed that the the TCV for the sum can be expressed as sum of tail covariances of individual components with the sum. In particular, we have

$$TCV_q(X) = \sum_{i=1}^{n} Cov\left(X_k - \mu_k, S - \mu_S \left| S > s_q \right.\right),$$

where $\mu_S = \sum_{k=1}^{n} \mu_k$ and s_q refers to the q-th quantile of the distribution of S. The term $Cov\left(X_k - \mu_k, S - \mu_S \left| S > s_q \right.\right)$ is called in this paper the *tail covariance* and we are able to extend the following explicit form of the tail covariance for Elliptical:

$$TCC_q\left(X_k \left| S \right.\right) = \left[\frac{\overline{F}_{Z^*}\left(z_{S,q}\right)}{\overline{F}_Z\left(z_{S,q}\right)} + \frac{T\left(z_{S,q}\right)}{\overline{F}_Z\left(z_{S,q}\right)} z_{S,q}\right] \times \rho_{k,S}\sigma_k\sigma_S, \tag{7}$$

where $\rho_{k,S} = \dfrac{\sigma_{k,S}}{\sigma_k\sigma_S}$ is the correlation coefficient of X_k and S, and $z_{S,q} = \left(s_q - \mu_S\right)/\sigma_S$. Here Z and Z^* are spherical random variables similarly defined as above.

In the rest of the paper, we have organized it as follows. In Section 2, we provide a brief discussion about elliptically contoured distributions and discuss the tail generator. We also give examples of known multivariate distributions belonging to this class. In Section 3, we develop the tail conditional variance formula for univariate elliptical distributions. The tail generator plays a key role in this formulation. In Section 4, we exploit the properties of elliptical distributions which allow us to derive explicit forms of the decomposition of TCV of sums of elliptical risks into individual component risks. We give concluding remarks in Section 5.

2 Elliptically Contoured Distributions and their Tail Generators

The class of elliptically-contoured distributions (or elliptical distributions, for short) provides a generalization of the multivariate normal distributions. Elliptical distributions therefore share many of the tractable and nice statistical properties of the Normal. The class contains many other non-Normal multivariate distributions such as the multivariate Student-t, multivariate Cauchy, multivariate Logistic, and multivariate Symmetric Stable, to name a few. This class of distributions has been introduced in the statistical literature by Kelker (1970) and widely discussed in Fang, et al. (1987) and Gupta and Varga (1993). See also Landsman and Valdez (2003), Valdez and Dhaene (2003), and Valdez and Chernih (2003) for applications in insurance and actuarial science. Embrechts, et al. (2001) also provides a fair amount of discussion of this important class as a tool for modelling risk dependencies.

There are a number of equivalent ways to define random vectors belonging to the class of elliptical distributions. Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ be an n-dimensional random vector.

Definition 1 *The n-dimensional vector $\mathbf{X}$ has a multivariate elliptical distribution, written as $\mathbf{X} \sim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$, if its characteristic function has the form*

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(it^T\mu)\psi\left(\tfrac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right) \tag{8}$$

for some column-vector $\boldsymbol{\mu}$, $n \times n$ *positive-definite matrix* $\boldsymbol{\Sigma}$*, and for some function* $\psi(t)$ *called the characteristic generator.*

The characteristic generator $\psi(\cdot)$ may explicitly depend on n, the dimension of the vector $\mathbf{X}$. A few observations about elliptical distributions are worth noting at this point. First, in general, it does not follow that $\mathbf{X}$ has a joint density $f_{\mathbf{X}}(\mathbf{x})$, but if its density exists, it can be expressed as

$$f_{\mathbf{X}}(\mathbf{x}) = c_n |\boldsymbol{\Sigma}|^{-1/2} g_n \left[\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right], \tag{9}$$

for some function $g_n(\cdot)$ called the density generator which, like the characteristic generator, may also explicitly depend on n. In cases where this density does not depend on n, we may drop the subscript n. Using the density generator, an alternative way to write $\mathbf{X}$ belongs to the family of elliptical distributions is $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. From (9), again if the density exists, the normalizing constant c_n can be explicitly determined using

$$c_n = (2\pi)^{-n/2} \Gamma(n/2) \left[\int_0^\infty x^{n/2-1} g_n(x) dx \right]^{-1}. \tag{10}$$

A detailed proof of (10) can be found in Landsman and Valdez (2003).

Second, in the univariate case, elliptical distributions consist of the class of symmetric distributions. However, they provide greater flexibility than the normal density because they allow for heavier (or even lighter) tails. If the elliptical density exists, their probability contours have the shape of an ellipsoidal [Valdez and Chernih, 2003].

Third, the mean and covariance of vector $\mathbf{X}$ do not necessarily exist. However, if the mean exists, it will be the vector $E(\mathbf{X}) = \boldsymbol{\mu}$ and if the covariance exists, it will be

$$Cov(\mathbf{X}) = -\psi'(0) \boldsymbol{\Sigma}. \tag{11}$$

A clear condition for this covariance to exist is $|\psi'(0)| < \infty$. Notice that in some cases where the characteristic generator satisfies $\psi'(0) = -1$, the covariance becomes $Cov(\mathbf{X}) = \boldsymbol{\Sigma}$.

Lastly, we observe that any marginal distribution of $\mathbf{X}$ is also elliptical with the same characteristic generator. If we take a subset of $\mathbf{X}$, say $\mathbf{X}_m = (X_1, X_2, ..., X_m)^T$ with $m \leq n$, then it follows that $\mathbf{X}_m$ is again elliptical. And in particular for the univariate marginals, for $k = 1, 2, ..., n$, we have $X_k \sim E_1(\mu_k, \sigma_k^2, g_1)$ and therefore its density can be expressed as

$$f_{X_k}(x) = \frac{c_1}{\sigma_k} g_1 \left[\frac{1}{2} \left(\frac{x - \mu_k}{\sigma_k} \right)^2 \right].$$

Furthermore, any linear combination of elliptical distributions is another elliptical distribution with the same characteristic generator ψ or from the same sequence of density generators $g_1, ...g_n$, corresponding to ψ. Suppose B is some $m \times n$ matrix of rank $m \leq n$ and b, some m-dimensional column-vector, then

$$B\mathbf{X} + b \sim E_m \left(B\boldsymbol{\mu} + b, B\boldsymbol{\Sigma}B^T, g_m \right). \tag{12}$$

One of the important concepts necessary for developing the tail conditional variance for this family is the *tail generator*. We now examine this function.

Definition 2 *Let the n-dimensional vector $\mathbf{X}$ have a multivariate elliptical distribution, that is $\mathbf{X} \sim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ where we assume the density generator g_n exists. We define the tail generator by*

$$T_n(u) = \int_{u^2/2}^{\infty} c_n g_n(x)\, dx, \tag{13}$$

provided the integral exists.

It will be understood that we drop the subscript n in the univariate case. In the multivariate normal case, it can be shown that the density generator has the form $g_n(x) = \exp(-x)$ which clearly does not depend on the dimension of the vector and that the corresponding tail generator has the expression

$$T_n(u) = \int_{u^2/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x} dx = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} = \varphi(u). \tag{14}$$

We notice from (14) that the tail generator has the form of a density function, in this case normal density. In general, we observe that with proper normalization, we can transform the tail generator into a density. Notice that

$$\begin{aligned}
\int_{-\infty}^{\infty} T_n(u)\, du &= \int_{-\infty}^{\infty} \int_{u^2/2}^{\infty} c_n g_n(x)\, dx du = \int_{-\infty}^{\infty} \int_{z}^{\infty} c_n z g_n\left(z^2/2\right) dz du \\
&= \int_{0}^{\infty} \int_{-z}^{z} c_n z g_n\left(z^2/2\right) du dz = 2\int_{0}^{\infty} c_n z^2 g_n\left(z^2/2\right) dz \\
&= \int_{-\infty}^{\infty} c_n z^2 g_n\left(z^2/2\right) dz = E\left(Z^{*2}\right)
\end{aligned}$$

where Z^* is an elliptical random variable with mean 0. But, we know from (11), this is equal to

$$\int_{-\infty}^{\infty} T_n(u)\, du = -\psi'(0).$$

Thus, we see

$$T^*(u) = -\frac{1}{\psi'(0)} T(u)$$

defines a proper density of a standard elliptical random variable.

The condition

$$\int_0^\infty x^{n/2-1} g_n(x)dx < \infty \tag{15}$$

guarantees $g_n(x)$ to be density generator (Fang, et al. 1987, Ch 2.2). As pointed out in Landsman and Valdez (2003), it can be shown by a simple transformation in the integral for the mean that

$$\int_0^\infty g_1(x)dx < \infty \tag{16}$$

guarantees the existence of the mean, and then the mean vector for $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ is $E(\mathbf{X}) = \boldsymbol{\mu}$. Because the existence of the mean, that is if $E(X) < \infty$, also guarantees the existence of the tail generator $T_n(u) < \infty$, it follows that a similar condition to (16) also guarantees existence of the tail generator. We now state and prove a lemma that will be useful later when we derive the tail conditional variance for elliptical distritbutions.

Lemma 1 *Suppose $X \sim E(\mu, \sigma^2, g)$ where its mean and variance exist. Then the following holds:*

$$\lim_{z \longrightarrow \infty} z \cdot T(z) = 0$$

where $T(\cdot)$ is the tail generator defined in (6).

Proof. When the mean exists, it immediately follows that $T(z)$ exists. Now, suppose the variance exists. Then, since

$$\int_{-\infty}^\infty \frac{c}{\sigma} (x-\mu)^2 g\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx = \sigma^2 \int_{-\infty}^\infty cz^2 g\left(z^2/2\right) dx,$$

the following integral must be finite:

$$\int_{-\infty}^\infty cz^2 g\left(z^2/2\right) dx < \infty.$$

But notice that, for any real $r \in \mathbb{R}$, we have

$$\int_{-\infty}^\infty cz^2 g\left(z^2/2\right) dx \geq \int_r^\infty cz^2 g\left(z^2/2\right) dx \geq rT(r) + \int_r^\infty T(x)\, dx \geq rT(r).$$

Thus, we see that

$$|rT(r)| < \infty$$

for any $r \in \mathbb{R}$ and therefore the result follows. ■

In this section, we only briefly describe the elliptical distributions and their properties useful for this paper. We next consider some important families of elliptical distributions and derive their tail generators. Many of the properties described below have appeared in Landsman and Valdez (2003). Please see that paper for some of the derivation, except those relating to the tail generator, a concept introduced in Valdez and Chernih (2003).

2.1 Multivariate Normal Distributions

A vector $\mathbf{X}$ belongs to the multivariate Normal family with the density generator

$$g(u) = e^{-u} \tag{17}$$

that does not depend on the dimension n. We shall write $\mathbf{X} \sim \mathbf{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The joint density of $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right].$$

The normalizing constant is given by $c_n = (2\pi)^{-n/2}$. It is well-known that its characterictic function is

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp\left(i\mathbf{t}^T\boldsymbol{\mu} - \tfrac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right)$$

so that the characteristic generator is

$$\psi(t) = e^{-t}.$$

Notice that choosing the density generator in (17) automatically gives $\psi'(0) = -1$ and hence $\boldsymbol{\Sigma} = Cov(\mathbf{X})$. To derive the tail generator, notice that

$$T(z) = c\int_{z^2/2}^{\infty} g(x)\,dx = c\int_{z^2/2}^{\infty} e^{-x}dx = (2\pi)^{-1/2} e^{-z^2/2} = \varphi(z). \tag{18}$$

2.2 Multivariate Student-t Distributions

A vector $\mathbf{X}$ is said to have a multivariate Student-t distribution if its density generator can be expressed as

$$g_n(u) = \left(1 + \frac{u}{k_p}\right)^{-p} \tag{19}$$

where the parameter $p > n/2$ and k_p is some constant that may depend on p. We write $\mathbf{X} \sim \mathbf{t}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}; p)$ if $\mathbf{X}$ belongs to this family. Its joint density has therefore the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}}\left[1 + \frac{(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2k_p}\right]^{-p}. \tag{20}$$

The normalizing constant is

$$c_n = \frac{\Gamma(p)}{\Gamma(p-n/2)}(2\pi k_p)^{-n/2}.$$

This is the most general form of a multivariate Student-t distribution. A similar form to it was considered in Gupta and Varga (1993) where they called this family

"Symmetric Multivariate Pearson Type VII" distributions. Taking for example $p = (n+m)/2$ where n and m are integers, and $k_p = m/2$, we get the traditional form of the multivariate Student t distribution with density

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma((n+m)/2)}{(\pi m)^{n/2}\Gamma(m/2)\sqrt{|\mathbf{\Sigma}|}}\left[1+\frac{(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{m}\right]^{-(n+m)/2}. \tag{21}$$

In the univariate case where $n = 1$, Bian and Tiku (1997) and MacDonald (1996) suggested to put $k_p = (2p-3)/2$ if $p > 3/2$ to get the so-called Generalized Student-t (GST) univariate distribution with density. This normalization leads to the important property that $Var(X) = \sigma^2$. Extending this to the multivariate case, we suggest to keep $k_p = (2p-3)/2$ if $p > 3/2$, then this multivariate GST has the advantage that

$$Cov(\mathbf{X}) = \mathbf{\Sigma}.$$

In particular, for $p = (n+m)/2$, we suggest instead of (21) to consider

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma((n+m)/2)}{[\pi(n+m-3)]^{n/2}\Gamma(m/2)\sqrt{|\mathbf{\Sigma}|}}\left[1+\frac{(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{n+m-3}\right]^{-(n+m)/2}$$

because it also has the property that the covariance is $Cov(\mathbf{X}) = \mathbf{\Sigma}$. If $1/2 < p \le 3/2$, the variance does not exist and we have a heavy-tailed multivariate distribution. If $1/2 < p \le 1$, even the expectation does not exist. In the case where $p = 1$, we have the multivariate Cauchy distribution with density

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma(\frac{n+1}{2})\pi^{-(n+1)/2}}{\sqrt{|\mathbf{\Sigma}|}}\left[1+(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]^{-(n+1)/2}.$$

Let X belong to the univariate Student-t family with density generator expressed as in (19). The tail generator can be shown to be

$$T(z) = c_p\int_{z^2/2}^{\infty} g(x)\,dx = c_p\int_{z^2/2}^{\infty}\left(1+\frac{x}{k_p}\right)^{-p}du = c_p\frac{k_p}{p-1}\left(1+\frac{z^2}{2k_p}\right)^{-(p-1)}, \tag{22}$$

provided $p > 1$. Here we denote the normalizing constant by c_p with the subscript p to emphasize that it depends on the parameter p. Recall that c_p can be expressed as

$$c_p = \frac{\Gamma(p)}{\sqrt{2k_p}\Gamma(1/2)\Gamma(p-1/2)} = \frac{\Gamma(p)}{\sqrt{2\pi k_p}\Gamma(p-1/2)}. \tag{23}$$

Note that the case where $p = 1$ gives the Cauchy distribution for which the mean does not exist and therefore its tail generator also does not exist.

Now, if we put $p = (m+1)/2$ and $k_p = m/2$, $m = 1, 2, 3, \ldots$, we obtain the classical version of the univariate Student-t distribution with m degrees of freedom. Then for $m > 2$, we obtain from (22) that

$$\begin{aligned} T(z) &= \frac{\Gamma[(m+1)/2]}{\sqrt{m\pi}\Gamma(m/2)}\frac{m/2}{(m-1)/2}\left(1+\frac{z^2}{m}\right)^{-(m-1)/2} \\ &= \frac{\sqrt{m}}{(m-1)\sqrt{\pi}}\frac{\Gamma[(m+1)/2]}{\Gamma(m/2)}\left(1+\frac{z^2}{m}\right)^{-(m-1)/2}. \end{aligned} \tag{24}$$

If $m = 2$, the variance does not exist and but the tail generator gives

$$T(z) = \left(1 + \frac{z^2}{2}\right)^{-1/2}.$$

The case where $m = 1$ represents the Cauchy distribution for which its tail generator does not exist.

2.3 Multivariate Logistic Distributions

An elliptical vector $\mathbf{X}$ belongs to the family of multivariate logistic distributions if its density generator has the form

$$g(u) = \frac{e^{-u}}{(1 + e^{-u})^2}. \tag{25}$$

If $\mathbf{X}$ belongs to the family of multivariate logistic distributions, we shall write $\mathbf{X} \sim ML_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Its joint density has the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \frac{\exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right]}{\left\{1 + \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right]\right\}^2}$$

where the normalizing constant can be evaluated using (10) as follows

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[\int_0^\infty x^{n/2-1} \frac{e^{-x}}{(1 + e^{-x})^2} dx\right]^{-1} = (2\pi)^{-n/2} \left[\sum_{j=1}^\infty (-1)^{j-1} j^{1-n/2}\right]^{-1}.$$

When $n = 1$, it can be verified that the normalizing constant is $c = 1/2$. To derive the tail generator, notice that

$$T(z) = c \int_{z^2/2}^\infty g(x)\, dx = \frac{1}{2} \int_{z^2/2}^\infty \frac{e^{-x}}{(1 + e^{-x})^2} dx = \frac{1}{2\left(1 + e^{-z^2/2}\right)}. \tag{26}$$

2.4 Multivariate Exponential Power Distributions

An elliptical vector $\mathbf{X}$ is said to have a multivariate exponential power distribution if its density generator has the form

$$g(u) = e^{-ru^s}, \text{ for } r, s > 0. \tag{27}$$

The joint density of $\mathbf{X}$ can be expressed in the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left\{-\frac{r}{2}\left[(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right]^s\right\}$$

where the normalizing constant can be expressed as

$$c_n = \frac{s\Gamma(n/2)}{(2\pi)^{n/2}\, \Gamma(n/(2s))} r^{n/(2s)}.$$

When $r = s = 1$, this family of distributions clearly reduces to the multivariate normal family. When $s = 1$ alone, this family reduces to the original Kotz multivariate distribution suggested by Kotz (1975). If $s = 1/2$ and $r = \sqrt{2}$, we have the family of Double Exponential or Laplace distributions.

When $n = 1$, one can determine the normalizing constant to be

$$c = \frac{sr^{1/(2s)}}{\sqrt{2}\Gamma\left[1/(2s)\right]}. \tag{28}$$

The tail generator can be derived as

$$\begin{aligned} T(z) &= c\int_{z^2/2}^{\infty} g(x)\,dx = \frac{sr^{1/(2s)}}{\sqrt{2}\Gamma\left[1/(2s)\right]}\int_{z^2/2}^{\infty} e^{-rx^s}dx \\ &= \frac{sr^{1/(2s)}}{\sqrt{2}\Gamma\left[1/(2s)\right]}\left(sr^{1/s}\right)^{-1}\int_{rx^s}^{\infty} w^{1/s-1}e^{-w}dw \\ &= \frac{r^{-1/(2s)}}{\sqrt{2}\Gamma\left[1/(2s)\right]}\left\{\Gamma(1/s) - \Gamma\left[r\left(\tfrac{1}{2}z_q^2\right)^s; 1/s\right]\right\}, \end{aligned} \tag{29}$$

where

$$\Gamma(z; 1/s) = \int_0^z w^{1/s-1}e^{-w}dw \tag{30}$$

denotes the incomplete Gamma function.

3 TCV in the Univariate Dimension

This section develops tail conditional variance formulas for the univariate elliptical distribution which as a matter of fact coincides with the class of symmetric distributions on the line $\mathbf{R}$. Recall that we denote by x_q the q-th quantile of the loss distribution $F_X(x)$. Because we are interested in considering the tails of symmetric distributions, we suppose that $q > 1/2$ so that clearly

$$x_q > \mu. \tag{31}$$

Now suppose $g(x)$ is a non-negative function on $[0, \infty)$ satisfying the condition that

$$\int_0^{\infty} x^{-1/2}g(x)dx < \infty.$$

Then as already discussed in Section 2, $g(x)$ can be a density generator of a univariate elliptical distribution of a random variable $X \sim E_1(\mu, \sigma^2, g)$ whose density is expressed as

$$f_X(x) = \frac{c}{\sigma}g\left[\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{32}$$

where c is the normalizing constant.

Note that because X has an elliptical distribution, the standardized random variable $Z = (X - \mu)/\sigma$ will have a standard elliptical (or spherical) distribution function

$$F_Z(z) = c\int_{-\infty}^{z} g\left(\tfrac{1}{2}u^2\right) du,$$

with mean 0 and variance

$$\sigma_Z^2 = 2c\int_0^{\infty} u^2 g\left(\tfrac{1}{2}u^2\right) du = -\psi'(0),$$

if the condition $|\psi'(0)| < \infty$ holds. Furthermore, if the generator of the elliptical family is chosen such that the condition $\psi'(0) = -1$ holds, then $\sigma_Z^2 = 1$.

Theorem 1 *Let $X \sim E_1(\mu, \sigma^2, g)$ and T be the cumulative generator defined in (13). Under the conditions that both the mean and the variance of X exist, the tail conditional variance of X can be expressed as*

$$TCV_q(X) = \left[\frac{\overline{F}_{Z^*}(z_q)}{\overline{F}_Z(z_q)} + \frac{T(z_q)}{\overline{F}_Z(z_q)} z_q\right]\sigma^2, \tag{33}$$

where $z_q = (x_q - \mu)/\sigma$ is the q-th quantile of the spherical Z and Z^* is another spherical random variable whose density is the tail generator.

Proof. Observe that

$$\begin{aligned} TCV_q(X) &= \frac{1}{\overline{F}_X(x_q)}\int_{x_q}^{\infty} (x - \mu_X)^2 f_X(x)\, dx \\ &= \frac{1}{\overline{F}_X(x_q)}\int_{x_q}^{\infty} (x - \mu_X)^2 \cdot \frac{c}{\sigma} g\left[\tfrac{1}{2}\left((x-\mu)/\sigma\right)^2\right] dx \end{aligned}$$

and by letting $z = (x - \mu)/\sigma$, we have

$$TCV_q(X) = \frac{\sigma^2}{\overline{F}_Z(z_q)}\int_{z_q}^{\infty} cz^2 g\left(\tfrac{1}{2}z^2\right) dz.$$

Now applying integration by parts with $u = z$ and $dv = czg\left(\frac{1}{2}z^2\right) dz$, we should be clear to have

$$\begin{aligned} TCV_q(X) &= \frac{\sigma^2}{\overline{F}_Z(z_q)}\left[-zT(z)|_{z_q}^{\infty} + \int_{z_q}^{\infty} T(z)\, dz\right] \\ &= \frac{\sigma^2}{\overline{F}_Z(z_q)}\left[z_q T(z_q) + \int_{z_q}^{\infty} T(z)\, dz\right] \end{aligned}$$

where we have applied Lemma (1) to evaluate $zT(z)$ at infinity. The result immediately follows because clearly

$$\int_{-\infty}^{\infty} T(z)\, dz = -\psi'(0),$$

which has been chosen to be 1 so that we see

$$f_{Z^*}(z) = -\frac{1}{\psi'(0)} T(z) = T(z)$$

defines a proper density of a standard elliptical random variable Z^*. The result immediately follows. ■

In the case of the Normal distribution, we know that the tail generator is the density of a standard Normal random variable so that $Z^* = Z$ in this case. Thus, using the result in the Theorem above, we have

$$\begin{aligned} TCV_q(X) &= \left[\frac{\overline{F}_{Z^*}(z_q)}{1-\Phi_Z(z_q)} + \frac{\varphi(z_q)}{1-\Phi_Z(z_q)} z_q\right]\sigma^2 \\ &= \left[\frac{1-\Phi_Z(z_q)}{1-\Phi_Z(z_q)} + \frac{\varphi(z_q)}{1-\Phi_Z(z_q)} z_q\right]\sigma^2 \\ &= \left[1 + \frac{\varphi(z_q)}{1-\Phi_Z(z_q)} z_q\right]\sigma^2. \end{aligned}$$

It is clear that (33) generalizes the tail conditional variance derived in Valdez (2004) for the class of normal distributions to the larger class of univariate symmetric distributions. For other members of the elliptical class, all that is needed is the distribution function of Z^* which can readily be determined by integrating the tail generator.

For illustration purpose, let us consider the classical version of the univariate Student-t distribution with m degrees of freedom and $\mu = 0$, $\sigma = 1$. Here, we put $p = (m+1)/2$ and $k_p = m/2$, for $m = 1, 2, 3, ...$, and $n = 1$ in (20), so that the density generator is

$$g(x) = \left(1 + \frac{x}{m/2}\right)^{-(m+1)/2}$$

and density is

$$f_X(x) = \frac{\Gamma[(m+1)/2]}{\Gamma(m/2)} \frac{1}{\sqrt{\pi m}} \left(1 + \frac{x^2}{m}\right)^{-(m+1)/2}.$$

A straightforward integration gives us the tail generator

$$\begin{aligned} T(z) &= \int_{z^2/2}^{\infty} cg(x)\,dx = \int_{z^2/2}^{\infty} c\left(1 + \frac{2}{m}x\right)^{-(m+1)/2} dx \\ &= c\frac{m}{m-1}\left(1 + \frac{z^2}{m}\right)^{-(m-1)/2} \end{aligned}$$

[Note: Two graphs on TCV of the Classical Student-t will be inserted here: one will be the quantile q vs TCV, and another will be the parameter m vs TCV.]

4 TCV and Multivariate Elliptical Distributions

Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ be a multivariate elliptical vector, that is, $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. Denote the (i,j)-th element of $\boldsymbol{\Sigma}$ by σ_{ij} so that $\boldsymbol{\Sigma} = (\sigma_{ij})$ for $i, j = 1, 2, ..., n$. Furthermore, let

$$F_Z(z) = c_1 \int_0^z g_1\left(\tfrac{1}{2}x^2\right) dx$$

be the standard one-dimensional distribution function corresponding to this elliptical family and

$$T(z) = c_1 \int_{z^2/2}^{\infty} g_1(x)\, dx \tag{34}$$

be its tail generator. From Theorem 1 and the fact that the marginals are also elliptical, we observe immediately that the formula for computing tail conditional variances for each component of the vector $\mathbf{X}$ can be expressed as

$$TCV_q(X_k) = \left[\frac{\overline{F}_{Z_k^*}(z_{k,q})}{\overline{F}_{Z_k}(z_{k,q})} + \frac{T(z_{k,q})}{\overline{F}_Z(z_{k,q})} z_{k,q}\right] \sigma_k^2,$$

where

$$z_{k,q} = \frac{x_q - \mu_k}{\sigma_k},$$

Z_k is the spherical random variable corresponding to the k-th component, and Z_k^* is another spherical random variable determined from integrating the tail generator. Notice that the form of the tail generator for each component will be similar.

4.1 Sums of Elliptical Risks

Suppose $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ and $\mathbf{e} = (1, 1, ..., 1)^T$ is the vector of ones with dimension n. Define

$$S = X_1 + \cdots + X_n = \sum_{k=1}^{n} X_k = \mathbf{e}^T \mathbf{X} \tag{35}$$

which is the sum of elliptical risks. We now state a theorem for finding the TCV for this sum.

Theorem 2 *The tail conditional expectation of S can be expressed as*

$$TCV_q(S) = \left[\frac{\overline{F}_{Z^*}(z_{S,q})}{\overline{F}_Z(z_{S,q})} + z_{S,q}\frac{T(z_{S,q})}{\overline{F}_Z(z_{S,q})}\right] \sigma_S^2 \tag{36}$$

where $\mu_S = \mathbf{e}^T \boldsymbol{\mu} = \sum_{k=1}^n \mu_k$, $\sigma_S^2 = \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e} = \sum_{i,j=1}^n \sigma_{ij}$ *and*

$$z_{S,q} = \frac{x_q - \mu_S}{\sigma_S}.$$

Here Z is the spherical random variable corresponding to the sum S and Z^* is another spherical random variable that corresponds to Z determined based on integrating the tail generator T.

Proof. It follows immediately from the fact that $S \sim E_n\left(\mathbf{e}^T\boldsymbol{\mu}, \mathbf{e^T\Sigma e}, g_1\right)$ and the result follows using Theorem 1. ■

4.2 Decomposition of the TCV for Sums

Consider the total loss or claim expressed as in (35) where one can think of each X_k as the claim arising from a particular line of business or product line in the case of insurance, or the loss resulting from a financial instrument or a portfolio of investments. As noticed by Valdez (2004), from the additivity of covariance, the tail conditional variance allows for a natural decomposition of the variability of the total loss along its right tail. Observe that

$$TCV_S(s_q) = Var\left(S - \mu_S \,|S > s_q\right) = \sum_{k=1}^{n} Cov\left[\left(X_k - \mu_k\right)\left(S - \mu_S\right)|S > s_q\right]. \quad (37)$$

Note that this is not in general equivalent to the sum of the tail conditional variances of the individual components. This is because

$$TCV_q(X_k) \neq Cov\left[\left(X_k - \mu_k\right)\left(S - \mu_S\right)|S > s_q\right].$$

Instead, we denote this the tail conditional covariance

$$TCC_q(X_k\,|S) = Cov\left[\left(X_k - \mu_k\right)\left(S - \mu_S\right)|S > s_q\right],$$

the contribution to the total risk attributable to risk k.

Formally, we define the tail covariance between two random variables.

Definition 3 *Consider a bivariate vector $\mathbf{Y}^T = (Y_1, Y_2)$ with mean vector $\boldsymbol{\mu}^T = (\mu_1, \mu_2)$. The tail covariance of $\mathbf{Y}^T$, conditional on $Y_2 > y_{2,q}$ is defined to be*

$$TCC_q(Y_1\,|Y_2) = Cov\left(Y_1 - \mu_1, Y_2 - \mu_2\,|Y_2 > y_{2,q}\right), \quad (38)$$

where $y_{2,q}$ denotes the q-th quantile of the distribution of Y_2.

In the following, we derive an expression for the tail covariance

$$TCC_q(X_k\,|S) = Cov\left(X_k - \mu_k, S - \mu_S\,|S > s_q\right)$$

in the case of Elliptical distribution. The conditional tail variance of S has been derived in Theorem 2. Write the correlation coefficient of X_k and S as

$$\rho_{k,S} = \frac{Cov\left(X_k, S\right)}{Var\left(X_k\right)Var\left(S\right)} = \frac{\sigma_{k,S}}{\sigma_k\sigma_S}.$$

The tail conditional covariance can be evaluated using

$$TCC_q(X_k|S) = \frac{1}{\overline{F}_S(s_q)} \int_{-\infty}^{\infty} \int_{s_q}^{\infty} (x_k - \mu_k)(s - \mu_S) f_{X_k,S}(x_k, s)\, ds dx_k \tag{39}$$

where $f_{X_k,S}(\cdot,\cdot)$ denotes the bivariate Elliptical density of (X_k, S).

To develop the formula for decomposition, first, we need the following lemma which has been proven in Landsman and Valdez (2003).

Lemma 2 *Let* $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. *Then for* $1 \le k \le n$, *the vector* $\mathbf{X}_{k,S} = (X_k, S)^T$ *has an elliptical distribution with the same generator, i.e.,* $\mathbf{X}_{k,S} \sim E_2(\boldsymbol{\mu}_{k,S}, \boldsymbol{\Sigma}_{k,S}, g_2)$, *where* $\boldsymbol{\mu}_{k,S} = \left(\mu_k, \sum_{j=1}^n \mu_j\right)^T$,

$$\boldsymbol{\Sigma}_{k,S} = \begin{pmatrix} \sigma_k^2 & \sigma_{kS} \\ \sigma_{kS} & \sigma_S^2 \end{pmatrix},$$

and $\sigma_k^2 = \sigma_{kk}, \sigma_{kS} = \sum_{j=1}^n \sigma_{kj}, \sigma_S^2 = \sum_{i,j=1}^n \sigma_{ij}$.

Proof. See Landsman and Valdez (2003). ■

Theorem 3 *Suppose that* $\mathbf{X}^T = (X_1, X_2, ..., X_n)$ *follows a multivariate Elliptical distribution with mean vector*

$$\boldsymbol{\mu}^T = (\mu_1, \mu_2, ..., \mu_n)$$

and variance-covariance matrix

$$\boldsymbol{\Sigma} = (\sigma_{ij}) \text{ for } i,j = 1,2,...,n.$$

Then the conditional tail covariance of (X_k, S), *for* $k = 1,2,...,n$, *can be expressed as*

$$TCC_q(X_k|S) = \rho_{k,S}\sigma_k\sigma_S \left[\frac{\overline{F}_{Z^*}(z_{S,q})}{\overline{F}_Z(z_{S,q})} + \frac{T(z_{S,q})}{\overline{F}_Z(z_{S,q})} z_{S,q}\right], \tag{40}$$

where $\rho_{k,S} = \dfrac{\sigma_{k,S}}{\sigma_k \sigma_S}$ *is the correlation coefficient of* X_k *and* S, *and* $z_{S,q} = (s_q - \mu_S)/\sigma_S$. *Here* Z *and* Z^* *are spherical random variables similarly defined as in Theorem 2.*

Proof. We know from lemma (2) that we can express the joint density of $\mathbf{X}_{k,S} = (X_k, S)^T$ as

$$f_{k,S}(x_k, s) = \frac{c_2}{\sqrt{|\boldsymbol{\Sigma}_{k,S}|}} g_2 \left[\tfrac{1}{2} \left(\mathbf{x}_{k,S} - \boldsymbol{\mu}_{k,S}\right)^T \boldsymbol{\Sigma}_{k,S}^{-1} \left(\mathbf{x}_{k,S} - \boldsymbol{\mu}_{k,S}\right)\right]$$

where c_2 denotes the normalizing constant. In the bivariate case, we have

$$|\boldsymbol{\Sigma}_{k,S}| = \begin{vmatrix} \sigma_k^2 & \sigma_{k,S} \\ \sigma_{k,S} & \sigma_S^2 \end{vmatrix} = \left(1 - \rho_{k,S}^2\right) \sigma_k^2 \sigma_S^2$$

and

$$
\begin{aligned}
& \left(\mathbf{x}_{k,S}-\boldsymbol{\mu}_{k,S}\right)^{T}\boldsymbol{\Sigma}_{k,S}^{-1}\left(\mathbf{x}_{k,S}-\boldsymbol{\mu}_{k,S}\right) \\
= & \frac{1}{\left(1-\rho_{k,S}^{2}\right)}\left[\left(\frac{x_k-\mu_k}{\sigma_k}\right)^{2}-2\rho_{k,S}\left(\frac{x_k-\mu_k}{\sigma_k}\right)\left(\frac{s-\mu_S}{\sigma_S}\right)+\left(\frac{s-\mu_S}{\sigma_S}\right)^{2}\right] \\
= & \frac{1}{\left(1-\rho_{k,S}^{2}\right)}\left\{\left[\left(\frac{x_k-\mu_k}{\sigma_k}\right)-\rho_{i,Z}\left(\frac{s-\mu_S}{\sigma_S}\right)\right]^{2}+\left(1-\rho_{k,S}^{2}\right)\left(\frac{s-\mu_S}{\sigma_S}\right)^{2}\right\}.
\end{aligned}
$$

Next, consider the tail conditional covariance

$$
\begin{aligned}
& TCC_q\left(X_k\left|S\right.\right) \\
= & \frac{1}{\overline{F}_S\left(s_q\right)}\int_{-\infty}^{\infty}\int_{s_q}^{\infty}\frac{\left(x_k-\mu_k\right)\left(s-\mu_S\right)}{\sqrt{\left|\boldsymbol{\Sigma}_{k,S}\right|}}c_2g_2\left[\tfrac{1}{2}\left(\mathbf{x}_{i,Z}-\boldsymbol{\mu}_{i,Z}\right)^{T}\boldsymbol{\Sigma}_{i,Z}^{-1}\left(\mathbf{x}_{i,Z}-\boldsymbol{\mu}_{i,Z}\right)\right]dsdx_k
\end{aligned}
$$

Denote the double integral by I. Using the transformations

$$
u=\frac{x_k-\mu_k}{\sigma_k}\text{ and }v=\frac{s-\mu_S}{\sigma_S},
$$

and the property that the marginal distributions of multivariate elliptical are again elliptical distributions with the same generator, we then have

$$
\begin{aligned}
I & = \frac{\sigma_k\sigma_S}{\sqrt{1-\rho_{k,S}^{2}}}\int_{-\infty}^{\infty}\int_{v_q}^{\infty}uv\cdot c_2g_2\left[\frac{1}{2}\frac{\left(u-\rho_{k,S}v\right)^{2}}{1-\rho_{k,S}^{2}}+\frac{1}{2}v^{2}\right]dvdu \\
& = \frac{\sigma_k\sigma_S}{\sqrt{1-\rho_{k,S}^{2}}}\int_{v_q}^{\infty}\int_{-\infty}^{\infty}uv\cdot c_2g_2\left[\frac{1}{2}\frac{\left(u-\rho_{k,S}v\right)^{2}}{1-\rho_{k,S}^{2}}+\frac{1}{2}v^{2}\right]dudv \\
& = \frac{\sigma_k\sigma_S}{\sqrt{1-\rho_{k,S}^{2}}}\int_{v_q}^{\infty}v\left\{\int_{-\infty}^{\infty}u\cdot c_2g_2\left[\frac{1}{2}\left(\frac{u-\rho_{k,S}v}{\sqrt{1-\rho_{k,S}^{2}}}\right)^{2}+\frac{1}{2}v^{2}\right]du\right\}dv \\
& = \frac{\sigma_k\sigma_S}{\sqrt{1-\rho_{k,S}^{2}}}\int_{v_q}^{\infty}vI_1\left(v\right)dv
\end{aligned}
$$

where

$$
I_1\left(v\right)=\int_{-\infty}^{\infty}u\cdot c_2g_2\left[\frac{1}{2}\left(\frac{u-\rho_{k,S}v}{\sqrt{1-\rho_{k,S}^{2}}}\right)^{2}+\frac{1}{2}v^{2}\right]du.
$$

With one more transformation $u^{*}=\dfrac{u-\rho_{k,S}v}{\sqrt{1-\rho_{k,S}^{2}}}$, we have

$$
\begin{aligned}
I_1\left(v\right) & = \sqrt{1-\rho_{k,S}^{2}}\int_{-\infty}^{\infty}\left(\sqrt{1-\rho_{k,S}^{2}}u^{*}+\rho_{k,S}v\right)c_2g_2\left[\frac{1}{2}\left(u^{*2}+v^{2}\right)\right]du^{*} \\
& = \sqrt{1-\rho_{k,S}^{2}}\int_{-\infty}^{\infty}\left(\sqrt{1-\rho_{k,S}^{2}}u^{*}+\rho_{k,S}v\right)c_2g_2\left[\frac{1}{2}\left(u^{*2}+v^{2}\right)\right]du^{*}
\end{aligned}
$$

and noting that

$$\int_{-\infty}^{\infty} c_2 u^* g_2 \left[\frac{1}{2} \left(u^{*2} + v^2 \right) \right] du^* = 0$$

and that

$$\int_{-\infty}^{\infty} c_2 g_2 \left[\frac{1}{2} \left(u^{*2} + v^2 \right) \right] du^* = c_1 g_1 \left(\frac{1}{2} v^2 \right),$$

it follows that

$$I_1 (v) = \sqrt{1 - \rho_{k,S}^2} \rho_{k,S} v \cdot c_1 g_1 \left(\frac{1}{2} v^2 \right)$$

so that

$$\begin{aligned} I &= \frac{\sigma_k \sigma_S}{\sqrt{1 - \rho_{k,S}^2}} \int_{v_q}^{\infty} v I_1 (v) \, dv \\ &= \frac{\sigma_k \sigma_S}{\sqrt{1 - \rho_{k,S}^2}} \cdot \sqrt{1 - \rho_{k,S}^2} \rho_{k,S} \int_{-\infty}^{\infty} v^2 \cdot c_1 g_1 \left(\frac{1}{2} v^2 \right) dv \\ &= \rho_{k,S} \sigma_k \sigma_S \int_{-\infty}^{\infty} v^2 \cdot c_1 g_1 \left(\frac{1}{2} v^2 \right) dv. \end{aligned}$$

It can therefore be verified that

$$\begin{aligned} TCC_q \left(X_k \left| S \right. \right) &= \frac{\rho_{k,S} \sigma_k \sigma_S}{\overline{F}_S \left(s_q \right)} \left\{ \int_{-\infty}^{\infty} v^2 \cdot c_1 g_1 \left(\frac{1}{2} v^2 \right) dv \right\} \\ &= \rho_{k,S} \sigma_k \sigma_S \left[\frac{\overline{F}_{Z^*} \left(z_{S,q} \right)}{\overline{F}_Z \left(z_{S,q} \right)} + \frac{T \left(z_{S,q} \right)}{\overline{F}_Z \left(z_{S,q} \right)} z_{S,q} \right]. \end{aligned}$$

■

Notice that it is straightforward to verify that if we sum the tail covariance above from $k = 1$ to $k = n$,

$$\begin{aligned} \sum_{k=1}^{n} TCC_q \left(X_k \left| S \right. \right) &= \left[\frac{\overline{F}_{Z^*} \left(z_{S,q} \right)}{\overline{F}_Z \left(z_{S,q} \right)} + \frac{T \left(z_{S,q} \right)}{\overline{F}_Z \left(z_{S,q} \right)} z_{S,q} \right] \times \left(\sum_{k=1}^{n} \rho_{k,S} \sigma_k \sigma_S \right) \\ &= \left[\frac{\overline{F}_{Z^*} \left(z_{S,q} \right)}{\overline{F}_Z \left(z_{S,q} \right)} + \frac{T \left(z_{S,q} \right)}{\overline{F}_Z \left(z_{S,q} \right)} z_{S,q} \right] \times \sigma_S^2, \end{aligned}$$

which gives clearly $TCV_S \left(s_q \right)$, the correct tail conditional variance for the sum of the Elliptical as given in Theorem 2. This follows immediately from the fact that

$$\begin{aligned} \sum_{k=1}^{n} TCC_q \left(X_k \left| S \right. \right) &= \sum_{k=1}^{n} Cov \left(X_k - \mu_k, S - \mu_S \left| S > s_q \right. \right) \\ &= Cov \left(\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} \mu_k, S - \mu_S \left| S > s_q \right. \right) \\ &= Cov \left(S - \mu_S, S - \mu_S \left| S > s_q \right. \right) \\ &= Var \left(S - \mu_S \left| S > s_q \right. \right) \\ &= TCV_q (S). \end{aligned}$$

5 Final Remarks

The right tail risk has long been considered in insurance phenomenon, although only recently has there been an increasing emphasis on developing risk measures for tails. See also Goovaerts, et al. (2003) for discussion of the importance of risk measures for capital allocation. In this paper, we recommend use of the tail conditional variance as a measure of right-tail risk, particularly useful when one is concerned with variability along the right (or left) tail of the loss distribution. This paper develops an appealing way to characterize the tail conditional variance and tail covariances of elliptical distributions. Similar to the tail conditional expectations in Landsman and Valdez (2003), the results in this paper are motivated by exploiting the nice properties of the Elliptical distribution. In the univariate case, the class of elliptical distributions consists of the class of symmetric distributions which include familiar distributions like Normal and Student t. This class can easily be extended into the multivariate framework by simply characterizing them either in terms of the characteristic generator or the density generator. This paper studied this class of multidimensional distributions rather extensively to allow the reader to understand them more thoroughly particularly many of the properties of the multivariate Normal is shared by this larger class. Furthermore, this paper defines the *tail generator* resulting from the integration of the density generator, and uses this generator quite extensively to generate formulas for tail conditional variance and tail covariances. We also know that tail conditional variance naturally permits a decomposition of into individual components consisting of the individual risks making up the multivariate random vector. We extended TCV formulas developed for the univariate case into the case where there are several risks which when taken together behaves like an elliptical random vector. We further extended the results into the case where we then decompose the TCV into individual components making up the sum of the risks. We are able to verify, using the results developed in this paper, the formulas that were investigated and developed by Valdez (2004) in the case of the multivariate Normal distribution.

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