Wang's Capital Allocation Formula for Elliptically-Contoured Distributions

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Abstract

There is a growing interest among insurance companies to be able not only to compute total company capital requirements but also to allocate this total capital across its various business units. Wang (2002) recently recommended allocating the total cost of capital of an insurance company based on the idea of "exponential tilting". Under the assumption that the risks or losses follow a multivariate normal distribution, the resulting allocation formula will be a function of the variance-covariance structure. We extend Wang's idea into a larger class of multivariate risks called "elliptically-contoured" multivariate distributions, of which the multivariate normal is a special case. In addition, this paper develops three criteria of what constitutes a "fair allocation" between lines of business of an insurance company: no undercut, symmetry, and consistency. We prove that the covariance-based allocation principle satisfies the requirements of a fair allocation. Because the resulting allocation is quite similar to the covariance-based principle, it follows that Wang's allocation formula is also considered fair.

Keywords: capital allocation, fair allocation, and elliptical distributions.

1 Introduction and Motivation

It is well known that insurance companies are obligated by regulators to hold capital, the primary purpose of which is to maintain the financial soundness of the companies. Furthermore, holding sufficient capital helps reduce the probability that claims from policyholders are not met as and when they become due, protects the company from insolvency, and ensures the future of the company as a going-concern. In recent years, insurance regulators around the globe have adopted Risk-Based Capital (RBC) requirements as a tool and framework for computing capital. These requirements have been modified several times over the years to adopt to the changing nature of the financial market and economic environment in which insurance companies operate and to better reflect the inherent uncertainty of the insurance business.

It is important that the insurance company holds an appropriate amount of capital. While it is desirable that it hold large amounts of capital, usually this does not come without cost. Investors demand a premium for lending capital and this cost of capital can indirectly be passed to the policyholders in the form of a higher premium loading. The capital required by the insurer is often viewed by ratings agencies as a measure of the company's capacity to bear risks. Insurance company ratings from these agencies are viewed by investors and policyholders as a measure of their capacity to bear risks. There has been a recent surge in the literature in developing a framework of risk measurements for computing capital requirements which is an important part of the risk management process for the insurance companies.

In this paper, we address the issue of being able to fairly divide the total capital requirement of a diversified insurance company across its various business units. The term "capital allocation" has been used in finance literature where a similar concept of fair division of capital in a diversified portfolio of investments has been investigated. In particular, we examine the capital allocation formula proposed by Wang (2002) where he derived the explicit form in the case where risks are multivariate normal. We extend his ideas in the larger class of multivariate elliptical risks which allows greater flexibility in modelling tails of distributions. This is particularly important for insurance companies which are often concerned with tails of loss distributions.

Given that most insurance companies write several lines of business, it is sometimes imperative to allocate capital requirements across these lines of business. Firstly, as previously stated, there is a cost associated with holding capital and the insurance company may wish to accurately determine this cost by line of business and thereby redistribute this cost equitably across the lines. Policyholders are therefore assessed with a more equitable premium. Secondly, again as previously stated, capital is often viewed as a measure of the level of risk inherent in the company and division of the capital therefore provides a division of the level of risks inherent across the business units. This division of total company risk can be useful to the insurance company wishing to allocate expenses across the lines of business, prioritizing new capital budgeting projects, or even deciding which lines of business to expand or to contract. Last but not least, capital allocation formulas provide a useful device for fair assessment of performance of managers of various business units. Salaries and bonuses may be linked to performance. As partly explored in this paper, there may be gains that can be made through diversification and a "fair" allocation formula recognizes such diversification benefits. In summary, the richer information often derived from capital allocation improves management of the insurance enterprise. Furthermore, the same principles equally apply to an insurance company which may have a single major line of business, e.g. life, health or property, but have several product lines, e.g. traditional whole life, universal life or equity-linked products in the case of a life company, and motor insurance or homeowners insurance in the case of a non-life company.

While it is true that there may have been a surge of research in the area of developing capital standards, there appears to have been very little work done in the literature on capital allocation. As a matter of fact, the Casualty Actuarial Society distributed a request last year for proposals on "Value Creation through Enterprise Risk Management" to the actuarial community, recognizing the importance of capital allocation strategies. To the authors' knowledge, only the works by Myers and Read (2001), Wang (2002), and Panjer (2002) specifically address the issue of capital allocation. Myers and Read uses ideas from options pricing and develops the capital allocation on the basis of lines of business marginal contribution to the company's default risk. Wang's work is in response to the Casualty Actuarial Society's request for proposals, and his ideas are further explored in our paper.

In capital allocation, it is necessary to consider what constitutes a "proper" or "fair" allocation. Because of the importance of this issue, it is covered in more detail in the next section. For obvious reasons, we shall sometimes interchangeably refer to such allocation as "rational". The concept of fair allocation has been partly investigated in the finance literature. See, for example, Denault (2001) and Hesselager and Andersson (2002) from which principles of rational allocation developed in this paper have been drawn. For example, diversification gains, the decline in total costs from multiple lines of business, need to be shared fairly. Myers and Read (2001) note that although the reduction in premiums due to default risk will be the same for all lines for a given firm, the marginal contribution to default risk does vary across lines. It is shown that these marginal contribution rates do "add up" and argue that capital should be allocated at the margin based on these contributions. Also, the reduction in risk and required capital due to diversification cannot be uniquely allocated back to the individual lines of business, however with two or more lines of business the lines' marginal capital allocations are unique and are additive. The additivity is referred in this paper as the full allocation principle and is a fundamental requirement of an allocation formula. Another work in finance related to capital risk contributions that is worth mentioning is the one by Tasche (2000).

The rest of this paper is organized as follows. Section 2 describes the requirements for an allocation to be "fair", and some possible capital allocation methodologies are considered. We state these requirements and some commonly used capital allocation formulas. Section 3 outlines Wang's capital allocation formula and then considers the case where losses follow a multivariate normal distribution. Section 4 briefly introduces elliptically-contoured distributions. Elliptical distributions are generalizations of the multivariate normal distributions and therefore share many of its tractable and mathematically-appealing properties. This class of properties was introduced by Kelker (1970) and has been widely discussed in Fang, et al. (1987) and also in Gupta and Varga (1993). This generalization of the normal family is beginning to gain importance as a tool for actuarial and financial risk management primarily because of the flexibility it offers in modelling the extremes or tails of distributions. See for example Embrechts, et al. (2001) and Bingham and Kiesel (2002). Section 5 considers Wang's allocation formula in the case of elliptically-contoured distributions and derives an explicit form for the allocation to each line. A numerical example follows to further clarify these results. Section 6 summarizes the key results in this paper and concludes.

2 Properties of "Fair Allocation"

To fix notations, we suppose that an insurance company has n different lines of business, each of which faces the risk of losing $X_1, X_2, ..., X_n$ at the end of a single period. Suppose that these random claims have a dependency structure characterized by the joint distribution of the vector $\mathbf{X}^T = (X_1, X_2, ..., X_n)$. Clearly, the total company loss is the random variable

$$Z = X_1 + X_2 + \cdots + X_n.$$

We suppose that the total capital required for the company, denoted by K, to hold can be determined by the risk measure

$$\rho : Z \to R$$

which maps out the risk X to the set of real numbers R. In short, we have $K = \rho(Z) \in R$. After determining the value K for the total company, we are now interested in the contribution to total capital of each line of business. For discussion of risk measures such as requirements of a coherent risk measures, see Artzner, et al. (1999).

Definition 1 *Let $\mathbf{X}^T = (X_1, X_2, ..., X_n)$ denote the vector of losses. We define an allocation A to be a mapping*

$$A : \mathbf{X}^T \to R^n$$

such that $A\left(\mathbf{X}^T\right) = (K_1, K_2, ..., K_n)^T$ where $\sum_{i=1}^{n} K_i = K = \rho(Z)$.

Each component K_i of the allocation can therefore be viewed as the i-th line of business contribution to the total capital. In theory, it is supposed to be the amount that reflects the riskiness inherent in that line of business. This will further be imposed when we later define the requirement for allocation to be considered fair. Furthermore, because the allocation must also reflect the fact that each line of business operates in the presence of the other lines of business, the notation

$$A(X_i \,| X_1, ..., X_n) = K_i$$

is well-suited for this purpose. The contribution to total capital of each line is therefore an amount in the presence of the other lines of business.

Notice also that the requirement $\sum_{i=1}^{n} K_i = \rho(Z)$ is sometimes called the "full allocation" requirement.

We are now ready to define when an allocation is considered fair.

Definition 2 *Let $N = \{1, 2, ..., n\}$ be the set of the first n positive integers. An allocation A is said to be a fair allocation if for every possible value of $\mathbf{X}^T = (X_1, X_2, ..., X_n)$, the following three properties are satisfied:*
(1) ***No Undercut:*** *For any subset $M \subseteq N$, we have*

$$\sum_{i \in M} A\left(X_i \left| X_1, ..., X_n\right.\right) \leq \rho\left(\sum_{i \in M} X_i\right).$$

(2) ***Symmetry:*** *Let $N^* = N - \{i_1, i_2\}$. If $M \subset N^*$ with $|M| = m$, $\mathbf{X}_m^T = (X_{j_1}, ..., X_{j_m})$ and that*

$$A\left(X_{i_1} \left| \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right.\right) = A\left(X_{i_2} \left| \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right.\right),$$

for every $M \subset N^*$, then we must have $K_{i_1} = K_{i_2}$.
(3) ***Consistency:*** *For any subset $M \subseteq N$, we have*

$$\sum_{i \in M} A\left(X_i \left| X_1, ..., X_n\right.\right) = A\left(\sum_{i \in M} X_i \left| X_1, ..., \sum_{i \in M} X_i, ..., X_n\right.\right).$$

The properties of "symmetry" and "consistency" have been separately considered by Denault (2001) and Hesselager and Andersson (2002). These properties can be understood intuitively. The "no undercut" criterion recognizes the benefits of diversification, in that the risk allocated to a unit or a group of units cannot exceed the risk allocated if these were offered on a stand-alone basis. The capital to hold for a subset of the company therefore must not be greater than the capital required if that subset were operating as a separate company. This criterion is analogous to the sub-additivity requirement of a coherent risk measure (Artzner, et al., 1999). The "symmetry" criterion states that two lines of business that equally contribute to the risk within the firm must have equal allocation. In effect, the allocation must therefore recognize only the level of contribution to risk within the firm, and nothing else. Risks considered equivalent must therefore have equal allocation. For obvious reasons, this criterion could have been more properly termed as the "equitability" property. Lastly, the "consistency" property ensures that a unit's allocation cannot depend on the level at which allocation occurs. Following the illustration made by Hesselager and Andersson (2002), consider a company structured with lines of business at the top level and sub-lines of business at lower levels. According to the consistency criterion, the capital allocation for a line of business at the top level must be the same whether the allocation of its sub-lines has been directly made to the company or indirectly first to the top level and later aggregated for the company.

Example 1: Relative Allocation A popular method for practical use is the Relative Allocation method, which uses the stand-alone risk $\rho(Y_i)$ to determine a weight of the total risk to allocate to the i-th line of business. This gives the allocation formula according to

$$A(X_i|X_1,\cdots,X_n) = \rho(Z)\frac{\rho\left(X_i\right)}{\rho\left(X_1\right)+\cdots+\rho\left(X_n\right)}.$$

Its simple nature is appealing, however this allocation method fails two of the criteria for rational allocation. To prove that this does not provide a fair allocation, consider three independent risks X_1, X_2, and X_3 with common zero mean $E\left(X_i\right)=0$ and variances $Var\left(X_i\right)=\sigma^2\left(X_i\right)$ for $i=1,2,3$. Define the capital risk measure to be of the form

$$\rho\left(X_i\right) = F_i^{-1}\left(1-\alpha\right)\cdot\sigma\left(X_i\right)$$

where F_i denote the distribution function of risk X_i so that $F_i^{-1}\left(1-\alpha\right)$ is the $(1-\alpha)$-th quantile of the distribution where $0\leq\alpha\leq 1$. Now suppose a life insurance company has four lines of business each facing the risks X_1, $-X_1$, X_2, and X_3 so that total risk is $Z=X_2+X_3$. Consider the subset M consisting of the risks $\{X_1,-X_1,X_2\}$ and observe that

$$\rho\left(\sum_{i\in M}X_i\right) = \rho(X_2) = F_2^{-1}\left(1-\alpha\right)\cdot\sigma\left(X_2\right).$$

Furthermore, since

$$\sum_{i\in M}A\left(X_i\left|X_1,-X_1,X_2,X_3\right.\right) = \rho(X_2+X_3)\frac{\rho\left(X_1\right)+\rho(-X_1)+\rho(X_2)}{\rho\left(X_1\right)+\rho(-X_1)+\rho(X_2)+\rho(X_3)},$$

the "no undercut" cannot be satisfied unless the risks have symmetric distributions. The "consistency" property is also not satisfied because

$$A\left(\sum_{i\in M}X_i\left|\sum_{i\in M}X_i,X_3\right.\right) = A\left(X_2\left|X_2,X_3\right.\right) = \rho(X_2+X_3)\frac{\rho(X_2)}{\rho(X_2)+\rho(X_3)}$$

and hence

$$\sum_{i \in M} A\left(X_i \left| X_1, -X_1, X_2, X_3\right.\right) \neq A\left(\sum_{i \in M} X_i \left| \sum_{i \in M} X_i, X_3\right.\right).$$

However, it can be shown that the "symmetry" property is satisfied for this allocation formula. Consider for example the case where

$$A\left(X_1 \left| X_1, -X_1, X_2\right.\right) = A\left(X_2 \left| X_1, -X_1, X_2\right.\right)$$

and it is straightforward to show that in this case

$$\rho\left(X_1\right) = \rho\left(X_2\right)$$

so that

$$A\left(X_1 \left| X_1, -X_1, X_2, X_3\right.\right) = A\left(X_2 \left| X_1, -X_1, X_2, X_3\right.\right)$$

and symmetry is satisfied.

Example 2: Covariance-based Allocation According to the covariance allocation principle, the formula is based on

$$A(X_i|X_1, \cdots, X_n) = \lambda_i \rho(Z)$$

where $\boldsymbol{\lambda}^T = (\lambda_1, ..., \lambda_n)$ denotes a vector of weights that add up to one so that full allocation is satisfied. Denote the mean vector of $\mathbf{X}$ by $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ so that the mean of the sum of the risks is $\mu_Z = \sum_{i=1}^{n} \mu_i$. To determine these weights λ_i, we minimize the following quadratic loss function

$$\ell\left(\boldsymbol{\lambda}\right) = E\left[\left(\left(\mathbf{X} - \boldsymbol{\mu}\right) - \boldsymbol{\lambda}\left(Z - \mu_Z\right)\right)^T \mathbf{W}\left(\left(\mathbf{X} - \boldsymbol{\mu}\right) - \boldsymbol{\lambda}\left(Z - \mu_Z\right)\right)\right]$$

where the weight-matrix $\mathbf{W}$ is assumed to be positive definite. Differentiating with respect to $\boldsymbol{\lambda}$ and equating to zero yields the expression:

$$\lambda_i = \frac{E\left[\left(X_i - \mu_i\right)\left(Z - \mu_Z\right)\right]}{E\left[\left(Z - \mu_Z\right)^2\right]} = \frac{Cov\left(X_i, Z\right)}{Var\left(Z\right)} \tag{1}$$

for $i = 1, 2, ..., n$. Observe that the solutions do not depend on the weight matrix $\mathbf{W}$. We now prove the result that the allocation formula above satisfies the properties of a rational allocation. We state this result as a theorem. In this theorem, we additionally assume that the risk measure ρ satisfies $\rho\left(X\right) \leq c \cdot \sigma\left(X\right)$ for some fixed constant c.

Theorem 3 *The capital allocation formula in (1) satisfies the three properties of a fair allocation: (a) no undercut, (b) symmetry, and (c) consistency.*

Proof. Let $N = \{1, 2, ..., n\}$ be the set of the first n positive integers. To prove the "no undercut" property, we consider any subset $M \subseteq N$ and notice that we have the following:

$$\begin{aligned}
\sum_{i \in M} K_i &= \sum_{i \in M} A\left(X_i \left| X_1, ..., X_n\right.\right) = \sum_{i \in M} \frac{Cov\left(X_i, Z\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&\leq \sum_{i \in M} \frac{\sigma\left(X_i\right)}{\sigma\left(Z\right)} \rho\left(Z\right) \leq \frac{\sigma\left(\sum_{i \in M} X_i\right)}{\sigma\left(Z\right)} \rho\left(Z\right) \leq \rho\left(\sum_{i \in M} X_i\right),
\end{aligned}$$

where we have used Cauchy-Schwarz inequality in the second line. To prove "symmetry", denote by $N^* = N - \{i_1, i_2\}$ and let $M \subset N^*$ with $|M| = m$, $\mathbf{X}_m^T = \left(X_{j_1}, ..., X_{j_m}\right)$. Suppose $A\left(X_{i_1} \left| \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right.\right) = A\left(X_{i_2} \left| \mathbf{X}_m^T, X_{i_1}, X_{i_2}\right.\right)$ for every $M \subset N^*$. Consider

$$\begin{aligned}
K_{i_1} &= \frac{Cov\left(X_{i_1}, Z\right)}{Var\left(Z\right)} \rho\left(Z\right) = \frac{Cov\left(X_{i_1}, Z - X_l + X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&= \frac{Cov\left(X_{i_1}, Z - X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) + \frac{Cov\left(X_{i_1}, X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&= \frac{Cov\left(X_{i_2}, Z - X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) + \frac{Cov\left(X_{i_1}, X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&= \frac{Cov\left(X_{i_2}, Z\right)}{Var\left(Z\right)} \rho\left(Z\right) + \frac{Cov\left(X_{i_1} - X_{i_2}, X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&= K_{i_2} + \frac{Cov\left(X_{i_1} - X_{i_2}, X_l\right)}{Var\left(Z\right)} \rho\left(Z\right)
\end{aligned}$$

and this is true for all $l = 1, 2, ..., n$. Summing both sides over all the values of l, we get

$$\begin{aligned}
nK_{i_1} &= nK_{i_2} + \sum_{l=1}^{n} \frac{Cov\left(X_{i_1} - X_{i_2}, X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&= nK_{i_2} + \frac{Cov\left(X_{i_1} - X_{i_2}, \sum_{l=1}^{n} X_l\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&= nK_{i_2} + \frac{Cov\left(X_{i_1} - X_{i_2}, Z\right)}{Var\left(Z\right)} \rho\left(Z\right) \\
&= nK_{i_2} + K_{i_1} - K_{i_2}.
\end{aligned}$$

But this statement is true if and only if $K_{i_1} = K_{i_2}$. Finally, to prove "consistency", consider any subset $M \subseteq N$ and note that

$$\begin{aligned} \sum_{i \in M} A(X_i | X_1, ..., X_n) &= \sum_{i \in M} \frac{Cov(X_i, Z)}{Var(Z)} \rho(Z) \\ &= \frac{Cov\left(\sum_{i \in M} X_i, Z\right)}{Var(Z)} \rho(Z) \\ &= A\left(\sum_{i \in M} X_i \middle| X_1, ..., \sum_{i \in M} X_i, ..., X_n\right). \end{aligned}$$

Therefore, the covariance allocation principle is considered fair.■

3 Wang's Capital Decomposition Formula

This section now examines Wang's proposal for allocating capital across different lines of business. To fix ideas, denote by X_i the loss random variable for the i-th business unit, $i = 1, 2, ..., n$. The insurance company has n different lines of business or business units. We shall assume that X_i is a random variable on a well-defined probability space $(\Omega, \mathcal{F}, P)$. Furthermore, assume that the loss random vector $\mathbf{X}^T = (X_1, ..., X_n)$ has a joint multivariate distribution which describes the dependency structure. For each possible state $\omega \in \Omega$, the aggregate loss to the company is the sum of the losses from all the business units, that is

$$Z(\omega) = \sum_{i=1}^{n} X_i(\omega). \tag{2}$$

Next, define the "exponential tilting" of X_i induced by the aggregate loss Z as

$$X_{i,Q}(\omega) = X_i(\omega) \frac{\exp(\lambda Z(\omega))}{E[\exp(\lambda Z(\omega))]}. \tag{3}$$

Here, λ is a parameter. Correspondingly, define the aggregate loss induced by this "exponential tilting"

$$Z_Q(\omega) = \sum_{i=1}^{n} X_{i,Q}(\omega).$$

It is straightforward to show that

$$Z_Q(\omega) = Z(\omega) \frac{\exp(\lambda Z(\omega))}{E[\exp(\lambda Z(\omega))]} \tag{4}$$

The subscript Q has been conveniently used for the transformed variables to denote a similar concept of a "change of measure."

Preserving the notation used by Wang (2002), denote the expectation of $X_{i,Q}$ by

$$H_\lambda[X_i, Z] = E(X_{i,Q}) = \frac{E[X \cdot \exp(\lambda Z)]}{E[\exp(\lambda Z)]} \tag{5}$$

and the expectation of the aggregate loss Z_Q by

$$H_\lambda[Z, Z] = E(Z_Q) = \frac{E[Z \cdot \exp(\lambda Z)]}{E[\exp(\lambda Z)]}. \tag{6}$$

This exactly gives the Esscher transform of Z. The idea of "exponential tilting" or Esscher transformation has its roots in economics whereby a market of agents willing to exchange random payments or risks come in agreement with prices at equilibrium. Buhlmann (1980, 1984) demonstrated that under an exponential utility function, the equilibrium market price can be expressed as the expectation of the random payment under the Esscher transform. At equilibrium, the risk exchange is optimal in the sense that all participants would have simultaneously maximized their utility.

Notice then that the price of a random payment of X_i presumably traded in the market is (5) so that one can think of the difference with its expectation

$$\begin{aligned} \rho(X_i) &= E(X_{i,Q}) - E(X_i) \\ &= H_\lambda[X_i, Z] - E(X_i) \end{aligned} \tag{7}$$

as the risk premium. See Gerber and Pafumi (1998). For the aggregate payment Z, its risk premium is given by

$$\rho(Z) = \rho\left(\sum_{i=1}^{n} X_i\right) = H_\lambda[Z, Z] - E(Z). \tag{8}$$

It is rather straightforward to show that we can respectively write equations (7) and (8) as

$$\rho(X_i) = \frac{Cov(X_i, \exp(\lambda Z))}{E[\exp(\lambda Z)]}$$

and

$$\rho(Z) = \frac{Cov(Z, \exp(\lambda Z))}{E[\exp(\lambda Z)]}.$$

Wang proposes computing the allocation of capital to individual business unit i based on the following formula:

$$K_i = H_\lambda[X_i, Z] - E(X_i). \tag{9}$$

Assuming an aggregate capital of K for the insurance company as a whole, the parameter λ in (9) can be computed using

$$K = H_\lambda[Z, Z] - E(Z).$$

One can therefore think of this as either a risk premium or the cost of capital. Consequently, for $i = 1, 2, ..., n$, it can readily be shown that

$$K = \sum_{i=1}^{n} K_i.$$

To see this, note that

$$\begin{aligned} \sum_{i=1}^{n} K_i &= \sum_{i=1}^{n} \{H_\lambda[X_i, Z] - E(X_i)\} \\ &= \sum_{i=1}^{n} H_\lambda[X_i, Z] - \sum_{i=1}^{n} E(X_i) \\ &= H_\lambda[Z, Z] - E(Z) = K. \end{aligned}$$

Thus, we see that Wang's capital allocation formula satisfies our definition (Definition 1) of an allocation.

Example 3: Multivariate Normal Distribution Under the assumption that the losses $X_1, ..., X_n$ follow a multivariate normal distribution, we have that Wang's allocation method reduces to the covariance method. Some straightforward calculation similar to the proof of Lemma (6) yields the results:

$$\begin{aligned} E\left(Ze^{\lambda Z}\right) &= \exp\left(\lambda\mu_Z + \frac{\lambda^2\sigma_Z^2}{2}\right) \cdot (\mu + \lambda\sigma_Z^2) \\ E\left(X_i e^{\lambda Z}\right) &= \exp\left(\lambda\mu_Z + \frac{\lambda^2\sigma_Z^2}{2}\right) \cdot (\mu_i + \lambda\sigma_{i,Z}) \end{aligned}$$

where μ_Z and μ_i are the mean losses of the aggregate Z and the i^{th} line of business respectively, and $\sigma_Z^2 = \sum_{i,j=1}^n \rho_{ij}\sigma_i\sigma_j$ and $\sigma_{i,Z} = \sum_{j=1}^n \rho_{ij}\sigma_i\sigma_j$ for $j = 1, 2, \cdots, n$. Then it follows that

$$K = \lambda\sigma_Z^2$$

and

$$K_i = \lambda\sigma_{i,Z}$$

which is clearly equivalent to the covariance method outlined earlier.

4 Elliptically-Contoured Distributions: Definitions and Properties

The class of elliptically-contoured distributions (or elliptical distributions, for short) provides a generalization of the multivariate normal distributions. Elliptical distributions therefore share many of the tractable and nice statistical properties of the normal. The class contains many other non-normal multivariate distributions such as the multivariate student-t, multivariate cauchy, multivariate logistic, and multivariate symmetric stable, to name a few. This class of distributions has been introduced in the statistical literature by Kelker (1970) and widely discussed in Fang, et al. (1987) and Gupta and Varga (1993). See also Landsman and Valdez (2002) and Valdez and Dhaene (2002) for applications in actuarial science.

There are a number of equivalent ways to define random vectors that belong to the class of elliptical distributions. Let $\mathbf{X} = (X_1, X_2, ..., X_n)^T$ be an n-dimensional random vector.

Definition 4 *The random vector* $\mathbf{X}$ *has a multivariate elliptical distribution, written as* $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$*, if its characteristic function has the form*

$$\varphi_{\mathbf{X}}(\mathbf{t}) = \exp(it^T\mu)\psi\left(\tfrac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}\right) \tag{10}$$

for some column-vector $\boldsymbol{\mu}$, $n \times n$ *positive-definite matrix* $\boldsymbol{\Sigma}$*, and for some function* $\psi(t)$ *called the characteristic generator.*

The characteristic generator $\psi(\cdot)$ may depend on n, the dimension of the vector $\mathbf{X}$. A few observations about elliptical distributions are worth noting at this point. First, in general, it does not follow that $\mathbf{X}$ has a joint density $f_{\mathbf{X}}(\mathbf{x})$, but if its density exists, it can be expressed as

$$f_{\mathbf{X}}(\mathbf{x}) = c_n \left|\mathbf{\Sigma}\right|^{-1/2} g_n \left[\tfrac{1}{2} \left(\mathbf{x} - \boldsymbol{\mu}\right)^T \mathbf{\Sigma}^{-1} \left(\mathbf{x} - \boldsymbol{\mu}\right)\right], \tag{11}$$

for some function $g_n(\cdot)$ called the density generator which may depend on n. In cases where this density does not depend on n, we may drop the subscript n. Using the density generator, an alternative way to write $\mathbf{X}$ belongs to the family of elliptical distributions is $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \mathbf{\Sigma}, g_n)$. From (11), again if the density exists, the normalizing constant c_n can be explicitly determined using

$$c_n = (2\pi)^{-n/2} \Gamma(n/2) \left[\int_0^\infty x^{n/2-1} g_n(x) dx\right]^{-1}. \tag{12}$$

A detailed proof of (12) can be found in Landsman and Valdez (2002).

Second, in the univariate case, elliptical distributions consist of the class of symmetric distributions. However, they provide greater flexibility than the normal density because they allow for heavier (or even lighter) tails. If the elliptical density exists, their probability contours have the shape of an ellipsoidal. Figure 1 demonstrates these contours in the bivariate case ($n = 2$) for selected members of the elliptical family. For the purpose of this illustration, we have selected $\boldsymbol{\mu} = (0,0)^T$ and $\mathbf{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity matrix. When $\boldsymbol{\mu}$ is the vector of zeroes and Σ is the identity matrix, we have a standard elliptical distribution or commonly called spherical distributions. For other $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$, the contours remain in shape except it would simply have tilted and shifted them.

Third, the mean and covariance of vector $\mathbf{X}$ do not necessarily exist. However, if the mean exists, it will be the vector $E(\mathbf{X}) = \boldsymbol{\mu}$ and if the covariance exists, it will be

$$Cov(\mathbf{X}) = -\psi'(0)\,\mathbf{\Sigma}. \tag{13}$$

A clear condition for this covariance to exist is $|\psi'(0)| < \infty$. Notice that in some cases where the characteristic generator satisfies $\psi'(0) = -1$, the covariance becomes $Cov(\mathbf{X}) = \mathbf{\Sigma}$.

Lastly, we observe that any marginal distribution of $\mathbf{X}$ is also elliptical with the same characteristic generator. If we take a subset of $\mathbf{X}$, say $\mathbf{X}_m = (X_1, X_2, ..., X_m)^T$ with $m \leq n$, then it follows that $\mathbf{X}_m$ is again elliptical.

And in particular for the univariate marginals, for $k = 1, 2, ..., n$, we have $X_k \sim E_1\left(\mu_k, \sigma_k^2, g_1\right)$ and therefore its density can be expressed as

$$f_{X_k}(x) = \frac{c_1}{\sigma_k} g_1 \left[\tfrac{1}{2}\left(\frac{x - \mu_k}{\sigma_k}\right)^2\right].$$

Furthermore, any linear combination of elliptical distributions is another elliptical distribution with the same characteristic generator ψ or from the same sequence of density generators $g_1, ...g_n$, corresponding to ψ. Suppose B is some $m \times n$ matrix of rank $m \leq n$ and b, some m-dimensional column-vector, then

$$B\mathbf{X} + b \sim E_m\left(B\boldsymbol{\mu} + b, B\boldsymbol{\Sigma}B^T, g_m\right). \tag{14}$$

In this section, we only briefly described the elliptical distributions and their properties useful for this paper. In the Appendix, we provide a list of some members of the elliptical family.

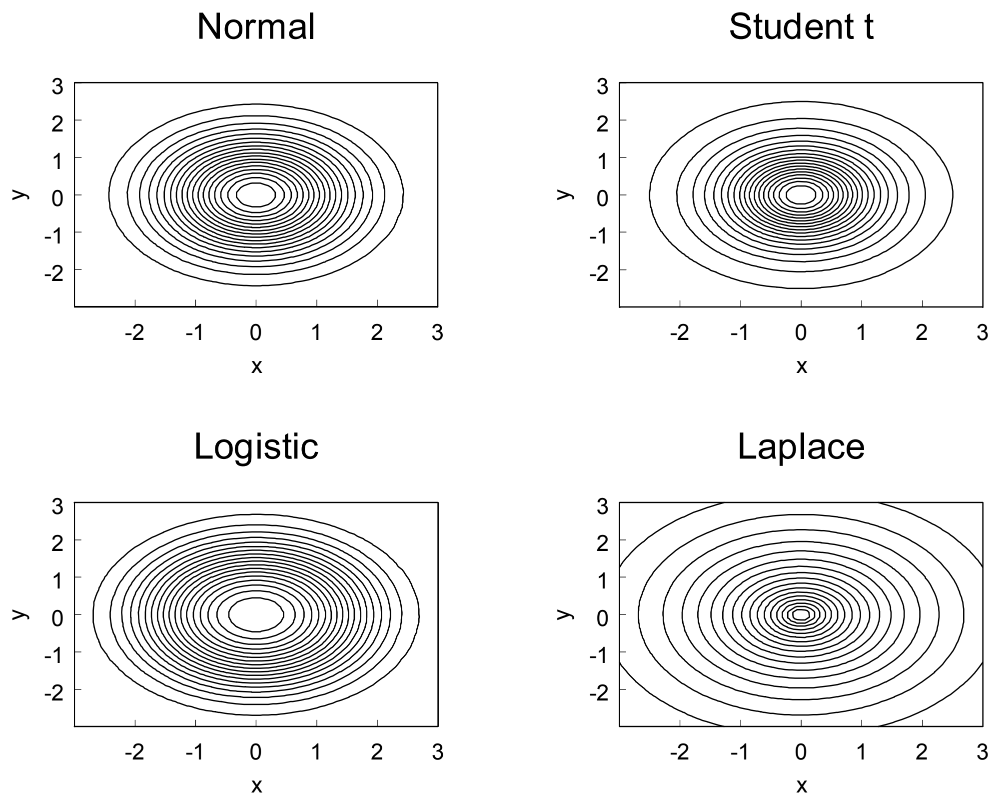

Figure 1: Contours of Bivariate Densities for Selected Elliptical Distributions

5 Application of Wang's Allocation Formula for Elliptical Distributions

In this section, we generalize Wang's capital allocation formula when the losses from an insurance company's different lines of business are jointly elliptical. As noted in the previous section, any linear combination of elliptical random variables will again belong to the class of elliptical distributions. Specifically then, the sum of elliptical random variables is again elliptical. Suppose $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ and $\mathbf{e} = (1, 1, ..., 1)^T$ is the vector of ones with dimension n. We assume henceforth that the density generator g_n exists. This is equivalent to saying that the condition

$$\int_0^\infty x^{n/2-1} g_n(x) dx < \infty \tag{15}$$

holds. See Fang, et al. (1987) and Landsman and Valdez (2002). Now define

$$Z = X_1 + \cdots + X_n = \sum_{k=1}^{n} X_k = \mathbf{e}^T \mathbf{X} \tag{16}$$

which is the sum of elliptical risks. We know from (14) that it follows that

$$Z \sim E_1\left(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e}^\mathbf{T} \boldsymbol{\Sigma} \mathbf{e}, g_1\right).$$

Denote by $\mu_Z = \mathbf{e}^T \boldsymbol{\mu} = \sum_{j=1}^n \mu_j$ and $\sigma_Z^2 = \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e} = \sum_{i,j=1}^n \sigma_{ij}$. For a density generator g_n of any elliptical random variable, we define the *tail generator* by

$$T_n(u) = \int_{\frac{1}{2}u^2}^\infty c_n g_n(x) \, dx. \tag{17}$$

It will be understood that we drop the subscript n in the univariate case. Before we state a theorem useful for finding Wang's capital allocation formula, we state a useful lemma that was proved in Landsman and Valdez (2002).

Lemma 5 *Let $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. Then for $1 \leq i \leq n$, the vector $\mathbf{X}_{i,Z} = (X_i, Z)^T$ has an elliptical distribution with the same generator, i.e., $\mathbf{X}_{i,Z} \sim E_2\left(\boldsymbol{\mu}_{i,Z}, \boldsymbol{\Sigma}_{i,Z}, g_2\right)$, where $\boldsymbol{\mu}_{i,Z} = \left(\mu_i, \sum_{j=1}^n \mu_j\right)^T$,*

$$\boldsymbol{\Sigma}_{i,Z} = \begin{pmatrix} \sigma_i^2 & \sigma_{i,Z} \\ \sigma_{i,Z} & \sigma_Z^2 \end{pmatrix},$$

and $\sigma_i^2 = \sigma_{ii}, \sigma_{i,Z} = \sum_{j=1}^{n} \sigma_{ij}, \sigma_Z^2 = \sum_{j,k=1}^{n} \sigma_{jk}.$

Lemma 6 *Let* $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ *and assume condition (15) holds. Let* T *be the tail generator defined in (17) associated with* Z *as defined in (16). Then*

$$H_\lambda[X_i, Z] = \mu_i + \lambda\rho_{i,Z}\sigma_i\sigma_Z \frac{\exp(\lambda\mu_Z)}{M_Z(\lambda)} \int_{-\infty}^{\infty} T_n(w) \exp(\lambda\sigma_Z w)\, dw \qquad (18)$$

Proof. We know from lemma (5) that we can express the joint density of $\mathbf{X}_{i,Z} = (X_i, Z)^T$ as

$$f_{i,Z}(x_i, z) = \frac{c_2}{\sqrt{|\boldsymbol{\Sigma}_{i,Z}|}} g_2\left[\tfrac{1}{2}\left(\mathbf{x}_{i,Z} - \boldsymbol{\mu}_{i,Z}\right)^T \boldsymbol{\Sigma}_{i,Z}^{-1}\left(\mathbf{x}_{i,Z} - \boldsymbol{\mu}_{i,Z}\right)\right]$$

where c_2 denotes the normalizing constant. In the bivariate case, we have

$$|\boldsymbol{\Sigma}_{i,Z}| = \begin{vmatrix} \sigma_i^2 & \sigma_{i,Z} \\ \sigma_{i,Z} & \sigma_Z^2 \end{vmatrix} = \left(1 - \rho_{i,Z}^2\right)\sigma_i^2\sigma_Z^2$$

and

$$\begin{aligned}
&\left(\mathbf{x}_{i,Z} - \boldsymbol{\mu}_{i,Z}\right)^T \boldsymbol{\Sigma}_{i,Z}^{-1}\left(\mathbf{x}_{i,Z} - \boldsymbol{\mu}_{i,Z}\right) \\
= &\frac{1}{\left(1 - \rho_{i,Z}^2\right)}\left[\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 - 2\rho_{i,Z}\left(\frac{x_i - \mu_i}{\sigma_i}\right)\left(\frac{z - \mu_Z}{\sigma_Z}\right) + \left(\frac{z - \mu_Z}{\sigma_Z}\right)^2\right] \\
= &\frac{1}{\left(1 - \rho_{i,Z}^2\right)}\left\{\left[\left(\frac{x_i - \mu_i}{\sigma_i}\right) - \rho_{i,Z}\left(\frac{z - \mu_Z}{\sigma_Z}\right)\right]^2 + \left(1 - \rho_{i,Z}^2\right)\left(\frac{z - \mu_Z}{\sigma_Z}\right)^2\right\}.
\end{aligned}$$

Next, consider

$$\begin{aligned}
&E\left[X_i \exp(\lambda Z)\right] \\
= &\frac{c_2}{\sqrt{|\boldsymbol{\Sigma}_{i,Z}|}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i e^{\lambda z} g_2\left[\tfrac{1}{2}\left(\mathbf{x}_{i,Z} - \boldsymbol{\mu}_{i,Z}\right)^T \boldsymbol{\Sigma}_{i,Z}^{-1}\left(\mathbf{x}_{i,Z} - \boldsymbol{\mu}_{i,Z}\right)\right] dx_i dz.
\end{aligned}$$

Denote this by I. Using the transformations

$$u = \frac{x_i - \mu_i}{\sigma_i} \text{ and } v = \frac{z - \mu_Z}{\sigma_Z},$$

and the property that the marginal distributions of multivariate elliptical are again elliptical distributions with the same generator, we then have

$$\begin{aligned} I &= \frac{c_2}{\sqrt{1-\rho_{i,Z}^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mu_i + \sigma_i u)\, e^{\lambda(\mu_Z + \sigma_Z v)} g_2 \left[\frac{1}{2} \frac{\left(u - \rho_{i,Z} v\right)^2}{1-\rho_{i,Z}^2} + \frac{1}{2} v^2 \right] du dv \\ &= \frac{c_2}{\sqrt{1-\rho_{i,Z}^2}} \int_{-\infty}^{\infty} e^{\lambda(\mu_Z + \sigma_Z v)} \left\{ \int_{-\infty}^{\infty} (\mu_i + \sigma_i u)\, g_2 \left[\frac{1}{2} \left(\frac{u - \rho_{i,Z} v}{\sqrt{1-\rho_{i,Z}^2}} \right)^2 + \frac{1}{2} v^2 \right] du \right\} dv \\ &= \frac{c_2}{\sqrt{1-\rho_{i,Z}^2}} \int_{-\infty}^{\infty} e^{\lambda(\mu_Z + \sigma_Z v)} I_1(v)\, dv \end{aligned}$$

where

$$I_1(v) = \int_{-\infty}^{\infty} (\mu_i + \sigma_i u)\, g_2 \left[\frac{1}{2} \left(\frac{u - \rho_{i,Z} v}{\sqrt{1-\rho_{i,Z}^2}} \right)^2 + \frac{1}{2} v^2 \right] du.$$

With one more transformation $u^* = \dfrac{u - \rho_{i,Z} v}{\sqrt{1-\rho_{i,Z}^2}}$, we get

$$\begin{aligned} I_1(v) &= \sqrt{1-\rho_{i,Z}^2} \int_{-\infty}^{\infty} \left[\mu_i + \sigma_i \left(\sqrt{1-\rho_{i,Z}^2} u^* + \rho_{i,Z} v \right) \right] g_2 \left[\frac{1}{2} \left(u^{*2} + v^2 \right) \right] du^* \\ &= \frac{\sqrt{1-\rho_{i,Z}^2}}{c_2} \int_{-\infty}^{\infty} c_2 \left[\left(\mu_i + \rho_{i,Z} \sigma_i v \right) + \sigma_i \sqrt{1-\rho_{i,Z}^2} u^* \right] g_2 \left[\frac{1}{2} \left(u'^2 + v^2 \right) \right] du^* \end{aligned}$$

and noting that

$$\int_{-\infty}^{\infty} c_2 u^* g_2 \left[\frac{1}{2} \left(u^{*2} + v^2 \right) \right] du^* = 0$$

and that

$$\int_{-\infty}^{\infty} c_2 g_2 \left[\frac{1}{2} \left(u^{*2} + v^2 \right) \right] du^* = c_1 g_1 \left(\frac{1}{2} v^2 \right),$$

it follows that

$$\begin{aligned} I &= \int_{-\infty}^{\infty} c_1 \left(\mu_i + \rho_{i,Z} \sigma_i v \right) e^{\lambda(\mu_Z + \sigma_Z v)} g_1 \left(\frac{1}{2} v^2 \right) dv \\ &= \mu_i M_Z(\lambda) + \rho_{i,Z} \sigma_i \int_{-\infty}^{\infty} c_1 v e^{\lambda(\mu_Z + \sigma_Z v)} g_1 \left(\frac{1}{2} v^2 \right) dv. \end{aligned}$$

Applying integration by parts, it can be shown that

$$\int_{-\infty}^{\infty} c_1 v e^{\lambda(\mu_Z+\sigma_Z v)} g_1\left(\frac{1}{2}v^2\right) dv = \lambda\sigma_Z \int_{-\infty}^{\infty} T(w)\, e^{\lambda(\mu_Z+\sigma_Z w)} dw$$

and the result immediately follows.■

It follows immediately that

$$H_\lambda[Z,Z] = \mu_Z + \lambda\sigma_Z^2 \frac{\exp(\lambda\mu_Z)}{M_Z(\lambda)} \int_{-\infty}^{\infty} T(w)\exp(\lambda\sigma_Z w)\, dw. \tag{19}$$

In the multivariate normal case, it can be shown that the density generator has the form $g(x) = \exp(-x)$ so that the corresponding tail generator has the expression

$$T(u) = \int_{\frac{1}{2}u^2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x} dx = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}. \tag{20}$$

Thus, from Theorem (6), we have

$$\begin{aligned} H_\lambda[X_i,Z] &= \mu_i + \lambda\rho_{i,Z}\sigma_i\sigma_Z \frac{\exp(\lambda\mu_Z)}{M_Z(\lambda)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \exp(\lambda\sigma_Z w)\, dw \\ &= \mu_i + \lambda\rho_{i,Z}\sigma_i\sigma_Z \frac{\exp(\lambda\mu_Z)}{M_Z(\lambda)} \exp\left(\frac{1}{2}\lambda^2\sigma_Z^2\right) \\ &= \mu_i + \lambda\rho_{i,Z}\sigma_i\sigma_Z = \mu_i + \lambda\sigma_{i,Z} \end{aligned}$$

and the covariance-based allocation as given in Example (3) follows. We notice from (20) that the tail generator has the form of a density function, in this case normal density. In general, we observe that with proper normalization, we can transform the tail generator into a density. Notice that

$$\begin{aligned} \int_{-\infty}^{\infty} T(w)\, dw &= \int_{-\infty}^{\infty}\int_{\frac{1}{2}w^2}^{\infty} c_n g_n(x)\, dx dw = \int_{-\infty}^{\infty}\int_{y}^{\infty} c_n y g_n\left(\frac{1}{2}y^2\right) dy dw \\ &= \int_{0}^{\infty}\int_{-y}^{y} c_n y g_n\left(\frac{1}{2}y^2\right) dw dy = 2\int_{0}^{\infty} c_n y^2 g_n\left(\frac{1}{2}y^2\right) dy \\ &= \int_{-\infty}^{\infty} c_n y^2 g_n\left(\frac{1}{2}y^2\right) dy = E\left(Z^{*2}\right) \end{aligned}$$

where Z^* is an elliptical random variable with mean 0. But, we know from (13), this is equal to

$$\int_{-\infty}^{\infty} T(w)\, dw = -\psi'(0).$$

Thus, we see

$$T^*(w) = -\frac{1}{\psi'(0)} T(w)$$

defines a proper density of a standard elliptical random variable. We can therefore express

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp\left[\lambda\left(\mu_Z + \sigma_Z w\right)\right] T_n(w)\, dw &= -\psi'(0) \int_{-\infty}^{\infty} \exp\left[\lambda\left(\mu_Z + \sigma_Z w\right)\right] T^*(w)\, dw \\
&= -\psi'(0)\, M_Z(\lambda)
\end{aligned}$$

where $Z \sim E_1\left(\mu_Z, \sigma_Z^2, g_1\right)$. Thus, we have the following result.

Theorem 7 *Let* $\mathbf{X} \sim E_n\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n\right)$ *and assume conditions (15) and* $|\psi'(0)| < \infty$ *hold. Then Wang's capital allocation formula in (9) can be expressed as*

$$K_i = -\lambda \psi'(0)\, \rho_{i,Z} \sigma_i \sigma_Z \tag{21}$$

Proof. The result immediately follows from the previous lemma:

$$\begin{aligned}
K_i &= H_\lambda\left[X_i, Z\right] - E\left(X_i\right) \\
&= \lambda \rho_{i,Z} \sigma_i \sigma_Z \frac{1}{M_Z(\lambda)} \int_{-\infty}^{\infty} \exp\left[\lambda\left(\mu_Z + \sigma_Z w\right)\right] T_n(w)\, dw \\
&= \lambda \rho_{i,Z} \sigma_i \sigma_Z \frac{1}{M_Z(\lambda)} \left[-\psi'(0)\, M_Z(\lambda)\right] \\
&= -\lambda \psi'(0)\, \rho_{i,Z} \sigma_i \sigma_Z.
\end{aligned}$$

■

Notice that by definition $\sigma_{i,Z} = \rho_{i,Z} \sigma_i \sigma_Z$ and that $-\psi'(0)\, \sigma_{i,Z} = Cov\left(X_i, Z\right)$ so that the allocation yields the covariance-based allocation principle. Wang's allocation formula is therefore considered fair for risks that belong to the family of elliptical distributions.

Example 4: Panjer's Example We now use Panjer's (2002) example to illustrate the capital allocation principle developed in this section. In that example, an insurance company with 10 lines of business is faced with the risks represented by the random vector $\mathbf{X}^T = \left(X_1, ..., X_{10}\right)$ each X_i represents the present value of losses over a specified time horizon. The estimated variance-covariance structure, $\widehat{\boldsymbol{\Sigma}}$, (in millions-squared) is given by

$$\begin{bmatrix} 7.24 & 0.00 & 0.07 & -0.07 & 0.28 & -2.71 & -0.51 & 0.28 & 0.23 & -0.21 \\ 0.00 & 20.16 & 0.05 & 1.60 & 0.05 & 1.39 & 1.14 & -0.91 & -0.81 & 1.74 \\ 0.07 & 0.05 & 0.04 & 0.00 & -0.01 & 0.08 & 0.01 & -0.02 & -0.02 & -0.07 \\ -0.07 & 1.60 & 0.00 & 1.74 & 0.17 & 0.26 & 0.19 & -0.14 & 0.18 & -0.79 \\ 0.28 & 0.05 & -0.01 & 0.17 & 0.32 & -0.24 & 0.01 & -0.02 & 0.08 & -0.01 \\ -2.71 & 1.39 & 0.08 & 0.26 & -0.24 & 14.98 & 0.43 & -0.33 & -1.89 & -1.60 \\ -0.51 & 1.14 & 0.01 & 0.19 & 0.01 & 0.43 & 2.53 & -0.38 & 0.13 & 0.58 \\ 0.28 & -0.91 & -0.02 & -0.14 & -0.02 & -0.33 & -0.38 & 0.92 & -0.16 & -0.40 \\ 0.23 & -0.81 & -0.02 & 0.18 & 0.08 & -1.89 & 0.13 & -0.16 & 1.12 & 0.58 \\ -0.21 & -1.74 & -0.07 & -0.79 & -0.01 & -1.60 & 0.58 & -0.40 & 0.58 & 6.71 \end{bmatrix}$$

and the estimated mean vector (in millions) is given by

$$\widehat{\boldsymbol{\mu}}^T = (25.69, 37.84, 0.85, 12.70, 0.15, 24.05, 14.41, 4.49, 4.39, 9.56)\,.$$

Panjer (2002) reported the correlation matrix, but we report here the variance-covariance structure which is what we practically need to compute the allocation based on principles developed in this paper. Given this variance-covariance structure and assuming for example multivariate normal, we are able to calculate appropriate capital allocations by lines of business by making use of equation (21). The resulting allocation is given by

$$\mathbf{K}^T = (2.72, 12.55, 0.08, 1.92, 0.37, 6.27, 2.51, -0.70, -0.30, 1.89)\,,$$

expressed in millions, with the final total capital equal to 27.31 million. For example, of the 27.31 million total capital, 12.55 million is being allocated for the second line of business, comprising approximately 46% of capital. It is interesting to note that not all allocations have to be positive, and in fact, in this example the allocations assigned to lines 8 and 9 are negative. This is primarily due to the negative covariances between each of these lines of business and the company as a whole. It is instructive to view this as a form of diversification benefit, in that these lines of business may reduce capital requirements for the firm as a whole. We report the percentage of allocation for each line of business as a dot-plot in Figure 2. Now using the same estimates of $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ but a different family belonging to the class of distributions, for example Student-t or logistic (see Appendix for other members of elliptical distributions), we will end up with the same resulting capital allocation. However, we caution the reader that in practice, if we assume individual losses or claims from individual lines of business come from a different distribution, the resulting estimates of $\mathbf{\Sigma}$ and $\boldsymbol{\mu}$ can vary so that the resulting capital allocation will be different.

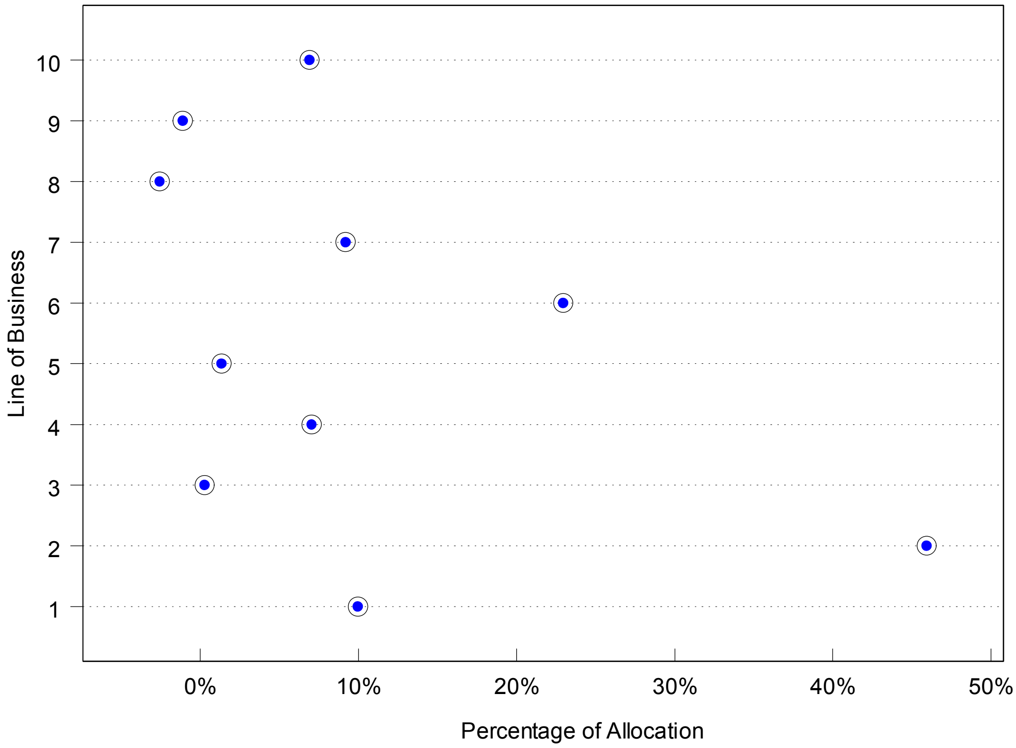

Figure 2: Percentage of Allocation based on Panjer's Example

6 Conclusion

This article examines the issue of "fair allocation" which consists of fairly dividing the total capital of a diversified insurance company across its various business units. The term "capital allocation" has been used in finance literature where a similar concept of fair division of capital in a diversified portfolio of investments has been investigated. In particular, we examine the capital allocation formula proposed by Wang (2002) where he derived the explicit form in the case where risks are multivariate normal and we extend this result to a larger class of multivariate elliptical risks allowing for a greater flexibility in modelling tails of distributions. This is particularly important for insurance companies which are often concerned with tails of loss distributions. Members of this class consist of distributions considered symmetric in the univariate case. This paper develops the theory of what constitutes a "fair allocation" between lines of business of an insurance company and we stated that there are generally three criteria that must be satisfied: no undercut, symmetry, and consistency. We justified why each of these criteria may be important. We then considered a number of possible allocation methodologies, most notably the covariance-based allocation principle, and proved that it provides a fair or rational allocation. We then examined the allocation methodology recommended in Wang(2002) within the context of the family of elliptical distributions and proved that the resulting allocation is quite similar to the covariance-based principle. Finally, an example is given using the data in Panjer (2002). There remains much research to be done into other possible capital allocation methodologies, and generalizing them to other distributions.

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APPENDIX
Some Members of the Class of Elliptical Distributions with their Density/Characteristic Generators

Family	Density $g_n(u)$ or characteristic $\psi(u)$ generators
Bessel	$g_n(u) = (2u/b)^{a/2} K_a\left[(2u/b)^{1/2}\right]$, $a > -n/2, b > 0$ where $K_a(\cdot)$ is the modified Bessel function of the 3rd kind
Cauchy	$g_n(u) = (1+2u)^{-(n+1)/2}$
Exponential Power	$g_n(u) = \exp\left[-r(2u)^s\right]$, $r, s > 0$
Laplace	$g_n(u) = \exp(-2\lvert u\rvert)$
Logistic	$g_n(u) = \dfrac{\exp(-2u)}{\left[1+\exp(-2u)\right]^2}$
Normal	$g_n(u) = \exp(-u)$; $\psi(u) = \exp(-u)$
Stable Laws	$\psi(u) = \exp\left[-r(2u)^{s/2}\right]$, $0 < s \leq 2, r > 0$
Student t	$g_n(u) = \left(1+\dfrac{2u}{m}\right)^{-(n+m)/2}$, $m > 0$ an integer