On Retirement Income Replacement Ratios

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ABSTRACT. Individuals all over the world are finding it more important to plan for retirement because of increased longevity and higher cost of living. Retirement income replacement ratio can be an effective tool for analyzing retirement needs. It is defined to be the amount of retirement income expressed as a percentage of the final salary just prior to retirement. This paper develops formulas for replacement ratios showing how they can be affected by factors such as wage increases, returns on investments, pre-retirement and post-retirement mortality, and inflation and allowing for these components to be modelled stochastically. To numerically illustrate the concepts presented in the paper, we fitted time series models to Australian data published by the Australian Bureau of Statistics in order to obtain estimates and a distribution of the replacement ratio. The results offer insight into future financial planning for retirement and the adequacy of retirement preparation as a whole.

1. Introduction. Retirement planning is a complicated process and yet is a fundamental and important part of everyone's life, especially in today's world where we are witnessing unprecendeted longevity. First, there are several aspects or dimensions that go with retirement planning. Together with this, there are many variables that altogether contribute to the uncertainty. For example, an individual's level of income is partly difficult to predict because wage increases are never certain nor guaranteed. There may even be wage decreases in periods where the individual may be faced with the problem of unemployment or even underemployment. Second, the process of planning for retirement is a long and continuous process that takes several years of patience and skills to be able to thoroughly manage retirement. Even at retirement, the process of being able to ensure that finances will flow as originally expected is an extension to the entire planning process.

It is important to have a device or tool that can singularly assist in the process for retirement. Part of the goal for retirement is to be able to maintain a long term standard of living that extends well beyond retirement. That level of standard of living is usually a function of the individual's level of income just prior to retirement. It is therefore natural to assume that the required level of post-retirement income would be related to the level of income just prior to retirement. In this paper, we analyze and examine the usefulness of the replacement ratio defined to be the ratio of annual post-retirement income derived from savings and other sources of income, to final salary. In effect, we have

$$RR(x,r) = \frac{I(x,r)}{S(x,r)} \tag{1}$$

where $RR(x,r)$ is the retirement income replacement for an individual currently age x who expects to retire at age r, $r > x$, and $I(x,r)$ is his post-retirement annual income and $S(x,r)$ is his projected salary just prior to retirement. This replacement ratio has been examined in Lian, Valdez, and Low (2000) where they analyzed the factors affecting the ratio on a deterministic framework, that is, for example, interest rates as well as wage rate increases were assumed to be constant. This paper provides an extension to that analysis by relaxing this deterministic framework and therefore considering the factors like interest rates and wage rate increases to be stochastic therefore making the results and analysis more realistic. While there is no definite 'rule of thumb' for what a suitable replacement ratio is at retirement, individuals will vary according to their needs and the lifestyle prior to retirement. Some will argue that the replacement ratio should be smaller than 100% because come retirement, there are no longer expenses associated with employment such as regular transportation and clothes. On the other hand, one can argue that there may be a need to have more than the usual salary because there is increase in health problems for example associated with old age. This paper does not make any recommendation as to the level of replacement ratio suitable for retirement because this certainly varies widely according to the individual. Instead, this paper provides recommending the use of the retirement income replacement ratio as a tool for retirement planning and hereby provides a framework for analyzing the various factors that can affect it.

2. A Simple Calculation of the Replacement Ratio. Suppose that for a person currently age x who is expected to retire at age r will have accumulated a total savings of say $TS(x,r)$ which can consist of accumulation of personal past personal savings and future personal savings. Assume that he or she saves a proportion s_k of salary in period k and all savings are invested at the rate of i_k in the same period. Assuming yearly wage rate increases[1] are w_1 in the first year, w_2 in the second year, and so on, we have the person's total savings at retirement:

$$\begin{aligned} TS(x,r) &= PS_x \cdot \prod_{m=1}^{r-x} (1+i_m) + AS_x \cdot \sum_{k=0}^{r-x-1} s_k \left[\prod_{l=0}^{k} (1+w_l) \right] \left[\prod_{m=k+1}^{r-x} (1+i_m) \right] \\ &= PS(x,r) + FS(x,r) \end{aligned}$$

where the first term $PS(x,r)$ can be thought of as the accumulation of past savings to retirement and the second term $FS(x,r)$ the accumulation of future savings arising

[1] For purposes of our notation, it is obvious that $w_0 = 0$.

from a proportion of future salaries. Writing the interest accumulation factor as

$$A(n;i) = (1+i_1)(1+i_2)\cdots(1+i_n) = \prod_{m=1}^{n}(1+i_m),$$

the wage rate increase factor as

$$A(n;w) = (1+w_0)(1+w_1)\cdots(1+w_n) = \prod_{l=0}^{n}(1+w_l),$$

and the savings, with interest, accumulation factor as

$$\begin{aligned}
\sum_{k=0}^{n-1} s_k \left[\prod_{l=0}^{k}(1+w_l)\right]\left[\prod_{m=k+1}^{r-x}(1+i_m)\right] &= \sum_{k=0}^{n-1} s_k \cdot A(k;w)\left[\frac{\prod_{m=1}^{r-x}(1+i_m)}{\prod_{m=1}^{k}(1+i_m)}\right] \\
&= \sum_{k=0}^{n-1} s_k \cdot A(k;w)\cdot\frac{A(r-x;i)}{A(k;i)} \\
&= \left[\sum_{k=0}^{n-1} s_k \cdot \frac{A(k;w)}{A(k;i)}\right] A(r-x;i) \\
&= G(n;s,i,w)\cdot A(r-x;i)
\end{aligned}$$

then we can write the total savings at retirement as

$$\begin{aligned}
TS(x,r) &= PS_x \cdot A(r-x;i) + AS_x \cdot G(r-x;s,i,w)\,.\cdot A(r-x;i) \\
&= \left[PS_x + AS_x \cdot G(r-x;s,i,w)\right] A(r-x;i) \qquad (2)
\end{aligned}$$

Here, we are denoting

$$G(n;s,i,w) = \sum_{k=0}^{n-1} s_k \cdot \frac{A(k;w)}{A(k;i)}.$$

The primary advantage of breaking the total savings at retirement into the components of past and future savings is that an individual can at any time during his lifetime re-calculate replacement ratio. Equation (2) therefore has the interpretation that total savings at retirement consist of interest accumulation of current past savings and savings made from future income.

The amount $TS(x,r)$ in equation (2) will be used at retirement age r to buy a life annuity with constant annual payments of $I(x,r)$ which is clearly given by

$$I(x,r) = \frac{TS(x,r)}{\ddot{a}_r}$$

where the denominator denotes the actuarial present value of a life annuity due that pays one dollar (or a unit) at the beginning of each year for the rest of the retiree's life.[2] Milevsky (2001) offers an analysis of other possible ways of annuitizing retirement savings to receive optimal benefits. The retirement income replacement ratio at retirement age r then for a person who is currently age x can be expressed as

$$RR(x,r) = \frac{I(x,r)}{S(x,r)} = \frac{TS(x,r)/\ddot{a}_r}{AS_x \cdot A(r-x;w)}. \tag{3}$$

Manipulating equation (3), we can see that

$$\begin{aligned} RR(x,r) &= \frac{PS_x \cdot A(r-x;i) + AS_x \cdot G(r-x;s,i,w)}{AS_x \cdot A(r-x;w)\,\ddot{a}_r} \\ &= \frac{PS_x \cdot A(r-x;i)}{AS_x \cdot A(r-x;w)\,\ddot{a}_r} + \frac{AS_x \cdot G(r-x;s,i,w) \cdot A(r-x;i)}{AS_x \cdot A(r-x;w)\,\ddot{a}_r}. \end{aligned}$$

Thus, we see that the retirement income replacement ration can be written as

$$RR(x,r) = \left[\frac{PS_x}{AS_x} + G(r-x;s,i,w)\right] \times \frac{A(r-x;i)}{A(r-x;w)} \times \frac{1}{\ddot{a}_r}. \tag{4}$$

All the terms in equation (4) have an interpretation. The ratio is intuitively interpreted as the sum of two components: (1) the current value of past personal savings expressed as a percentage of current salary, $\frac{PS_x}{AS_x}$, and (2) the accumulation of future savings rate (already expressed as a percentage of salary) discounted with interest today, $G(r-x;s,i,w)$. The sum of these two components are therefore accumulated with inflation-adjusted (inflation is measured with wage rate increases) interest rates, $\frac{A(r-x;i)}{A(r-x;w)}$, at retirement, and are spread evenly across the future lifetime of the individual, $\frac{1}{\ddot{a}_r}$. We shall sometimes denote the proportion of past savings to current salary as

$$c_x = \frac{PS_x}{AS_x}. \tag{5}$$

[2] In short, this is equal to

$$\ddot{a}_r = \sum_{t=0}^{\infty} (1+j)^{-t} \; {}_tp_r$$

assuming a yearly constant interest rate j. See Bowers, et al. (1997).

3. The Deterministic Framework. It is clear that the retirement income replacement ratio so developed in the previous section consists of some uncertain quantities and is therefore truly stochastic or random. To simplify initial analysis and to gain a little bit more understanding of how factors affect this replacement ratio, it may be necessary to develop the ratio in a deterministic framework. Most of the analysis done in this section are more comprehensively described in Lian, et al. (2000). Please see the article for further details.

Assume for the moment that interest rates, wage rate increases and savings rate are all constant in the future. In effect, we are assuming that

$$i_m = i \text{ for } m = 1, 2, ...$$

$$w_l = w \text{ for } l = 1, 2, ...$$

and

$$s_k = s \text{ for } k = 1, 2, ...$$

First, we note that

$$\frac{A\,(r-x;i)}{A\,(r-x;w)} = \frac{\prod_{m=1}^{r-x}(1+i_m)}{\prod_{l=0}^{r-x}(1+w_l)} = \left(\frac{1+i}{1+w}\right)^{r-x}$$

and that

$$G\,(r-x;s,i,w) = \sum_{k=0}^{r-x-1} s_k \cdot \frac{A\,(k;w)}{A\,(k;i)} = s \cdot \sum_{k=0}^{r-x-1} \left(\frac{1+w}{1+i}\right)^k .$$

Application of fundamental result in elementary theory of interest will lead us to

$$G\,(r-x;s,i,w) = s \cdot \begin{cases} \ddot{a}_{\overline{r-x}|j_1} & \text{if } i > w \\ (r-x) & \text{if } i = w \\ \ddot{s}_{\overline{r-x}|j_2} & \text{if } i < w \end{cases} \tag{6}$$

where

$$j_1 = \left(\frac{1+i}{1+w}\right) - 1$$

and

$$j_2 = \left(\frac{1+w}{1+i}\right) - 1.$$

The replacement ratio reduces to

$$RR\left(x,r\right)=\left[c_x+G\left(r-x;s,i,w\right)\right]\times\left(\frac{1+i}{1+w}\right)^{r-x}\times\frac{1}{\ddot{a}_r}.$$

It is interesting to note that in the case where $i=w$, that is, interest rates are equal to wage rate increases, the replacement ratio further simplifies to

$$RR\left(x,r\right)=\left[c_x+s\left(r-x\right)\right]\cdot\frac{1}{\ddot{a}_r}$$

and becomes independent of interest rates and wage rate increases. In this case, the replacement ratio will be affected only by how much has already been saved at age x and the rate of savings from income to be made in the future. If we hold savings s and wage increases w fixed or constant, it can be shown that

$$\frac{\partial RR}{\partial i}\geq 0.$$

See Lian, et al. (2000) for a proof. This is intuitively appealing result because if our savings are earning higher interest rates, we would therefore expect our replacement ratio to increase. This is true even so if the rate of wage increases is larger than interest rate.

On the other hand, if we hold savings s and interest rate i constant, the result may not be quite as intuitively appealing. First re-write

$$RR\left(x,r\right)=\left[\frac{c_x}{\left(1+w\right)^{r-x}}+\frac{G\left(r-x;s,i,w\right)}{\left(1+w\right)^{r-x}}\right]\cdot\frac{\left(1+i\right)^{r-x}}{\ddot{a}_r}$$

and note that

$$\begin{aligned}
\frac{\partial}{\partial w}\left(\frac{G\left(r-x;s,i,w\right)}{\left(1+w\right)^{r-x}}\right) &= s\left(1+i\right)^{r-x}\cdot\frac{\partial}{\partial w}\left[\sum_{k=0}^{r-x-1}\left(\frac{1+w}{1+i}\right)^{k-r+x}\right] \\
&= s\left(1+i\right)^{r-x}\cdot\frac{\partial}{\partial w}\left[\sum_{k=1}^{r-x}\left(\frac{1+i}{1+w}\right)^{k}\right] \\
&= -s\frac{\left(1+i\right)^{r-x}}{1+w}\cdot\sum_{k=1}^{r-x}k\left(\frac{1+i}{1+w}\right)^{k} \\
&= -s\frac{\left(1+i\right)^{r-x}}{1+w}\cdot\begin{cases}\left(Is\right)_{\overline{r-x}|j_1} & \text{if } i>w \\ \frac{1}{2}\left(r-x\right)\left(r-x+1\right) & \text{if } i=w \\ \left(Ia\right)_{\overline{r-x}|j_2} & \text{if } i<w\end{cases}
\end{aligned}$$

where it is clear that $\dfrac{\partial}{\partial w}\left(\dfrac{G\left(r-x;s,i,w\right)}{\left(1+w\right)^{r-x}}\right)\leq 0$ and that

$$\frac{\partial}{\partial w}\left(\frac{c_x}{\left(1+w\right)^{r-x}}\right)=\frac{-c_x\left(r-x\right)}{\left(1+w\right)^{r-x+1}}\leq 0.$$

The result

$$\frac{\partial RR}{\partial i}\leq 0$$

clearly follows. In Figure 1, we show these relationships: the replacement ratio is an increasing function of interest rate and is a decreasing function of wage rate increases. Here in this figure we are assuming the following: $x=25$, $r=65$, a life annuity factor based in the British A1967-70 ultimate Mortality Table, current salary and savings-to-date of $20,000$. For the figure of replacement ratio as a function of interest rate, we held the rate of wage increases at $w=5\%$; for the figure of replacement ratio against rate of wage increases, interest rate i was held fixed at 4%.

[**Insert Figure 1 here.**]

4. Decomposing Change in Replacement Ratio. In this section, we develop a recursive formula for calculating replacement ratios. However, beginning with the previous year at age x, if all previous assumptions were materialized in the year, then the total savings anticipated at retirement age r should be the same as when calculated at age $x+1$. Begin with equation (2) applied to age $x+1$:

$$\begin{aligned}
& TS\left(x+1,r\right) \\
= \;& PS_{x+1}\cdot A\left(r-x-1;i\right)+AS_{x+1}\cdot G\left(r-x-1;s,i,w\right).\cdot A\left(r-x-1;i\right) \\
= \;& \left[PS_x\left(1+i_1\right)+AS_x\cdot s_0\left(1+i_1\right)\right]\cdot\frac{A\left(r-x;i\right)}{\left(1+i_1\right)} \\
& +AS_x\left(1+w_1\right)\cdot\left[\frac{G\left(r-x;s,i,w\right)-s_0}{\left(1+w_1\right)}\cdot A\left(r-x;i\right)\right] \\
= \;& PS_x\cdot A\left(r-x;i\right)+AS_x\cdot G\left(r-x;s,i,w\right).\cdot A\left(r-x;i\right) \\
= \;& TS\left(x,r\right).
\end{aligned}$$

The replacement ratio should therefore remain unchanged:

$$\begin{aligned}
RR\,(x+1,r) &= \frac{TS\,(x+1,r)}{AS_{x+1}\cdot A\,(r-x-1;w)}\times\frac{1}{\ddot{a}_r}\\
&= \frac{TS\,(x,r)}{AS_x\cdot(1+w_1)\cdot\left[\dfrac{A\,(r-x;w)}{(1+w_1)}\right]}\times\frac{1}{\ddot{a}_r}\\
&= \frac{TS\,(x,r)}{AS_x\cdot A\,(r-x;w)}\times\frac{1}{\ddot{a}_r}\\
&= RR\,(x,r)\,.
\end{aligned}$$

However, in reality, the assumptions generally do not materialize and therefore things do not remain the same. Let us examine the case only in the first year,or the period from age x to $x+1$. For purposes of our notation, we shall denote the "new" terms with a prime. To simplify the illustration, let us just consider the case where the interest rate in the first year was $i_1^{'}$ and the wage rate increase in the first year was $w_1^{'}$ so that the new total accumulation of savings at retirement becomes

$$\begin{aligned}
TS^{'}\,(x+1,r) &= \left[PS_x\left(1+i_1^{'}\right)+AS_x\cdot s_0\left(1+i_1^{'}\right)\right]\cdot\frac{A\,(r-x;i)}{(1+i_1)}\\
&\quad +AS_x\left(1+w_1^{'}\right)\left[\frac{G\,(r-x;s,i,w)-s_0}{(1+w_1)}\cdot A\,(r-x;i)\right].
\end{aligned}$$

Manipulating this equation, we see that

$$\begin{aligned}
& TS^{'}\,(x+1,r)\\
= & \left[PS_x\,(1+i_1)+AS_x\cdot s_0\,(1+i_1)\right]\cdot\frac{A\,(r-x;i)}{(1+i_1)}\\
& +AS_x\,(1+w_1)\left[\frac{G\,(r-x;s,i,w)-s_0}{(1+w_1)}\cdot A\,(r-x;i)\right]\\
& +\left(i_1^{'}-i_1\right)\cdot(PS_x+AS_x\cdot s_0)\,\frac{A\,(r-x;i)}{(1+i_1)}\\
& +\left(w_1^{'}-w_1\right)\cdot AS_x\left[\frac{G\,(r-x;s,i,w)-s_0}{(1+w_1)}\cdot A\,(r-x;i)\right].\\
= & \; TS\,(x,r)+\left(i_1^{'}-i_1\right)\cdot(PS_x+AS_x\cdot s_0)\,\frac{A\,(r-x;i)}{(1+i_1)}\\
& +\left(w_1^{'}-w_1\right)\cdot AS_x\left[\frac{G\,(r-x;s,i,w)-s_0}{(1+w_1)}\cdot A\,(r-x;i)\right].
\end{aligned}$$

Therefore, the one-period change in the total savings accumulation at retirement will be due to a change in interest rate

$$\left(i_1^{'}-i_1\right)\cdot\left(PS_x+AS_x\cdot s_0\right)\frac{A\left(r-x;i\right)}{\left(1+i_1\right)} \tag{7}$$

and a change in the rate of wage increase

$$\left(w_1^{'}-w_1\right)\cdot AS_x\left[\frac{G\left(r-x;s,i,w\right)-s_0}{\left(1+w_1\right)}\cdot A\left(r-x;i\right)\right]. \tag{8}$$

We shall denote the term in (7) as

$$\Delta TS^{i}\left(x,r\right)$$

the change attributable to a change in interest rate,and the term in (8) as

$$\Delta TS^{w}\left(x,r\right)$$

the change attributable to rate of wage increase. Because of the change in the total savings accumulated, the replacement ratio will also change. The "new" replacement ratio is now computed as

$$\begin{aligned}
& RR^{'}\left(x+1,r\right)\\
= \;& \frac{TS^{'}\left(x+1,r\right)}{AS_{x+1}^{'}\cdot A\left(r-x-1;w\right)}\times\frac{1}{\ddot{a}_r}\\
= \;& \frac{TS\left(x,r\right)+\Delta TS^{i}\left(x,r\right)+\Delta TS^{w}\left(x,r\right)}{AS_x\cdot\left(1+w_1^{'}\right)\cdot\left[\dfrac{A\left(r-x;w\right)}{\left(1+w_1\right)}\right]}\times\frac{1}{\ddot{a}_r}\\
= \;& \left[\frac{TS\left(x,r\right)}{AS_x\cdot A\left(r-x;w\right)}+\frac{\Delta TS^{i}\left(x,r\right)+\Delta TS^{w}\left(x,r\right)}{AS_x\cdot A\left(r-x;w\right)}\right]\times\left(\frac{1+w_1}{1+w_1^{'}}\right)\times\frac{1}{\ddot{a}_r}\\
= \;& \left\{RR\left(x,r\right)+\left[\frac{\Delta TS^{i}\left(x,r\right)+\Delta TS^{w}\left(x,r\right)}{AS_x\cdot A\left(r-x;w\right)}\times\frac{1}{\ddot{a}_r}\right]\right\}\left(\frac{1+w_1}{1+w_1^{'}}\right).
\end{aligned}$$

5. Post-retirement Replacement Ratio. An individual wants to have a sufficient income during the period of his retirement. A primary concern that can erode the value of his retirement income is commonly rapid increases in consumer prices over time. Hsiao (1984) provides insight into the possible impact of inflation on retirement benefits. Particularly troubling to the aged has been the rapid increase in food and medical care which are two of the primary items of expenditures for the retired and older people. Australia is no exception to this. In Figure 2, we show the quarterly changes in consumer prices in Australia over the period from December 1969 to December 2001. The top part of the figure shows that the Consumer Price Index (CPI) during this period has been steadily increasing. The bottom part of the figure shows the quarterly percentage changes, and the average percentage increase over the said period has been about 1.63%, although this percentage increase has started to decline in the past decade. It also shows a smaller variability in the more recent years. Nevertheless, according to these statistics, a dollar back in 1969 will be worth about 8 times more in year 2001. For the past decade, a dollar in 1990 will be worth 1.30 in year 2001. For the retired individual, this can be a major concern particularly when the sources of income are not directly tied to inflation.

[**Insert Figure 2 here.**]

To compensate for the possible deterioration of standard of living past the retirement age as a result of price inflation, the benefits received at retirement must also increase annually. The replacement ratio, which provides an excellent measure of maintaining post-retirement standard of living, is generally expressed as a percentage of salary. However, after retirement, the retiree is no longer working nor is receiving annual wages. To develop a post-retirement replacement ratio, we can only hypothetically assume that he continues to receive salary, with wage increases, throughout the retirement period. At the same time, we can assume that he selects benefit increases that may or may not be directly linked to inflation throughout the retirement period.

To fix notations, denote by $ES_x(r)$ the expected salary of a person currently age x retiring at age r. Assuming wage increases continue after retirement, then salary at age $r+z$, for $z>0$, becomes

$$\begin{aligned} ES_x(r+z) &= ES_x(r) \cdot \prod_{l=1}^{z} (1+w_{r+l}) \\ &= ES_x(r) \cdot A_r(z;w) \end{aligned} \tag{9}$$

where the accumulation of wage factor past retirement is defined as

$$A_r(z;w) = \prod_{l=1}^{z} (1+w_{r+l}).$$

On the retirement benefit side, assume that $I(x,r+z)$ is the annual retirement annuity income at age $r+z$ so that

$$TS(x,r) = \sum_{t=0}^{\infty} I(x,r+t) \cdot (1+j)^{-t} \ {}_tp_r. \tag{10}$$

Typically, there is a constant rate of increase for benefits, say b. Then

$$I(x,r+z) = I(x,r) \cdot (1+b)^z$$

so that Equation (10) becomes

$$\begin{aligned} TS(x,r) &= \sum_{t=0}^{\infty} I(x,r+t) \cdot (1+j)^{-t} \ {}_tp_r \\ &= I(x,r) \cdot \sum_{t=0}^{\infty} (1+b)^t \cdot (1+j)^{-t} \ {}_tp_r \\ &= I(x,r) \cdot \sum_{t=0}^{\infty} \left(\frac{1+j}{1+b}\right)^{-t} \ {}_tp_r \\ &= I(x,r) \cdot \ {}^{j'}\ddot{a}_r \end{aligned}$$

where the post-retirement rate of annuity interest is given by

$$j' = \frac{1+j}{1+b} - 1 = \frac{j-b}{1+b}.$$

The post-retirement replacement ratio at age $r+z$, where $z>0$, is thus defined as follows:

$$PRR(x,r+z) = \frac{I(x,r+z)}{ES_x(r+z)} = \frac{I(x,r+z)}{ES_x(r) \cdot A_r(z;w)}. \tag{11}$$

Assuming an annual constant rate of benefit increase b and an annual constant rate of post-retirement wage increase w, then the post-retirement replacement ratio would simplify to

$$PRR(x, r+z) = \frac{I(x,r) \cdot (1+b)^z}{ES_x(r) \cdot (1+w)^z} = \frac{I(x,r)}{ES_x(r)} \left(\frac{1+b}{1+w} \right)^z .$$

It is clear then that to achieve constant replacement ratio during retirement, one must have $b = w$, that is, benefit increases must keep up with wage increases. In general, we have

$$PRR(x, r+z) = \frac{I(x,r)}{ES_x(r)} \prod_{t=1}^{z} \left(\frac{1+b_t}{1+w_{r+t}} \right)$$

where b_t and w_{r+t} are the annual rates of changes in benefits and wages, respectively, during retirement.

Analyzing constant replacement ratios at retirement has been studied in Berin and Richter (1982). Using our notation, the analysis was as follows. First, express the post-retirement benefit $I(x, r+z)$ as the sum of benefits derived from personal savings $I_a(x, r+z)$, benefits derived from private pensions $I_b(x, r+z)$, and benefits derived from a superannuation (or social security) program $I_c(x, r+z)$. Assume that these are subject to annual increases at the rate of a, b, and c, respectively. In effect, we have the post-retirement replacement ratio as expressed by

$$\begin{aligned} & PRR(x, r+z) \\ = \; & \frac{I_a(x,r) \prod_{t=1}^{z}(1+a_t) + I_b(x,r) \prod_{t=1}^{z}(1+b_t) + I_c(x,r) \prod_{t=1}^{z}(1+c_t)}{ES_x(r) \prod_{t=1}^{z}(1+w_{r+t})}. \end{aligned} \tag{12}$$

Therefore, to maintain a constant replacement ratio, we must have

$$RR(x, r+z) = RR(x, r)$$

for all $z > 0$. So that, we have

$$\begin{aligned} & \frac{I_a(x,r) \prod_{t=1}^{z}(1+a_t) + I_b(x,r) \prod_{t=1}^{z}(1+b_t) + I_c(x,r) \prod_{t=1}^{z}(1+c_t)}{ES_x(r) \prod_{t=1}^{z}(1+w_{r+t})} \\ = \; & \frac{I_a(x,r) + I_b(x,r) + I_c(x,r)}{ES_x(r)} \end{aligned}$$

which implies that

$$\begin{aligned} & I_a(x,r)\prod_{t=1}^{z}(1+a_t)+I_b(x,r)\prod_{t=1}^{z}(1+b_t)+I_c(x,r)\prod_{t=1}^{z}(1+c_t) \\ = & \prod_{t=1}^{z}(1+w_{r+t})\left[I_a(x,r)+I_b(x,r)+I_c(x,r)\right]. \end{aligned} \tag{13}$$

This clearly shows that it no longer depends on the level of final salary prior to retirement. To keep up with post-retirement inflation, the retiree must simply have pension, superannuation or social security benefits, and income from personal savings that can reasonably cover hypothetically developed wage increases.

Post-retirement inflation can therefore be a burden dealt with from either the retiree himself through his own personal accumulation of wealth or savings, or the employer through the provision of private pension benefits, or the government in the form of social security benefits. Re-writing equation (13) as

$$\begin{aligned} & I_a(x,r)\prod_{t=1}^{z}(1+a_t)+I_b(x,r)\prod_{t=1}^{z}(1+b_t)+I_c(x,r)\prod_{t=1}^{z}(1+c_t) \\ = & \prod_{t=1}^{z}(1+w_{r+t})\left[I_a(x,r)+I_b(x,r)+I_c(x,r)\right], \end{aligned}$$

and by letting

$$\prod_{t=1}^{z}(1+a_t)=1+a^*,\ \prod_{t=1}^{z}(1+b_t)=1+b^*,\ \prod_{t=1}^{z}(1+c_t)=1+c^*$$

and similarly

$$\prod_{t=1}^{z}(1+w_{r+t})=1+w^*,$$

then clearly we have

$$I_a(x,r)\cdot a^*+I_b(x,r)\cdot b^*+I_c(x,r)\cdot c^*=w^*\cdot\left[I_a(x,r)+I_b(x,r)+I_c(x,r)\right]$$

or

$$I_a(x,r)\cdot(a^*-w^*)+I_b(x,r)\cdot(b^*-w^*)+I_c(x,r)\cdot(c^*-w^*)=0.$$

Suppose that the retiree assumes the main responsibility of ensuring cover for post-retirement inflation, so that $b^* = 0$, we must have

$$a^* = w^* + \frac{I_b(x,r)}{I_a(x,r)} \cdot (w^* - b^*) + \frac{I_c(x,r)}{I_a(x,r)} \cdot (w^* - c^*) . \tag{14}$$

6. Numerical Illustration. In this section, we illustrate our methodology of calculating retirement income replacement ratios with wage rate increases, savings rates, and investment rates developed from models based on Australian historical data. In Australia, there is an increasing debate over the adequacy of superannuation benefits and standards of living at retirement. See Australian Senate Report on Superannuation (2002). Although Australia is believed to have one of the better retirement income systems among industrialized nations, the country continues to address the issues of providing adequate standards of living for its retired population. More recently, the Institute of Actuaries of Australia advised the Select Committee on Superannuation to use replacement ratios, similar to the one defined in this paper, as a measure of assessing adequacy of retirement incomes. According to the Institute, the replacement ratio is a measure that is "more robust and less subject to distortion by differences in modelling approaches". The numerical illustration provided in this section hopes to contribute to this continuing debate.

The data used in this example is based on information collected from the Australian Bureau of Statistics (ABS). Our data consisted of four time series, $\{i_t\}$, $\{s_t\}$, $\{w_t\}$, and $\{f_t\}$, denoting respectively the investment earnings rates, savings rates, wage rate increases, and inflation rates. These rates were available from ABS on a quarterly basis over the period from December 1969 to December 2001. We have a total $T = 129$ observations in each of our series. Table 1 summarizes basic descriptive statistics for each of these series and Figure 3 displays these series graphically. We place a superscript f to denote the series has been inflation-adjusted.

[**Insert Figure 3 here.**]

Table 1

Some Summary Statistics on the Four Time Series

	Count	Mean	Median	Std Dev	Minimum	Maximum
f_t	129	0.0163	0.0168	0.0121	-0.0046	0.0583
i_t	129	0.0221	0.0281	0.0371	-0.0987	0.1489
s_t	129	0.0854	0.0866	0.0401	0.0136	0.1638
w_t	129	0.0192	0.0164	0.0165	-0.0166	0.1063
i_t^f	129	0.0057	0.0096	0.0360	-0.1214	0.1110
s_t^f	129	0.0679	0.0695	0.0322	0.0078	0.1441
w_t^f	129	0.0029	0.0027	0.0135	-0.0359	0.0565

A discussion about the four time series is warranted. The inflation rate has been computed as the quarterly proportional change in the Consumer Price Index (CPI; see also previous section) as published by ABS. Although the rate of inflation does not directly come into the calculation of the replacement ratio, each of the other series that do is affected by rate of inflation. As we note later, we adjusted the investment earnings rate, the savings rate, and the changes in wage rate for inflation. Some basic summary statistics about these inflation-adjusted rates are also included in Table 1 above, and Figure 4 displays these inflation-adjusted rates graphically. The investment earnings rates have been computed based on a weighted average rate that could be earned from having a portfolio with a combination of shares, short-term liquid investments (cash), short and long-term bonds. These combinations have been judged subjectively and do not purport to be perfectly realistic. However, judging from the resulting time series, we believe that the investment earnings rates are within what one would have expected based on some prudent investments. The savings rates in our series are based on the ratio of household savings to household disposable income as published by the ABS. These rates are therefore to be interpreted with caution because these are aggregated for the households in the economy, and if one is using the series to predict an individual's savings, one should expect large variation from these aggregates. We suspect that a higher correlation between level of wages and savings rate would exist for an individual case.

[**Insert Figure 4 here.**]

Modelling the time series has been the challenging aspect of this paper. It was initially decided to model the above series by means of a Vector Autoregressive Moving Average model (or VARMA, for short). However, after several inspections and examinations of the data, we concluded that a VARMA is unsuitable for the data. Despite the fact that this may appear counter-intuitive, there were low correlations between the series to further justify fitting a VARMA model. Most of the possible interrelationships between savings, investments, and wages were explained by the presence of inflation. Therefore, we decided to fit univariate Autoregressive Moving Average (ARMA) models on each of investment earnings rate, savings rate, and changes in wage rates, but each of them was adjusted for inflation before fitting the models. Because when projecting the time series models of the inflation-adjusted rates we need to revert back to the unadjusted rates, we therefore needed to separately model the inflation rates. A univariate ARMA model was also fitted for this series.

Using the statistical package SAS, we selected the best ARMA model for each case. Using the standard notation for time series models, namely that ARMA(p, q) means an autoregressive process of order p and a moving average process of order q, we summarize the selected models in Table 2 below for each of the series. These models were then used to project each of the series into the future. Using as a hypothetical example for illustrative purposes, we considered an individual who is currently age 30 and has 35 years to retirement, has a total savings of $\$20,000$ to-date and currently earns an annual income of $\$75,000$ (assumed to be net of taxes). For the purposes of this illustration, we ignore mortality and probability of unemployment during the accumulation period. Together with this information, we projected this individual's total savings that will be accumulated to retirement age (assumed to be 65 in this case) and then a replacement ratio at retirement is computed using the principles so developed in the early sections of this paper. This projection was simulated and repeated a total of 5,000 times to be able to capture not only a location estimate of the replacement ratio, but also the resulting volatility. Because extremely large values of the replacement ratio can distort resulting statistics and the shape of the distribution will not be affected, we truncated the resulting distribution within some reasonable bound of the replacement ratio.

Table 2

Summary of Time Series Models and Parameter Estimates for the Data

Rates X_t	Model Description
Inflation * Δf_t	ARMA(1,2)
	$X_t = -3.98 \cdot 10^{-6} + 0.05005 X_{t-1} + Z_t - 0.7986 Z_{t-1} + 0.19047 Z_{t-2}$
	where $Z_t \sim N(0, \sigma^2)$ and $\widehat{\sigma}^2 = 0.0008$.
Investments i_t^f	ARMA(2,1)
	$X_t = 0.002452 + 0.36819 X_{t-1} + 0.24008 X_{t-2} + Z_t - 0.44764 Z_{t-1}$
	where $Z_t \sim N(0, \sigma^2)$ and $\widehat{\sigma}^2 = 0.001247$.
Savings s_t^f	ARMA(1,0)
	$X_t = 0.011005 - 0.34769 X_{t-1} + Z_t$
	where $Z_t \sim N(0, \sigma^2)$ and $\widehat{\sigma}^2 = 0.068114$.
Wages w_t^f	ARMA(2,2)
	$X_t = 0.008435 - 1.105929 X_{t-1} - 0.89544 X_{t-2} + Z_t$ $+ 1.17957 Z_{t-1} + 0.96558 Z_{t-2}$
	where $Z_t \sim N(0, \sigma^2)$ and $\widehat{\sigma}^2 = 0.000176$.

* Δ denotes first difference

Figure 5 provides the histogram of the resulting replacement ratio from our simulation. The resulting mean is 79% and the median is 24%, with a standard deviation of 116%. The distribution of the replacement ratio resulting from using the models derived from historical data indicates a highly positively skewed distribution, indicating a low probability of exceeding the mean and a low probability of falling below zero. A quantile may be better indicator of a location for such a skewed distribution. Our simulation of replacement ratios resulted with a first quartile of 4% and a third quartile of 181%. In addition, the 65-th percentile of the resulting distribution is a replacement ratio of 100%, which means that the replacement ratio was below 100% approximately 65% of the time.

[**Insert Figure 5 here.**]

7. Conclusion. In this paper, we advocate the use of the replacement ratio as a tool for financial planning for retirement. The replacement ratio is defined to be the proportion of retirement income expressed as a percentage of final salary. However, because the retiree must plan for possible effect of inflation on retirement income, we further developed the replacement ratio applicable even during the retirement period. We know that the ratio is affected by several factors including increases in wages, savings, returns on investments and inflation. We have developed the replacement ratio to accommodate for possible annual changes in these factors. Moreover, it is possible that these factors are interrelated as we have observed in the case of the Australian data used in this paper where inflation has an influence on savings, investments, and wage rate changes. The fitting of time series models to data series of wage rate increases, savings rates, investment returns and inflation permits us to forecast these series. As a result, replacement ratios can be simulated to illustrate the earlier sections of the paper and to provide additional information to the analysis of the adequacy of retirement preparation.

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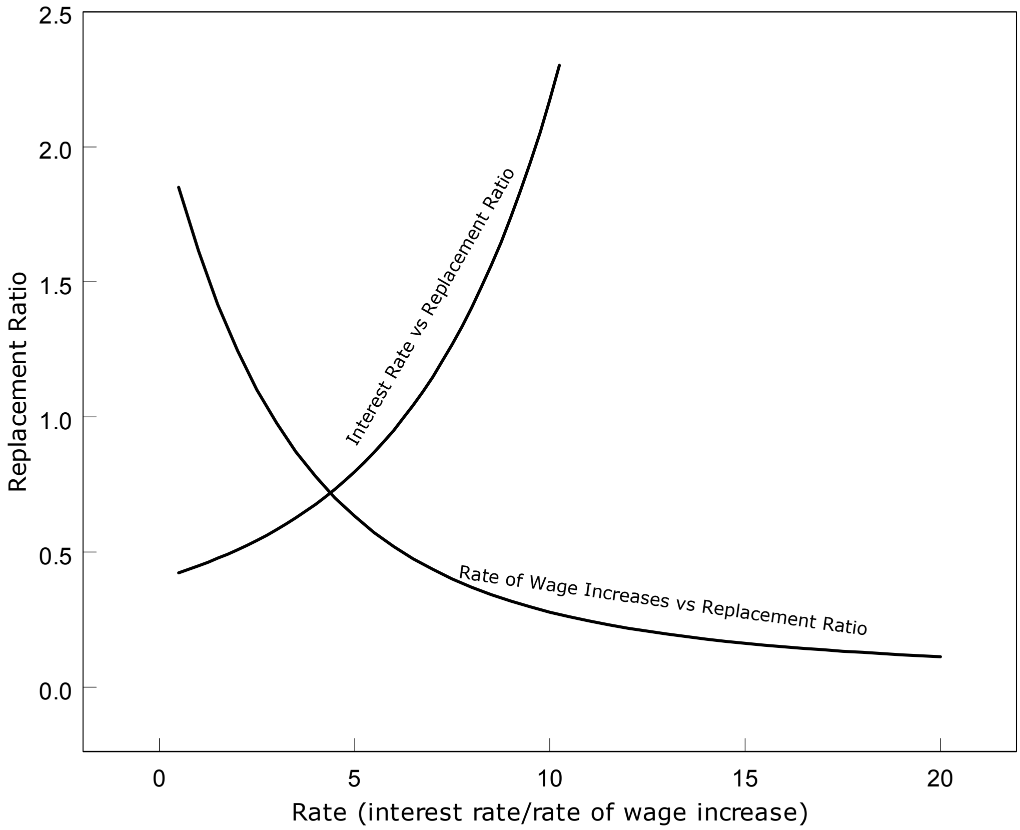

Figure 1: Replacement ratio as a function of interest rate and rate of wage increase

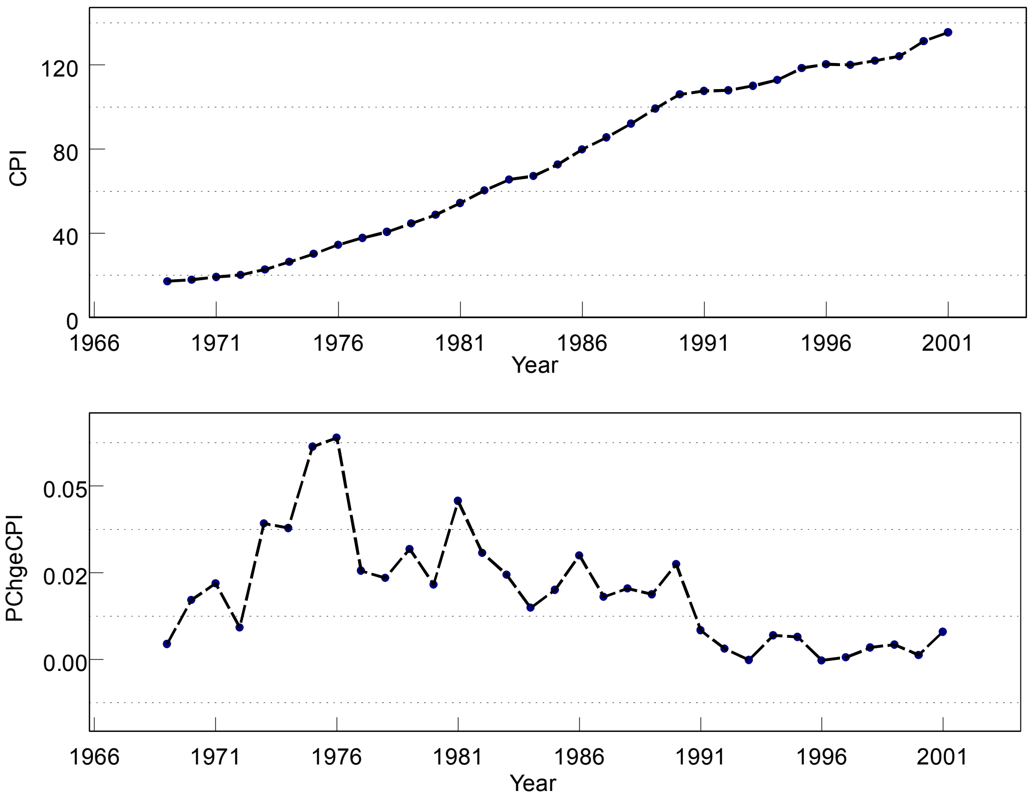

Figure 2: Australian quarterly CPI and Percent Changes from 1969 to 2001.

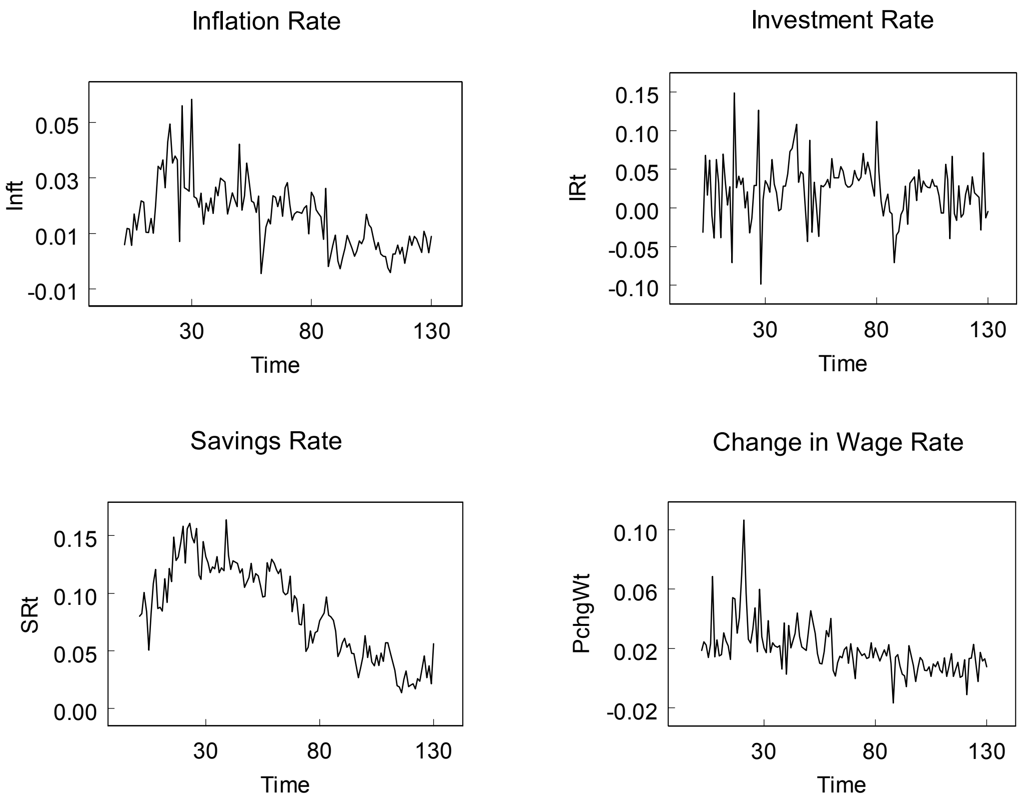

Figure 3: Plot of Four Time Series: Inflation, Investments, Savings, and Wages.

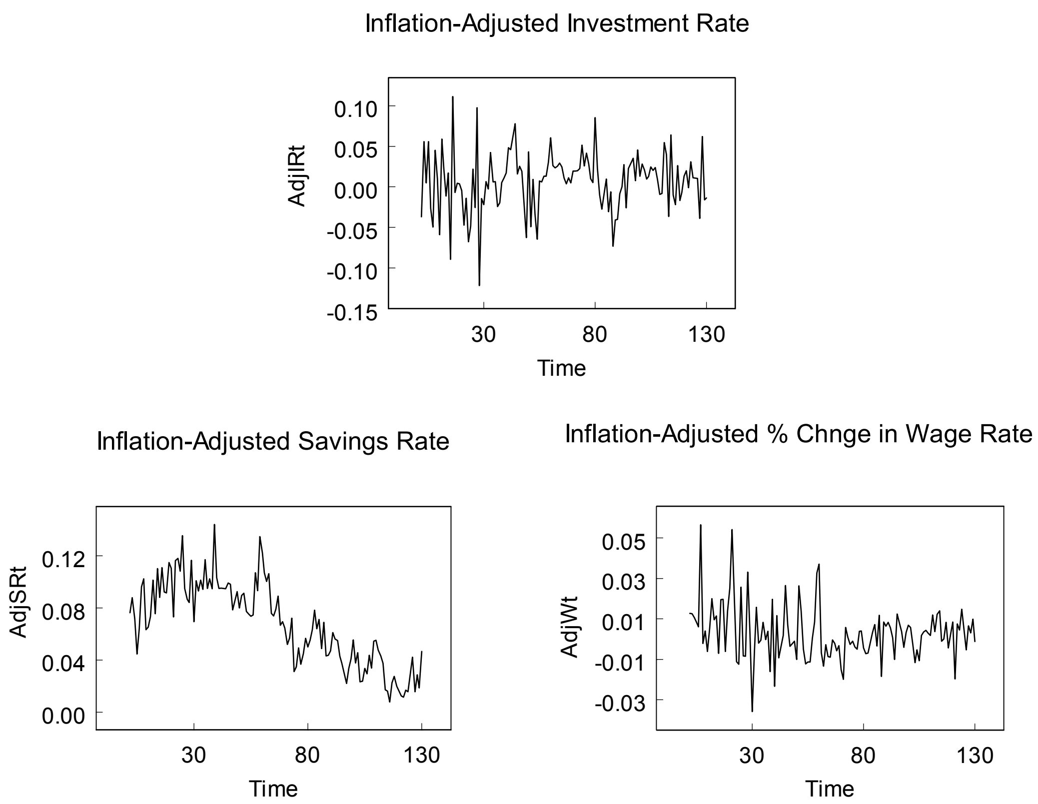

Figure 4: Time Series Plots of Inflation-Adjusted Rates

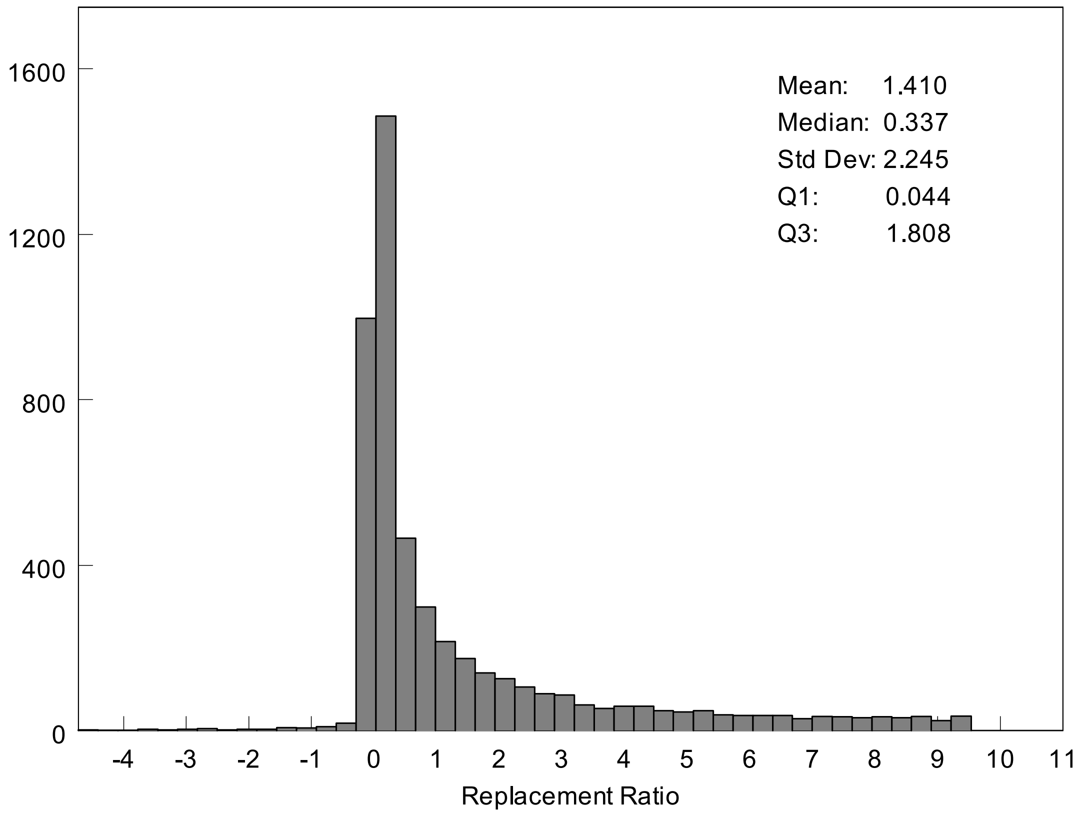

Figure 5: Histogram of the Replacement Ratios from a Simulation of 5,000 Runs.