

# The Iterated Tail Conditional Expectation for the Log-Elliptical Loss Process<sup>‡</sup>

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## Abstract

This paper derives the analytic form of the Iterated Tail Conditional Expectation (ITCE) risk measure in the case of a log-elliptical loss process. The ITCE has been suggested by Hardy and Wirch (2003) as a dynamic risk measure designed to cover multi-period loss processes. This risk measure allows for explicit allowance of losses to evolve over time and the idea is to apply a repeated calculation of the static Tail Conditional Expectation (TCE) risk measure by backwards induction. Hardy and Wirch (2003) derived the case of the log-Normal loss process. In this paper, we are able to exploit the properties of the elliptical distributions similar to the Normal distribution in order to derive an explicit form in computing the iterated TCE. In particular, the *cumulative generator* defined in an earlier paper by Landsman and Valdez (2003) plays an important role in the process of developing these explicit forms.

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# 1 Introduction

Consider an insurance company with a long term horizon  $T$  and assume it has a portfolio of insurance contracts with a random liability of  $X_T$  at the end of this horizon. Suppose this random liability will evolve over time according to the risk process  $\{X_t\}$  for  $t = 1, 2, \dots, T$ , where in some instances the process at  $t = 0$ , denoted by  $X_0$ , is fixed and known. Here  $t$  is a time period which may be measured in days, weeks, months, or years. In this paper, we will assume discrete time stochastic process and in particular, we will examine separately the additive and the multiplicative risk processes as described in Section 2.

Now, the insurer wants to determine the amount of economic capital required to cover for its potential liability  $X_T$ . According to the static approach of risk measurement, the company will determine today a fixed amount and hold that fixed amount at every valuation date in the future regardless of the history of the process that would have evolved over time. On one hand, it is possible that the risk would evolve in its favor so that excessive capital would have been held. On the other hand, it is possible that the risk would evolve in an unfavorable situation so that there would be inadequate amount being held. It is therefore important to capture the dynamic nature of the risk process so that a risk measure must be itself dynamic to the changing nature of the risk process according to time. In this paper, we will examine the iterated tail conditional expectation which is obtained by repeated calculation of the tail conditional expectation (or any other applicable static risk measure, for that matter) through backwards induction, a method suggested by Hardy and Wirch (2003). Notice also that although the set-up here is for an insurance company, the results in this paper are equally applicable to any other financial institutions faced with a random loss over a long term horizon.

For a random loss  $X$  with distribution function denoted by  $F_X(x)$  and tail function denoted by  $\overline{F}_X(x) = 1 - F_X(x)$ , the *tail conditional expectation* (TCE) is defined to be

$$TCE_q(X) = E(X | X > x_q) \tag{1}$$

and can be interpreted as the “average worst possible loss”. See Artzner, et al. (1999). Here, the expectation is conditional on exceeding a fixed particular value  $x_q$ , generally referred to as the  $q$ -th quantile with

$$\overline{F}_X(x_q) = 1 - q.$$

The value of  $q$  is chosen to be high, typically 95% or 99%, to ensure there is little chance for the insurance company to ruin. Artzner, et al. (1999) demonstrated that the TCE satisfies all requirements for a coherent risk measure. When compared to the traditional Value-at-Risk ( $VaR$ ) measure, the tail conditional expectation provides a more conservative measure of risk for the same level of degree of confidence  $q$ . To see this, note that  $VaR_q(X) = x_q$  and since we can re-write formula (1) as

$$TCE_q(X) = x_q + E(X - x_q | X > x_q)$$

then

$$TCE_q(X) \geq VaR_q(X)$$

because the second term is clearly non-negative. Artzner and his co-authors also showed that the Value-at-Risk does not satisfy all requirements of a coherent risk measure. In particular, it violates the sub-additivity requirement of a coherent risk measure. However, the sub-additivity requirement is satisfied only when continuous risks are concerned. In Dhaene, et al. (2003), there are situations when sub-additivity is not met in cases where the risk is either purely discrete or a mixed random variable. At any rate, the TCE continues to be a popular risk measure and according to Hardy and Wirch (2003), it is the one risk measure recommended by the Canadian Institute of Actuaries for determining capital requirements for insurance companies with portfolios of equity-linked insurance contracts.

Consider a normal random variable  $X$  with mean  $\mu$  and variance  $\sigma^2$  and we write  $X \sim N(\mu, \sigma^2)$ . Its density can be expressed as

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma}\varphi\left(\frac{x-\mu}{\sigma}\right),$$

and in this paper, we shall use the symbols  $\varphi(\cdot)$  and  $\Phi(\cdot)$  to denote, respectively, the density and distribution functions of a standard normal random variable. We shall also use the usual notation  $Z$  to denote the standard normal random variable. It has been demonstrated, for example, in Panjer (2002) and in Landsman and Valdez (2003), that the TCE for the normal distribution can be expressed as

$$TCE_q(X) = E(X | X > x_q) = \mu + \frac{\varphi(z_q)}{1 - \Phi(z_q)} \cdot \sigma, \quad (2)$$

where  $z_q = (x_q - \mu)/\sigma$  is the standardized  $q$ -th quantile of the distribution. Now suppose  $X$  has a log-normal distribution, that is,  $\log X \sim N(\mu, \sigma^2)$  is normal with mean  $\mu$  and variance  $\sigma^2$ . We can then write its density as

$$f_X(x) = \frac{1}{\sigma x} \varphi\left(\frac{\log x - \mu}{\sigma}\right), \text{ for } x > 0.$$

It has been shown in Dhaene, et al. (2003) that an expression for the TCE is given by

$$TCE_q(X) = \exp(\mu + \sigma^2/2) \cdot \frac{\Phi(\sigma - z_q)}{1 - q}. \quad (3)$$

The TCE for the log-normal can be extended to the case of the log-elliptical distribution. In the univariate case, we have  $X$  belonging to the class of elliptical distributions if its density can be expressed in the form

$$f_X(x) = \frac{c}{\sigma} g\left[\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

for some density generator  $g$ . Later in the paper, the definition of elliptical distributions is expressed in terms of its characteristic function because this function always exist for a distribution. It has been demonstrated in Landsman and Valdez (2003) that the TCE of  $X$  can be similarly expressed as

$$TCE_q(X) = \mu + \frac{\overline{G}(z_q^2)}{\overline{F}_Z(z_q)} \cdot \sigma,$$

where  $z_q = (x_q - \mu) / \sigma$ . Here,  $Z$  is the standardized elliptical random variable and  $G$  is called the cumulative generator and later defined in this paper. In the case where  $X$  is log-elliptical, the TCE can be shown to be

$$TCE_q(X) = \frac{e^\mu}{1-q} \cdot \psi(-\sigma^2) \cdot F_{Z^*}(F_Z^{-1}(1-q)),$$

where the density of  $Z^*$  is given by

$$f_{Z^*}(x) = \frac{e^{\sigma x}}{\psi(-\sigma^2)} \cdot f_Z(x),$$

with  $\psi$  denoting the elliptical distribution's characteristic generator.

The rest of the paper has been organized as follows. Section 2 explains the construction of the dynamic risk measure through backwards induction as has been proposed by Hardy and Wirch (2003). This is further examined when the risk process is additive and also when the risk process is multiplicative. Section 3 briefly discusses the rich class of elliptical distributions in the univariate dimension as well as the class of log-elliptical distributions. Section 4 gives several results about the static TCE in the case of elliptical as well as log-elliptical distributions. Here, we also derive the resulting iterated TCE formulas for these distributions. Section 5 considers the special case of the Laplace and the log-laplace distributions. Finally, in section 6, we give concluding remarks.

## 2 Constructing a Dynamic Risk Measure Through Backwards Induction

In this section, we show how to construct a dynamic risk measure through backwards induction and the result is an *iterated risk measure* as has been called in Hardy and Wirch (2003). The main ideas of this section have appeared in Hardy and Wirch (2004), but we present the results and ideas again in this paper for completeness and to fit our notation in the paper.

Again consider an insurance company with a long term horizon  $T$  and assume it has a portfolio of insurance contracts with a random liability of  $X_T$  at the end of this horizon. Assume that the history of information is denoted by  $\mathcal{F}_t$  for  $t = 1, 2, \dots, T$  and we denote the dynamic risk measure at time  $t$  by  $\rho(X_T | \mathcal{F}_t)$  given the information available at that time. Furthermore, assume a yearly discounting at the continuous

force of interest  $\delta$ . We determine then the risk measure at every point in time  $t$  through backwards induction beginning with time  $T - 1$  as follows:

$$\rho(X_T | \mathcal{F}_{T-1}) = E_D(e^{-\delta} X_T | \mathcal{F}_{T-1})$$

where  $E_D(\cdot)$  denotes the expectation under a distorted probability measure, which is actually any applicable static risk measure. Assume that this risk measure satisfies coherence as in the spirit of Artzner, et al. (1999). At time  $t = T - 2$ , the risk measure becomes

$$\begin{aligned} \rho(X_T | \mathcal{F}_{T-2}) &= \rho(\rho(X_T | \mathcal{F}_{T-1}) | \mathcal{F}_{T-2}) \\ &= E_D[e^{-\delta} \rho(X_T | \mathcal{F}_{T-1}) | \mathcal{F}_{T-2}] \\ &= E_D[e^{-\delta} E_D(e^{-\delta} X_T | \mathcal{F}_{T-1}) | \mathcal{F}_{T-2}] \\ &= E_D(e^{-2\delta} X_T | \mathcal{F}_{T-2}) \end{aligned}$$

applying the law of iterated expectations in the last step. Proceeding inductively, we find that in general, at time  $t = k$ , we have the risk measure being equal to

$$\rho(X_T | \mathcal{F}_k) = E_D(e^{-(T-k)\delta} X_T | \mathcal{F}_k) \quad (4)$$

for  $k = 1, 2, \dots, T$ . In the case where the risk measure selected is the TCE, characteristics and properties of the resulting iterated TCE has been discussed in Hardy and Wirch (2003). Furthermore, Riedel (2003) and Wang (2000) provide axiomatic treatment of dynamic risk measures. Let us now look at separately the cases where we have the additive as well as the multiplicative risk processes.

## 2.1 Additive Risk Process

Suppose that the final random liability  $X_T$  will evolve over time according to the risk process

$$X_t = X_{t-1} + \Delta X_t \quad (5)$$

for  $t = 1, 2, \dots, T$  where  $X_0 = 0$ . Here, the incremental loss in each period  $\Delta X$  is described by a certain probability model, and these increments  $\Delta X_t$  are independent and identically distributed random variables. For example, they may follow the familiar normal distribution, or any other member of the family of elliptical distributions as later described in the paper.

Applying backwards induction, we find that

$$\begin{aligned} \rho(X_T | X_{T-1}) &= E_D(e^{-\delta} X_T | X_{T-1}) \\ &= e^{-\delta} E_D(X_{T-1} + \Delta X_T | X_{T-1}) \\ &= e^{-\delta} X_{T-1} + E_D(e^{-\delta} \Delta X_T | X_{T-1}) \\ &= e^{-\delta} X_{T-1} + \rho_S(\Delta X_T). \end{aligned}$$

Extending this to time  $t = T - 2$ , we find that the iterated risk measure becomes

$$\begin{aligned}
\rho(X_T | X_{T-2}) &= \rho(\rho(X_T | X_{T-1}) | X_{T-2}) \\
&= \rho\{[e^{-\delta} X_{T-1} + \rho_S(\Delta X_T)] | X_{T-2}\} \\
&= e^{-\delta} \rho(X_{T-1} | X_{T-2}) + \rho_S(\Delta X_T) \\
&= e^{-\delta} [e^{-\delta} X_{T-2} + \rho_S(\Delta X_{T-1})] + \rho_S(\Delta X_T) \\
&= e^{-2\delta} X_{T-2} + [e^{-\delta} \rho_S(\Delta X_{T-1}) + \rho_S(\Delta X_T)]
\end{aligned}$$

applying the law of iterated expectations in the last step. Extending this to time  $t = T - 3$ , we find that the iterated risk measure becomes

$$\begin{aligned}
\rho(X_T | X_{T-3}) &= \rho(\rho(X_T | X_{T-2}) | X_{T-3}) \\
&= \rho\{[e^{-2\delta} X_{T-2} + [e^{-\delta} \rho_S(\Delta X_{T-1}) + \rho_S(\Delta X_T)]] | X_{T-3}\} \\
&= e^{-2\delta} \rho(X_{T-2} | X_{T-3}) + [e^{-\delta} \rho_S(\Delta X_{T-1}) + \rho_S(\Delta X_T)] \\
&= e^{-2\delta} [e^{-\delta} X_{T-3} + \rho_S(\Delta X_{T-2})] + [e^{-\delta} \rho_S(\Delta X_{T-1}) + \rho_S(\Delta X_T)] \\
&= e^{-3\delta} X_{T-3} + [e^{-2\delta} \rho_S(\Delta X_{T-2}) + e^{-\delta} \rho_S(\Delta X_{T-1}) + \rho_S(\Delta X_T)]
\end{aligned}$$

applying the law of iterated expectations in the last step. Proceeding inductively, we have the following result for the iterated risk measure.

**Theorem 1** *Suppose the final random liability  $X_T$  evolve according to the additive risk process in (5). Then, at time  $t = k$ , the iterated risk measure can be expressed as*

$$\rho(X_T | X_k) = e^{-(T-k)\delta} X_k + \left[ \sum_{i=0}^{T-k-1} e^{-i\delta} \rho_S(\Delta X_{T-i}) \right] \quad (6)$$

for  $k = 1, 2, \dots, T$  and where  $\delta$  is the force of interest and  $\rho_S(\cdot)$  represent any static risk measure.

**Proof.** The proof can be done inductively. Suppose this is true for  $t = k + 1$ . Then for  $t = k$ , we have

$$\begin{aligned}
\rho(X_T | X_k) &= \rho(\rho(X_T | X_{k+1}) | X_k) \\
&= \rho\left\{e^{-(T-k-1)\delta} X_{k+1} + \left[ \sum_{i=0}^{T-k-2} e^{-i\delta} \rho_S(\Delta X_{T-i}) \right] \middle| X_k \right\} \\
&= e^{-(T-k-1)\delta} \rho(X_{k+1} | X_k) + \left[ \sum_{i=0}^{T-k-2} e^{-i\delta} \rho_S(\Delta X_{T-i}) \right] \\
&= e^{-(T-k-1)\delta} [e^{-\delta} X_k + \rho_S(\Delta X_{T-k+1})] + \left[ \sum_{i=0}^{T-k-2} e^{-i\delta} \rho_S(\Delta X_{T-i}) \right] \\
&= e^{-(T-k)\delta} X_k + \left[ \sum_{i=0}^{T-k-1} e^{-i\delta} \rho_S(\Delta X_{T-i}) \right]
\end{aligned}$$

which proves the desired result. ■

The result in (6) can in fact be intuitively interpreted. At any time  $t = k$  in the future, the value of the risk measure is going to be the discounted value of the current value,  $X_k$ , of your future liability expressed as

$$e^{-(T-k)\delta} X_k$$

plus the discounted value of the risk measures for the remaining increments in the value of your future liability expressed as

$$\sum_{i=0}^{T-k-1} e^{-i\delta} \rho_S(\Delta X_{T-i}).$$

Thus, if for example, these increments are identically distributed (not necessarily independent), the portion accounting for the remaining increments simplifies to a simple annuity payout as follows:

$$\rho_S(\Delta X) \cdot \sum_{i=0}^{T-k-1} e^{-i\delta} = \rho_S(\Delta X) \cdot \ddot{a}_{\overline{T-k}|\delta},$$

where  $\ddot{a}_{\overline{T-k}|\delta}$  is a standard actuarial notation for the present value of annuity payment of 1 for a period of  $T - k$  years evaluated at the force of interest  $\delta$ .

**Example 2.1 Normal Distribution** Suppose that  $\Delta X_t$  are independent and identically distributed (i.i.d.) random variables with normal  $N(\mu, \sigma^2)$  distribution and consider the TCE as the static risk measure. Thus, since it is known that

$$\rho_S(\Delta X_t) = \mu + \frac{\varphi(z_q)}{1 - \Phi(z_q)} \cdot \sigma,$$

we have an expression for the iterated TCE:

$$\rho(X_T | X_k) = ITCE_q(X_T | X_k) = e^{-(T-k)\delta} X_k + \left( \sum_{i=0}^{T-k-1} e^{-i\delta} \right) \left[ \mu + \frac{\varphi(z_q)}{1 - \Phi(z_q)} \cdot \sigma \right]. \quad (7)$$

Observe that when the force of interest  $\delta$  is zero, then at  $k = 0$ , the iterated TCE in (7) becomes

$$\begin{aligned} ITCE_q(X_T | X_0) &= X_0 + T \left[ \mu + \frac{\varphi(z_q)}{1 - \Phi(z_q)} \cdot \sigma \right] \\ &= T\mu + \frac{\varphi(z_q)}{1 - \Phi(z_q)} \cdot T\sigma \end{aligned}$$

which clearly is larger than the resulting static TCE. This is because

$$X_T = \sum_{i=1}^T \Delta X_i,$$

the sum of  $T$  i.i.d. normal random variables, and is therefore  $N(T\mu, T\sigma^2)$  distribution. The resulting static TCE is given by

$$\rho_S(X_T) = T\mu + \frac{\varphi(z_q)}{1 - \Phi(z_q)} \cdot \sqrt{T}\sigma.$$

## 2.2 Multiplicative Risk Process

Suppose that the final random liability  $X_T$  will evolve over time according to the risk process

$$X_t = \Delta X_t \cdot X_{t-1} \quad (8)$$

for  $t = 1, 2, \dots, T$  where  $X_0 = 1$ . Here, the incremental loss in each period  $\Delta X$  is described by a certain probability model, and these increments  $\Delta X_t$  are independent and identically distributed random variables. For example, they may follow the familiar log-normal distribution, or any other member of the family of log-elliptical distributions as later described in the paper. Notice that if we take the logarithm of both sides of (8), then we are back to the case of the additive risk process. So, sometimes this process can be called the log-additive risk process.

Applying backwards induction, we find that the iterated risk measure can be expressed as

$$\begin{aligned} \rho(X_T | X_{T-1}) &= E_D(e^{-\delta} X_T | X_{T-1}) \\ &= e^{-\delta} E_D(\Delta X_T \cdot X_{T-1} | X_{T-1}) \\ &= e^{-\delta} X_{T-1} \cdot E_D(\Delta X_T | X_{T-1}) \\ &= e^{-\delta} X_{T-1} \cdot \rho_S(\Delta X_T). \end{aligned}$$

Extending this to time  $t = T - 2$ , we find that the iterated risk measure becomes

$$\begin{aligned} \rho(X_T | X_{T-2}) &= \rho(\rho(X_T | X_{T-1}) | X_{T-2}) \\ &= \rho\{[e^{-\delta} X_{T-1} \cdot \rho_S(\Delta X_T)] | X_{T-2}\} \\ &= e^{-\delta} \rho(X_{T-1} | X_{T-2}) \cdot \rho_S(\Delta X_T) \\ &= e^{-\delta} [e^{-\delta} X_{T-2} \cdot \rho_S(\Delta X_{T-1})] \cdot \rho_S(\Delta X_T) \\ &= e^{-2\delta} X_{T-2} \cdot [\rho_S(\Delta X_{T-1}) \cdot \rho_S(\Delta X_T)] \end{aligned}$$

applying the law of iterated expectations in the last step. Extending this to time  $t = T - 3$ , the iterated risk measure becomes

$$\begin{aligned} \rho(X_T | X_{T-3}) &= \rho(\rho(X_T | X_{T-2}) | X_{T-3}) \\ &= \rho\{[e^{-2\delta} X_{T-2} \cdot [\rho_S(\Delta X_{T-1}) \cdot \rho_S(\Delta X_T)]] | X_{T-3}\} \\ &= e^{-2\delta} \rho(X_{T-2} | X_{T-3}) \cdot [\rho_S(\Delta X_{T-1}) \cdot \rho_S(\Delta X_T)] \\ &= e^{-2\delta} [e^{-\delta} X_{T-3} \cdot \rho_S(\Delta X_{T-2})] \cdot [\rho_S(\Delta X_{T-1}) \cdot \rho_S(\Delta X_T)] \\ &= e^{-3\delta} X_{T-3} \cdot [\rho_S(\Delta X_{T-2}) \cdot \rho_S(\Delta X_{T-1}) \cdot \rho_S(\Delta X_T)] \end{aligned}$$

applying the law of iterated expectations in the last step. Proceeding inductively, we have the following result for the iterated risk measure.

**Theorem 2** *Suppose the final random liability  $X_T$  evolve according to the additive risk process in (8). Then, at time  $t = k$ , the iterated risk measure can be expressed as*

$$\rho(X_T | X_k) = e^{-(T-k)\delta} X_k \cdot \left[ \prod_{i=0}^{T-k-1} \rho_S(\Delta X_{T-i}) \right] \quad (9)$$



for  $k = 1, 2, \dots, T$  and where  $\delta$  is the force of interest and  $\rho_S(\cdot)$  represent any static risk measure.

**Proof.** The proof can be done inductively. Suppose this is true for  $t = k + 1$ . Then for  $t = k$ , we have

$$\begin{aligned}
\rho(X_T | X_k) &= \rho(\rho(X_T | X_{k+1}) | X_k) \\
&= \rho \left\{ e^{-(T-k-1)\delta} X_{k+1} \cdot \left[ \prod_{i=0}^{T-k-2} \rho_S(\Delta X_{T-i}) \right] \middle| X_k \right\} \\
&= e^{-(T-k-1)\delta} \rho(X_{k+1} | X_k) \cdot \left[ \prod_{i=0}^{T-k-2} \rho_S(\Delta X_{T-i}) \right] \\
&= e^{-(T-k-1)\delta} \left[ e^{-\delta} X_k \cdot \rho_S(\Delta X_{k+1}) \right] \cdot \left[ \prod_{i=0}^{T-k-2} \rho_S(\Delta X_{T-i}) \right] \\
&= e^{-(T-k)\delta} X_k \cdot \left[ \prod_{i=0}^{T-k-1} \rho_S(\Delta X_{T-i}) \right]
\end{aligned}$$

which proves the desired result. ■

**Example 2.2 Log-Normal Distribution** Suppose that  $\Delta X_t$  are i.i.d. random variables with log-normal  $LN(\mu, \sigma^2)$  distribution and consider the TCE as the static risk measure. We know that

$$\rho_S(\Delta X_t) = e^{(\mu + \sigma^2/2)} \cdot \frac{\Phi(\sigma - z_q)}{1 - q},$$

and therefore, we have an expression for the iterated TCE:

$$\rho(X_T | X_k) = ITCE_q(X_T | X_k) = e^{-(T-k)\delta} \cdot e^{(T-k)(\mu + \sigma^2/2)} X_k \cdot \left[ \frac{\Phi(\sigma - z_q)}{1 - q} \right]^{T-k} \quad (10)$$

which exactly replicates the formula derived in Hardy and Wirch (2003).

Next, consider the special case where  $\delta = 0$  and  $k = 0$ , then (10) becomes

$$ITCE_q(X_T | X_0) = e^{(T\mu + T\sigma^2/2)} \cdot \left[ \frac{\Phi(\sigma - z_q)}{1 - q} \right]^T.$$

Observe that

$$X_T = \prod_{i=1}^T \Delta X_i$$

which obviously is also log-normal in this case. This is because  $\log(\Delta X_i) \sim N(\mu, \sigma^2)$  and therefore  $\log\left(\prod_{i=1}^T \Delta X_i\right) \sim N(T\mu, T\sigma^2)$ . The static TCE in this case becomes

$$\rho_S(X_T) = e^{(T\mu + T\sigma^2/2)} \cdot \left[ \frac{\Phi(\sqrt{T}\sigma - z_q)}{1 - q} \right]$$

and therefore the two resulting TCE's are different in this case.

### 3 Elliptical and Log-Elliptical Distributions

The class of elliptical loss distribution models provides a generalization of the class of normal loss models. In the following, we will describe this class of models first in the univariate dimension, and briefly extending it to the multivariate dimension. The class of elliptical distributions has been introduced in the statistical literature by Kelker (1970) and widely discussed in Fang, et al. (1990) and Gupta and Varga (1993). See also Landsman and Valdez (2003), Valdez and Dhaene (2003), and Valdez and Chernih (2003) for applications in insurance and actuarial science. Embrechts, et al. (2001) also provides a fair amount of discussion of this important class as a tool for modelling risk dependencies.

#### 3.1 Definition of Elliptical Distributions

It is well-known that a random variable  $X$  with a normal distribution has the characteristic function expressed as

$$E[\exp(itX)] = \exp(it\mu) \cdot \exp\left(-\frac{1}{2}t^2\sigma^2\right), \quad (11)$$

where  $\mu$  and  $\sigma^2$  are respectively, the mean and variance of the distribution. We shall use the notation  $X \sim N(\mu, \sigma^2)$ . The class of elliptical distributions is a natural extension to the class of normal distributions.

**Definition 1** *The random variable  $X$  is said to have an elliptical distribution with parameters  $\mu$  and  $\sigma^2$  if its characteristic function can be expressed as*

$$E[\exp(itX)] = \exp(it\mu) \cdot \psi(t^2\sigma^2) \quad (12)$$

for some scalar function  $\psi$ .

If  $X$  has the elliptical distribution as defined above, we shall conveniently write  $X \sim E(\mu, \sigma^2, \psi)$  and say that  $X$  is elliptical. The function  $\psi$  is called the *characteristic generator* of  $X$  and therefore, for the normal distribution, the characteristic generator is clearly given by  $\psi(u) = \exp(-u/2)$ .

It is well-known that the characteristic function of a random variable always exists and that there is a one-to-one correspondence between distribution functions and characteristic functions. Note however that not every function  $\psi$  can be used to construct a characteristic function of an elliptical distribution. Obviously, this function  $\psi$  should fulfill the requirement that  $\psi(0) = 1$ . A necessary and sufficient condition for the function  $\psi$  to be a characteristic generator of an elliptical distribution can be seen in Theorem 2.2 of Fang, et al. (1990).

It is also interesting to note that the class of elliptical distributions consists mainly of the class of symmetric distributions which include well-known distributions like normal and Student- $t$ . The moments of  $X \sim E(\mu, \sigma^2, \psi)$  do not necessarily exist. However, it can be shown that if the mean,  $E(X)$ , exists, then it will be given by

$$E(X) = \mu \quad (13)$$

and if the variance,  $Var(X)$ , exists, then it will be given by

$$Var(X) = -2\psi'(0) \sigma^2, \quad (14)$$

where  $\psi'$  denotes the first derivative of the characteristic generator. A necessary condition for the variance to exist is

$$|\psi'(0)| < \infty, \quad (15)$$

see Cambanis et al. (1981).

In the case where  $\mu = 0$  and  $\sigma^2 = 1$ , we have what we call a spherical distribution and the random variable  $X$  is replaced by a standard random variable  $Z$ . That is, we have  $Z \sim E(0, 1, \psi)$  and the notation  $S(\psi)$  for  $E(0, 1, \psi)$  is more typically used and thus, we write  $Z \sim S(\psi)$ . It is clear that the characteristic function of  $Z$  has the form

$$E[\exp(itZ)] = \psi(t^2)$$

for any real number  $t$ .

Also, if we consider any random variable  $X$  satisfying

$$X \stackrel{d}{=} \mu + \sigma Z,$$

for some real number  $\mu$ , some positive real number  $\sigma$  and some spherical random variable  $Z \sim S(\psi)$ , then it can be shown that  $X \sim E(\mu, \sigma^2, \psi)$ . Similarly, for any elliptical random variable  $X \sim E(\mu, \sigma^2, \psi)$ , we can always define the random variable

$$Z = \frac{X - \mu}{\sigma}$$

which is clearly spherical.

### 3.2 Densities of Elliptical Distributions

An elliptically distributed random variable  $X \sim E(\mu, \sigma^2, \psi)$  does not necessarily possess a density function  $f_X(x)$ . In the case of a normal random variable  $X \sim N(\mu, \sigma^2)$ , its density is well-known to be

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right]. \quad (16)$$

For elliptical distributions, one can prove that if  $X \sim E(\mu, \sigma^2, \psi)$  has a density, it will have the form

$$f_X(x) = \frac{c}{\sigma} g\left[\left(\frac{x - \mu}{\sigma}\right)^2\right] \quad (17)$$

for some non-negative function  $g(\cdot)$  satisfying the condition

$$0 < \int_0^\infty z^{-1/2} g(z) dz < \infty \quad (18)$$

and a normalizing constant  $c$  given by

$$c = \left[ \int_0^\infty z^{-1/2} g(z) dz \right]^{-1}. \quad (19)$$

Also, the opposite statement holds: Any non-negative function  $g(\cdot)$  satisfying the condition (18) can be used to define a one-dimensional density

$$\frac{c}{\sigma} g \left[ \left( \frac{x - \mu}{\sigma} \right)^2 \right]$$

of an elliptical distribution, with  $c$  given by (19). The function  $g(\cdot)$  is called the *density generator*. One sometimes writes  $X \sim E(\mu, \sigma^2, g)$  for the one-dimensional elliptical distributions generated from the function  $g(\cdot)$ . A detailed proof of these results for the case of  $n$ -dimension, using spherical transformations of rectangular coordinates, can be found in Landsman & Valdez (2003).

From (16), one immediately finds that the density generators and the corresponding normalizing constants of the normal random variable  $X \sim N(\mu, \sigma^2)$  are given by

$$g(z) = \exp(-z/2) \quad (20)$$

and

$$c = \frac{1}{\sqrt{2\pi}}, \quad (21)$$

respectively.

As a special case, we find that from the results concerning elliptical distributions, if a spherical random variable  $Z \sim S(\psi)$  possess a density  $f_Z(z)$ , then it will have the form

$$f_Z(z) = cg(z^2),$$

where the density generator  $g$  satisfies the condition (18) and the normalizing constant  $c$  satisfies (19). Furthermore, the opposite also holds: any non-negative function  $g(\cdot)$  satisfying the condition (18) can be used to define a one-dimensional density  $cg(z^2)$  of a spherical distribution with the normalizing constant  $c$  satisfying (19). One often writes  $S(g)$  for the spherical distribution generated from the density generator  $g(\cdot)$ .

As yet another example, let us consider the elliptical Student- $t$  distribution  $E(\mu, \sigma^2, g)$ , with  $g(u) = \left(1 + \frac{u}{m}\right)^{-(m+1)/2}$ . We will denote this distribution (with  $m$  degrees of freedom) by  $t^{(m)}(\mu, \sigma^2)$ . Its density can be expressed as

$$f_X(x) = \frac{c}{\sigma} \left[ 1 + \frac{1}{m} \left( \frac{x - \mu}{\sigma} \right)^2 \right]^{-(n+m)/2}.$$

In order to determine the normalizing constant, first note from (19) that

$$c = \left[ \int_0^\infty z^{-1/2} g(z) dz \right]^{-1} = \left[ \int_0^\infty z^{-1/2} \left(1 + \frac{z}{m}\right)^{-(m+1)/2} dz \right]^{-1}.$$

Performing the substitution  $u = 1 + (z/m)$ , we find

$$\int_0^\infty z^{-1/2} \left(1 + \frac{z}{m}\right)^{-(m+1)/2} dz = m^{1/2} \int_1^\infty (1 - u^{-1})^{-1/2} u^{-m/2-1} du.$$

Making one more substitution  $v = 1 - u^{-1}$ , we get

$$\int_0^\infty z^{-1/2} \left(1 + \frac{z}{m}\right)^{-(m+1)/2} dz = m^{1/2} \frac{\Gamma(1/2) \Gamma(m/2)}{\Gamma((m+1)/2)},$$

from which we find

$$c = \frac{\Gamma((m+1)/2)}{(m\pi)^{1/2} \Gamma(m/2)}.$$

These results lead us to the familiar density of a univariate Student- $t$  random variable with  $m$  degrees of freedom

$$f_X(x) = \frac{\Gamma(\frac{m+1}{2})}{(m\pi)^{1/2} \Gamma(\frac{m}{2})} \frac{1}{\sigma} \left[ 1 + \frac{1}{m} \left( \frac{x - \mu}{\sigma} \right)^2 \right]^{-(m+1)/2}.$$

Its mean is

$$E(X) = \mu,$$

whereas it can be verified that its variance is given by

$$Var(X) = \frac{m}{m-2} \sigma^2,$$

provided the degrees of freedom  $m > 2$ . Note that  $\frac{m}{m-2} = -2\psi'(0)$ , where  $\psi$  is the characteristic generator of the family of Student- $t$  distributions with  $m$  degrees of freedom.

To derive the characteristic function of  $X$ , note that

$$\begin{aligned} E(e^{itX}) &= \frac{\Gamma(\frac{m+1}{2})}{(m\pi)^{1/2} \Gamma(\frac{m}{2})} e^{i\mu t} \int_{-\infty}^{\infty} e^{i\sigma tz} \left(1 + \frac{1}{m} z^2\right)^{-(m+1)/2} dz \\ &= 2 \frac{\Gamma(\frac{m+1}{2})}{(m\pi)^{1/2} \Gamma(\frac{m}{2})} \frac{e^{i\mu t}}{\sigma (m\sigma^2)^{-(m+1)/2}} \int_0^\infty \cos(tz) (m\sigma^2 + z^2)^{-(m+1)/2} dz. \end{aligned}$$

Hence, from Gradshteyn & Ryzhik (2000, p. 907), we find that the characteristic function of  $X \sim t^{(m)}(\mu, \sigma^2)$  is given by

$$E(e^{itX}) = e^{i\mu t} \frac{1}{2^{m/2-1} \Gamma(\frac{m}{2})} (t\sqrt{m}\sigma)^{m/2} K_{m/2}(t\sqrt{m}\sigma), \quad (22)$$

where  $K_\nu(\cdot)$  is the Bessel function of the second kind. For a similar derivation, see Witkovsky (2001). Observe that equation (22) can then be used to find the characteristic generator for the family of Student- $t$  distributions.

### 3.3 Log-Elliptical Distributions

Log-elliptical distributions are natural generalizations of log-normal distributions. Recall that a random variable  $X$  has a log-normal distribution if  $\log X$  has a normal distribution. In this case we have that  $\log X \sim N(\mu, \sigma^2)$ .

**Definition 2** *The random variable  $X$  is said to have a log-elliptical distribution with parameters  $\mu$  and  $\sigma^2$  if  $\log X$  has an elliptical distribution:*

$$\log X \sim E(\mu, \sigma^2, \psi).$$

In the sequel, we shall write  $\log X \sim E(\mu, \sigma^2, \psi)$  as  $X \sim LE(\mu, \sigma^2, \psi)$ . When  $\mu = 0$  and  $\sigma^2 = 1$ , we shall write  $X \sim LS(\psi)$ . Clearly, if  $Y \sim E(\mu, \sigma^2, \psi)$  and  $X = \exp(Y)$ , then  $X \sim LE(\mu, \sigma^2, \psi)$ .

If the density of  $X \sim LE(\mu, \sigma^2, \psi)$  exists, then the density of  $Y = \log X \sim E(\mu, \sigma^2, \psi)$  also exists. From (17), it follows that the density of  $X$  must be equal to

$$f_X(x) = \frac{c}{\sigma x} g \left[ \left( \frac{\log x - \mu}{\sigma} \right)^2 \right], \quad (23)$$

see Fang et al. (1990). If the mean of  $X$  exists, then it is given by

$$E(X) = e^\mu \cdot \psi(-\sigma^2).$$

For proof of this, see Valdez and Dhaene (2003).

## 4 The ITCE for Elliptical and Log-Elliptical Distribution Loss Models

In this section, we develop the iterated tail conditional expectation formulas for elliptical distribution models in the additive risk process and later for the log-elliptical distribution models suitable for the multiplicative risk process. But, first we state, without proof as the proofs are readily available in Landsman and Valdez (2003) and Valdez and Dhaene (2003), the result of the static tail conditional expectations (TCE) for both the univariate elliptical and the log-elliptical distributions. Recall that we denote by  $x_q$  the  $q$ -th quantile of the loss distribution  $F_X(x)$ . It can be conveniently assumed that  $q > 1/2$  so that clearly

$$x_q > \mu.$$

This is because we are generally interested in the tails of symmetric distributions. Now suppose  $g(\cdot)$  is a non-negative function on  $[0, \infty)$  satisfying the condition that

$$\int_0^\infty z^{-1/2} g(z) dz < \infty.$$

Then the function  $g$  can be a density generator of a univariate elliptical distribution of a random variable  $X \sim E(\mu, \sigma^2, g)$  whose density can be expressed as

$$f_X(x) = \frac{c}{\sigma} g \left[ \left( \frac{x - \mu}{\sigma} \right)^2 \right] \quad (24)$$

where  $c$  is the normalizing constant.

Note that because  $X$  has an elliptical distribution, the standardized random variable  $Z = (X - \mu) / \sigma$  will have a standard elliptical, or oftentimes called spherical, distribution function

$$F_Z(z) = c \int_{-\infty}^z g(u^2) du,$$

with mean 0 and variance

$$\sigma_Z^2 = 2c \int_0^{\infty} u^2 g(u^2) du = -2\psi'(0),$$

if condition (15) holds. Furthermore, if the generator of the elliptical family is chosen such that  $\psi'(0) = -1/2$  holds, then  $\sigma_Z^2 = 1$ .

Now, define the function

$$G(x) = c \int_0^x g(z) dz \quad (25)$$

which has been called the *cumulative generator* in Landsman and Valdez (2003). This function  $G$  plays an important role in the tail conditional expectations formula for the class of elliptical distributions. Furthermore, denote by

$$G(\infty) = \lim_{x \rightarrow \infty} G(x)$$

and

$$\bar{G}(x) = G(\infty) - G(x).$$

The condition

$$\int_0^{\infty} g(z) dz < \infty$$

guarantees the existence of the expectation and can equivalently be expressed as

$$G(\infty) < \infty.$$

**Theorem 3** *Let  $X \sim E(\mu, \sigma^2, g)$  and  $G$  be the cumulative generator defined in (25). Under the condition that*

$$\int_0^{\infty} g(z) dz < \infty,$$

the tail conditional expectation of  $X$  is given by

$$TCE_q(X) = \mu + \lambda \cdot \sigma^2 \quad (26)$$

where  $\lambda$  is expressed as

$$\lambda = \frac{1}{2\sigma} \cdot \frac{\overline{G}(z_q^2)}{\overline{F}_X(x_q)} = \frac{1}{2\sigma} \cdot \frac{\overline{G}(z_q^2)}{\overline{F}_Z(z_q)} \quad (27)$$

and  $z_q = (x_q - \mu) / \sigma$ . Moreover, if the variance of  $X$  exists, or equivalently if (15) holds, then  $\frac{1}{2\sigma^2} \overline{G}(z^2)$  is a density of another spherical random variable  $Z^*$  and  $\lambda$  has the form

$$\lambda = \frac{\sigma_Z^2}{\sigma} \cdot \frac{f_{Z^*}(z_q)}{\overline{F}_Z(z_q)}. \quad (28)$$

**Proof.** See Landsman and Valdez (2003). ■

It is clear that (26) generalizes the tail conditional expectation formula derived by Panjer (2002) for the class of normal distributions to the larger class of univariate symmetric distributions. For the normal distribution, we exactly replicate the formula developed by Panjer (2002) stated in (2).

The next theorem gives the TCE formula for log-elliptical distributions.

**Theorem 4** *Let  $X \sim LE(\mu, \sigma^2, \psi)$  and  $Z \sim S(\psi)$  with density  $f_Z(\cdot)$  and distribution function  $F_Z(\cdot)$ , then the tail conditional expectation of  $X$  can be expressed as*

$$TCE_q(X) = \frac{e^\mu}{1-q} \cdot \psi(-\sigma^2) \cdot F_{Z^*}(F_Z^{-1}(1-q)), \quad \text{for any } 0 < q < 1,$$

where the density of  $Z^*$  is given by

$$f_{Z^*}(x) = \frac{e^{\sigma x}}{\psi(-\sigma^2)} \cdot f_Z(x).$$

**Proof.** See Valdez and Dhaene (2003). ■

Note that the density of  $Z^*$  in Theorem 4 above can be interpreted as the Esscher transform with parameter  $\sigma$  of  $Z$ . We also have that if  $g$  is the normalized density generator of  $Z \sim S(\psi)$ , then

$$F_Z(x) = \int_{-\infty}^x g(z^2) dz.$$

In the case of the log-normal distribution, we know that the characteristic generator is  $\psi(u) = \exp(-u/2)$ . Thus, using the result in the Theorem above for computing the TCE of a log-normal random variable, it is easy to verify that

$$TCE_q(X) = e^{(\mu + \sigma^2/2)} \cdot \frac{\Phi(\sigma - z_q)}{1-q}.$$



**Example 5.1 Additive Risk Process** Consider the additive risk process discussed in Section 2. Suppose  $\Delta X_t$  are identically distributed random variables with elliptical  $E(\mu, \sigma^2, \psi)$  distribution and again, consider the TCE as the risk measure. Therefore, from the result on this section, we find the TCE

$$\rho_S(\Delta X_t) = \mu + \frac{\overline{G}(z_q^2)}{\overline{F}_Z(z_q)} \cdot \sigma = \mu + \frac{\overline{G}(z_q^2)}{1-q} \cdot \sigma$$

and we have an expression for the iterated TCE:

$$\begin{aligned} \rho(X_T | X_k) &= ITCE_q(X_T | X_k) \\ &= e^{-(T-k)\delta} X_k + \left( \sum_{i=0}^{T-k-1} e^{-i\delta} \right) \left[ \mu + \frac{\overline{G}(z_q^2)}{1-q} \cdot \sigma \right] \\ &= e^{-(T-k)\delta} X_k + \ddot{a}_{\overline{T-k}|\delta} \left[ \mu + \frac{\overline{G}(z_q^2)}{1-q} \cdot \sigma \right], \end{aligned}$$

where  $\ddot{a}_{\overline{T-k}|\delta}$  denotes the symbol for the annuity payout as defined in section 2. Observe that when the force of interest  $\delta$  is zero, then at  $k = 0$ , the iterated TCE becomes

$$ITCE_q(X_T | X_0) = T\mu + \frac{\overline{G}(z_q^2)}{1-q} \cdot T\sigma.$$

For the case of the static TCE, first observe that

$$X_T = \sum_{t=1}^T \Delta X_t.$$

If the random vector  $(\Delta X_1, \Delta X_2, \dots, \Delta X_T)$  has a joint elliptical distribution, it can be readily shown that  $X_T$  would have an elliptical  $E(T\mu, T\sigma^2, \psi)$  distribution. We would have, in this case,

$$\rho_S(X_T) = T\mu + \frac{\overline{G}(z_q^2)}{1-q} \cdot \sqrt{T}\sigma.$$

Note that we have not discussed the multivariate version of the class of elliptical distributions in this paper. Statistical properties of the multivariate version are readily found in most of the references on elliptical distributions cited in the paper including, for example, Fang, et al. (1990).

**Example 5.2 Multiplicative Risk Process** Consider the multiplicative risk process discussed in Section 2. Suppose that  $\Delta X_t$  are identically distributed log-elliptical  $LE(\mu, \sigma^2, \psi)$  random variables and also consider the TCE as the risk measure. Thus, from the result on this section, we have the TCE

$$\rho_S(\Delta X_t) = \frac{e^\mu}{1-q} \cdot \psi(-\sigma^2) \cdot F_{Z^*}(F_Z^{-1}(1-q))$$

and we have an expression for the iterated TCE:

$$\begin{aligned}\rho(X_T | X_k) &= ITCE_q(X_T | X_k) \\ &= e^{-(T-k)\delta} \cdot e^{(T-k)\mu} X_k \cdot \left[ \psi(-\sigma^2) \frac{F_{Z^*}(F_Z^{-1}(1-q))}{1-q} \right]^{T-k}.\end{aligned}\quad (29)$$

For the special case where  $\delta = 0$  and  $k = 0$ , then (29) becomes

$$ITCE_q(X_T | X_0) = e^{T\mu} \cdot \left[ \psi(-\sigma^2) \frac{F_{Z^*}(F_Z^{-1}(1-q))}{1-q} \right]^T.$$

Observe that

$$X_T = \prod_{t=0}^T \Delta X_t$$

so that clearly

$$\log X_T = \log \left( \prod_{t=0}^T \Delta X_t \right) = \sum_{t=1}^T \log \Delta X_t$$

which will be log-elliptical provided as explained previously that the random vector  $(\log \Delta X_1, \log \Delta X_2, \dots, \log \Delta X_T)$  has a joint elliptical distribution. In this instance, because  $\log(\Delta X_t) \sim E(\mu, \sigma^2, \psi)$ , then  $\log X_T \sim E(T\mu, T\sigma^2, \psi)$ . The static TCE in this case is seen to be

$$\rho_S(X_T) = e^{T\mu} \cdot \left[ \psi(-T\sigma^2) \frac{F_{Z^*}(F_Z^{-1}(1-q))}{1-q} \right] \quad (30)$$

and therefore the two resulting TCE's in (29) and (30) are different. Here, the density of  $Z^*$  is given by

$$f_{Z^*}(x) = \frac{e^{\sqrt{T}\sigma x}}{\psi(-T\sigma^2)} \cdot f_Z(x).$$

## 5 The Case of the Log-Laplace Distribution

As an illustration, let us consider the Laplace distribution whose density has the form

$$f_X(x) = \frac{1}{2\sigma} \exp[-|x - \mu|/\sigma] \text{ for } -\infty < x < \infty. \quad (31)$$

We shall write  $X \sim \text{Laplace}(\mu, \sigma)$ . See also Johnson, Kotz, and Balakrishnan (1995) for this special type of symmetric distribution. It is straightforward to show that the moment generating function (m.g.f.) of  $X$  can be written as

$$\begin{aligned}E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{2\sigma} \exp[-|x - \mu|/\sigma] dx \\ &= \frac{e^{\mu}}{(1 - \sigma t)(1 + \sigma t)}.\end{aligned}$$

Now, consider the case where  $Y = \log X$  has a Laplace distribution so that  $X$  has a log-Laplace distribution. The density of  $X$  can therefore be written as

$$f_X(x) = \frac{1}{2\sigma x} \exp[-|\log x - \mu|/\sigma] \text{ for } 0 < x < \infty. \quad (32)$$

It is rather straightforward to show that the quantile function of the standard Laplace can be expressed as

$$z_q = \begin{cases} \log(2q), & \text{for } 0 < q \leq 1/2 \\ \log[2(1-q)]^{-1}, & \text{for } 1/2 < q < 1 \end{cases}.$$

Thus, its TCE can be expressed as

$$TCE_q(X) = \frac{1}{1-q} \int_{x_q}^{\infty} \frac{1}{2\sigma} \exp[-|\log x - \mu|/\sigma] dx$$

With transformation  $z = (\log x - \mu)/\sigma$ , this TCE becomes

$$TCE_q(X) = \frac{e^\mu}{2(1-q)} \int_{z_q}^{\infty} e^{\sigma z} \cdot \exp[-|z|] dz.$$

In the case where  $0 < q \leq 1/2$  so that  $z_q \leq 0$ , it can be shown that

$$TCE_q(X) = \frac{e^\mu}{2(1-q)} \cdot \left[ \frac{2}{(1+\sigma)(1-\sigma)} - \frac{e^{(1+\sigma)z_q}}{1+\sigma} \right]$$

and in the case where  $1/2 < q < 1$  so that  $z_q > 0$ , it can be shown that

$$TCE_q(X) = \frac{e^\mu}{2(1-q)} \cdot \frac{e^{-(1-\sigma)z_q}}{1-\sigma}.$$

In both instances, the TCE exists provided  $\sigma < 1$  because otherwise the integration blows up to infinity and the TCE does not exist.

Consider next the multiplicative risk process and for the special case where  $\delta = 0$  and  $k = 0$ , then (29) becomes

$$\rho(X_T | X_0) = \left[ \frac{e^\mu}{2(1-q)} \right]^T \times \begin{cases} \left[ \frac{2}{(1+\sigma)(1-\sigma)} - \frac{e^{(1+\sigma)z_q}}{1+\sigma} \right]^T, & \text{for } 0 < q \leq 1/2 \\ \left[ \frac{e^{-(1-\sigma)z_q}}{1-\sigma} \right]^T, & \text{for } 1/2 < q < 1 \end{cases}. \quad (33)$$

Since

$$X_T = \prod_{t=1}^T \Delta X_t,$$

then  $X_T$  is also log-Laplace in this case. This follows immediately from the fact that  $\log(\Delta X_t) \sim \text{Laplace}(\mu, \sigma)$ . and therefore  $\log X_T \sim \text{Laplace}(T\mu, \sqrt{T}\sigma)$ . The static TCE in this case becomes

$$\rho_S(X_T) = \frac{e^{T\mu}}{2(1-q)} \times \begin{cases} \left[ \frac{2}{(1+\sqrt{T}\sigma)(1-\sqrt{T}\sigma)} - \frac{e^{(1+\sqrt{T}\sigma)z_q}}{1+\sqrt{T}\sigma} \right], & \text{for } 0 < q \leq 1/2 \\ \frac{e^{-(1-\sqrt{T}\sigma)z_q}}{1-\sqrt{T}\sigma}, & \text{for } 1/2 < q < 1 \end{cases} \quad (34)$$

and therefore the two resulting TCE's in (33) and (34) are again different.

For numerical illustration, we consider the case where  $\mu = 1/2$  and  $\sigma = 1/20$  and  $T = 10$  years. The difference between the dynamic and the static TCE is shown in Figures 1. In Figure 1, we show the dynamic TCE and the static TCE in the case of the log-Laplace distribution, for various levels of  $q$  and in Figure 2, we show this same comparison in the case of the log-normal distribution. Here the location and scale parameters are the same for both distributions:  $\mu = 1/2$  and  $\sigma = 1/20$ . The time horizon is also  $T = 10$  years in the case of the log-normal distribution. Table 1 also gives the numerical comparison between the two distributions. The results are not at all surprising. The log-Laplace generally exhibits fatter tail in the distribution than the log-normal. Thus, the TCE's for the log-Laplace are generally larger than those of the log-normal, as can be observed from the figures as well as the Table below.

Table 1  
The Iterated TCE and the Static TCE between  
the Log-Normal and the Log-Laplace Distributions<sup>1</sup>

$q$	Log-Normal			Log-Laplace		
	$z_q$	$STCE$	$ITCE$	$z_q$	$STCE$	$ITCE$
0.01	-2.32635	150.81	152.20	-3.91202	153.06	155.57
0.05	-1.64485	152.54	158.29	-2.30259	155.54	164.91
0.10	-1.28155	154.46	165.08	-1.60944	158.09	174.39
0.50	0.00000	169.16	222.19	0.00000	176.29	247.88
0.90	1.28155	196.30	357.68	1.60944	227.37	554.27
0.95	1.64485	206.01	417.01	2.30259	253.71	783.86
0.99	2.32635	226.48	563.30	3.91202	327.23	1,752.76

<sup>1</sup>Note:  $z_q$ =quantiles;  $STCE$ =static TCE;  $ITCE$ =iterated TCE.

## 6 Concluding Remarks

The tail conditional expectation has long been considered an important risk measure but it is generally not dynamic. Recently, there has been numerous discussion in the literature about dynamic risk measures because liabilities of insurance companies, or many other financial institutions are long-term in nature. See Reidel (2003) and Wang (2000), for example. In a static approach to risk measurement, companies determine today a fixed amount of economic capital and hold that fixed amount at every valuation date in the future regardless of the history of the process that could have evolved over time. As stated in the introduction, this can possibly lead to either holding excessive or deficient capital at some time in the future. This is because, on one hand, it is possible that the risk evolves in the company's favor so that excessive capital may then be held. On the other hand, it is possible that the risk move against the favor of the company so that there would be inadequate amount being held. It is therefore important to capture the dynamic nature of the risk process so that a risk measure must be itself dynamic to the changing nature of the risk process according to time. A dynamic risk measure allows one to update the value of the risk measure as additional information or history becomes available. In this paper, we examine the iterated tail conditional expectation which is obtained by repeated calculation of the tail conditional expectation (or any other applicable static risk measure, for that matter) through backwards induction, a method suggested by Hardy and Wirch (2003). Any other applicable static risk measure can be used in lieu of the TCE. We are able to extend the results in Hardy and Wirch (2003) in the case where risks follow either the elliptical distributions or the log-elliptical distributions.

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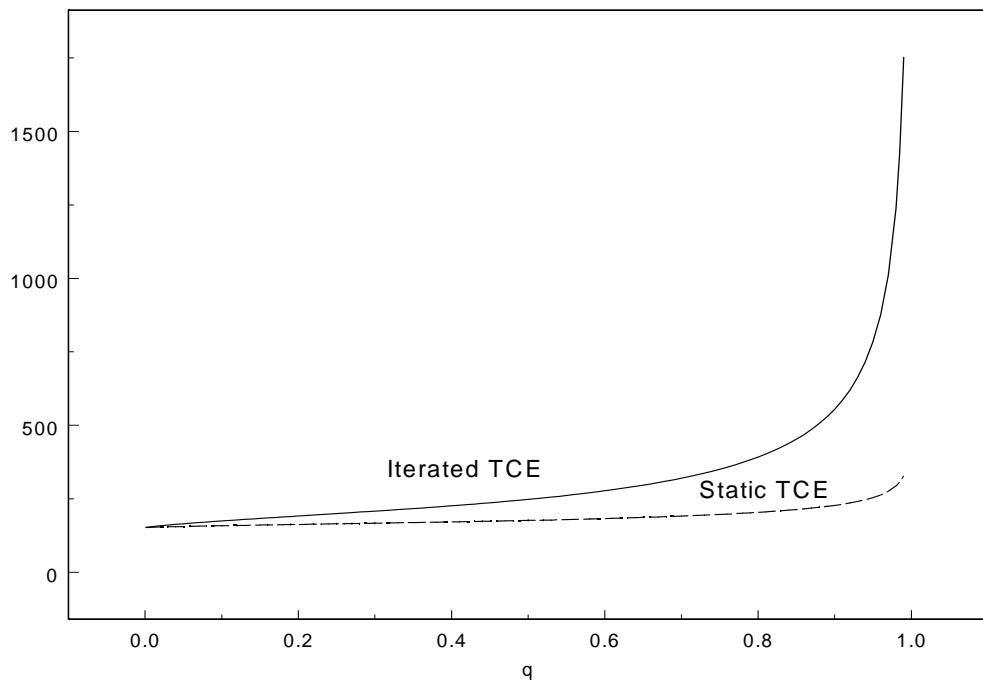


Figure 1: Iterated TCE and the Static TCE for the log-Laplace Distribution.

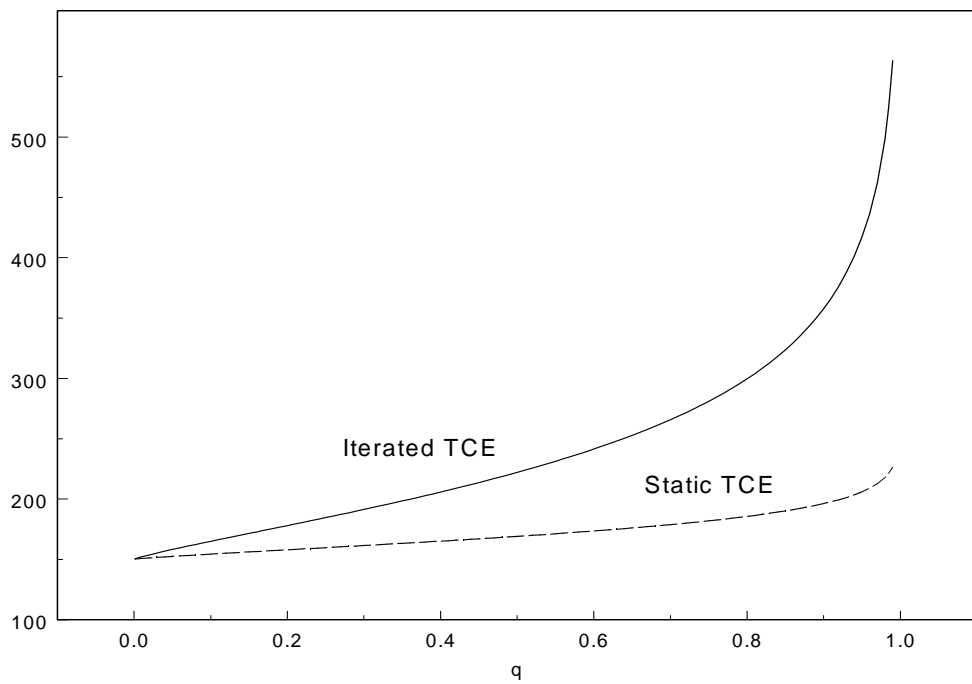


Figure 2: Iterated TCE and the Static TCE for the Log-Normal Distribution.