

# Bounds for Sums of Non-Independent Log-Elliptical Random Variables

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## ABSTRACT

In this paper, we construct upper and lower bounds for the distribution of a sum of non-independent log-elliptical random variables. These bounds are applications of the ideas developed in Kaas, Dhaene & Goovaerts (2000). The class of multivariate log-elliptical random variables is an extension of the class of multivariate log-normal random variables. Hence, the results presented here are natural extensions of the results presented in Dhaene, Denuit, Goovaerts, Kaas & Vyncke (2002a, 2002b), where bounds for sums of log-normal random variables were derived. The upper bound is based on the idea of replacing the sum of log-elliptical random variables by a sum of random variables with the same marginals, but with a dependency structure described by the comonotonic copula. Lower bounds and improved upper bounds are constructed by including additional information about the dependency structure by introducing some conditioning random variable.

## 1 Introduction

Sums of non-independent random variables occur in several situations in insurance and finance.

As a first example, consider a portfolio of  $n$  insurance risks  $X_1, X_2, \dots, X_n$ . The aggregate claims  $S$  is defined to be the sum of these individual risks:

$$S = \sum_{k=1}^n X_k, \tag{1}$$

where generally the risks are non-negative random variables, i.e.  $X_k \geq 0$ . Knowledge of the distribution of this sum provides essential information for the insurance company and can be used as an input in the calculation of premiums and reserves.

A particularly important problem is the determination of stop-loss premiums of the aggregate claims  $S$ . Suppose that the insurance company agrees to enter into a stop-loss reinsurance contract where total

claims beyond a pre-specified amount  $d$  will be covered by the reinsurer. The stop-loss premium with retention  $d$  is defined as

$$E[(S - d)_+] = \int_d^\infty \bar{F}_S(x) dx, \quad (2)$$

where  $\bar{F}_S(x) = 1 - F_S(x) = \Pr(S > x)$  and  $(s - d)_+ = \max(s - d, 0)$ .

In classical risk theory, the individual risks  $X_k$  are typically assumed to be mutually independent, mainly because computation of the aggregate claims becomes more tractable in this case. For special families of individual claim distributions, one may determine the exact form of the distribution for the aggregate claims. Several exact and approximate recursive methods have been proposed for computing the aggregate claims in case of discrete marginal distributions, see e.g. Dhaene & De Pril (1994) and Dhaene & Vandebroek (1995). Approximating the aggregate claims distribution by a normal distribution with the same first and second moment is often not satisfactory for the insurance practice, where the third central moment is often substantially different from 0. In this case, approximations based on a translated gamma distribution or the normal power approximation will perform better, see e.g. Kaas, Goovaerts, Dhaene & Denuit (2001).

It is important to note that all standard actuarial methods mentioned above for determining the aggregate claims distribution are only applicable in case the individual risks are assumed to be mutually independent. However, there are situations where the independence assumption is questionable, e.g. in a situation where the individual risks  $X_k$  are influenced by the same economic or physical environment.

In finance, a portfolio of  $n$  investment positions may be faced with potential losses  $L_1, L_2, \dots, L_n$  over a given reference period, e.g. 1 month. The total potential loss  $L$  for this portfolio is then given by

$$L = \sum_{k=1}^n L_k. \quad (3)$$

As the returns on the different investment positions will in general be non-independent, it is clear that  $L$  will be a sum of non-independent random variables. Quantities of interest are quantiles of the distribution of (3), which in finance are called Values-at-Risk. Regulatory bodies require financial institutions like banks and investment firms to meet risk-based capital requirements for their portfolio holdings. These requirements are often expressed in terms of a Value-at-Risk or some other risk measure which depends on the distribution of the sum in (3).

A related problem is determining an investment portfolio's total rate of return. Suppose  $R_1, R_2, \dots, R_n$  denote the random yearly rates of return of  $n$  different assets in a portfolio and suppose  $w_1, w_2, \dots, w_n$  denote the weights in the portfolio. Then the total portfolio's yearly rate of return is given by

$$R = \sum_{k=1}^n w_k R_k, \quad (4)$$

which is clearly a sum of non-independent random variables.

As pointed out by Dhaene et al. (2002a), a topic that is currently receiving increasing attention is the combination of actuarial and financial risks. To illustrate, consider random payments of  $X_k$  to be made at times  $k$  for the next  $n$  periods. Further, suppose that the stochastic discount factor over

the period  $(0, k)$  is the random variable  $Y_k$ . Hence, an amount of 1 at time 0 is assumed to grow to a stochastic amount  $Y_k^{-1}$  at time  $k$ . The present value random variable  $S$  is defined as the scalar product of the payment vector  $(X_1, X_2, \dots, X_n)$  and the discount vector  $(Y_1, Y_2, \dots, Y_n)$ :

$$S = \sum_{k=1}^n X_k Y_k. \quad (5)$$

The present value quantity in (5) is of considerable importance for computing reserves and capital requirements for long term insurance business. The random variables  $X_k Y_k$  will be non-independent not only because in any realistic model, the discount factors will be rather strongly positive dependent, but also because the claim amounts  $X_k$  can often not be assumed to be mutually independent.

As illustrated by the examples above, it is important to be able to determine the distribution function of sums of random variables in the case that the individual random variables involved are not assumed to be mutually independent. In general, this task is difficult to perform or even impossible because the dependency structure is unknown or too cumbersome to work with. In this paper, we develop approximations for sums involving non-independent log-elliptical random variables. The results presented here are natural extensions of the ideas developed in Dhaene et al. (2002a, 2002b) where they specifically constructed bounds for sums of dependent log-normal distributions.

In Sections 2 and 3, we introduce spherical, elliptical and log-elliptical distributions, as in Fang, Kotz & Ng (1990). We develop and repeat some results that will be used in later sections. Most results proved elsewhere are simply stated, but some basic useful results are also proved. In Section 4, we summarize the ideas developed in Dhaene et al. (2002a, 2002b) regarding the construction of upper and lower bounds for sums of non-independent random variables. In Section 5 we determine these bounds in the case of functions of sums involving log-elliptical distributions. Section 6 concludes the paper.

## 2 Elliptical and Spherical Distributions

### 2.1 Definition of Elliptical Distributions

It is well-known that a random vector  $\mathbf{Y} = (Y_1, \dots, Y_n)^T$  is said to have a  $n$ -dimensional normal distribution if

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \mathbf{A} \mathbf{Z}, \quad (6)$$

where  $\mathbf{Z} = (Z_1, \dots, Z_m)^T$  is a random vector consisting of  $m$  mutually independent standard normal random variables,  $\mathbf{A}$  is a  $n \times m$  matrix,  $\boldsymbol{\mu}$  is a  $n \times 1$  vector and  $\stackrel{d}{=}$  stands for ‘‘equality in distribution’’. Equivalently, one can say that  $\mathbf{Y}$  is normal if its characteristic function is given by

$$E \left[ \exp(it^T \mathbf{Y}) \right] = \exp(it^T \boldsymbol{\mu}) \exp\left(-\frac{1}{2} \mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}\right), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n). \quad (7)$$

for some fixed vector  $\boldsymbol{\mu}(n \times 1)$  and some fixed matrix  $\boldsymbol{\Sigma}(n \times n)$ . For random vectors belonging to the class of multivariate normal distributions with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$ , we shall use the notation  $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . It is well-known that the vector  $\boldsymbol{\mu}$  is the mean vector and that the matrix  $\boldsymbol{\Sigma}$  is the variance-covariance matrix. Note that the relation between  $\boldsymbol{\Sigma}$  and  $\mathbf{A}$  is given by  $\boldsymbol{\Sigma} = \mathbf{A} \mathbf{A}^T$ .

The class of multivariate elliptical distributions is a natural extension of the class of multivariate normal distributions.

DEFINITION 1. *The random vector  $\mathbf{Y} = (Y_1, \dots, Y_n)^T$  is said to have an elliptical distribution with parameters the vector  $\boldsymbol{\mu}(n \times 1)$  and the matrix  $\boldsymbol{\Sigma}(n \times n)$  if its characteristic function can be expressed as*

$$E[\exp(it^T \mathbf{Y})] = \exp(it^T \boldsymbol{\mu}) \phi(\mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n), \quad (8)$$

for some scalar function  $\phi$  and where  $\boldsymbol{\Sigma}$  is given by

$$\boldsymbol{\Sigma} = \mathbf{A} \mathbf{A}^T \quad (9)$$

for some matrix  $\mathbf{A}(n \times m)$ .

If  $\mathbf{Y}$  has the elliptical distribution as defined above, we shall write  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  and say that  $\mathbf{Y}$  is elliptical. The function  $\phi$  is called the *characteristic generator* of  $\mathbf{Y}$ . Hence, the characteristic generator of the multivariate normal distribution is given by  $\phi(u) = \exp(-u/2)$ .

It is well-known that the characteristic function of a random vector always exists and that there is a one-to-one correspondence between distribution functions and characteristic functions. Note however that not every function  $\phi$  can be used to construct a characteristic function of an elliptical distribution. Obviously, this function  $\phi$  should fulfill the requirement  $\phi(0) = 1$ . A necessary and sufficient condition for the function  $\phi$  to be a characteristic generator of an  $n$ -dimensional elliptical distribution is given in Theorem 2.2 of Fang, et al. (1990).

In the sequel, we shall denote the elements of  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  by

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T \quad (10)$$

and

$$\boldsymbol{\Sigma} = (\sigma_{kl}) \quad \text{for } k, l = 1, 2, \dots, n, \quad (11)$$

respectively. Note that (9) guarantees that the matrix  $\boldsymbol{\Sigma}$  is symmetric, positive definite and has positive elements on the first diagonal. Hence, for any  $k$  and  $l$ , one has that  $\sigma_{kl} = \sigma_{lk}$ , whereas  $\sigma_{kk} \geq 0$  which will often be denoted by  $\sigma_k^2$ .

It is interesting to note that in the one-dimensional case, the class of elliptical distributions consists mainly of the class of symmetric distributions which include well-known distributions like normal and student  $t$ .

An  $n$ -dimensional random vector  $\mathbf{Y} = (Y_1, \dots, Y_n)^T$  is normal with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  if and only if for any vector  $\mathbf{b}$ , the linear combination  $\mathbf{b}^T \mathbf{Y}$  of the marginals  $Y_k$  has a univariate normal distribution with mean  $\mathbf{b}^T \boldsymbol{\mu}$  and variance  $\mathbf{b}^T \boldsymbol{\Sigma} \mathbf{b}$ . It is straightforward to generalize this result into the case of multivariate elliptical distributions.

THEOREM 1. *The  $n$ -dimensional random vector  $\mathbf{Y}$  is elliptical with parameters  $\boldsymbol{\mu}(n \times 1)$  and  $\boldsymbol{\Sigma}(n \times n)$ , notation  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ , if and only if for any vector  $\mathbf{b}(n \times 1)$ , one has that*

$$\mathbf{b}^T \mathbf{Y} \sim E_1(\mathbf{b}^T \boldsymbol{\mu}, \mathbf{b}^T \boldsymbol{\Sigma} \mathbf{b}, \phi).$$

From Theorem 1, we find in particular that for  $k = 1, 2, \dots, n$ ,

$$Y_k \sim E_1(\mu_k, \sigma_k^2, \phi). \quad (12)$$

Hence, the marginal components of a multivariate elliptical distribution have an elliptical distribution with the same characteristic generator.

As

$$S = \sum_{k=1}^n Y_k = \mathbf{e}^T \mathbf{Y}$$

where  $\mathbf{e}(n \times 1) = (1, 1, \dots, 1)^T$ , it follows that

$$S \sim E_1(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e}, \phi) \quad (13)$$

where  $\mathbf{e}^T \boldsymbol{\mu} = \sum_{k=1}^n \mu_k$  and  $\mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e} = \sum_{k=1}^n \sum_{l=1}^n \sigma_{kl}$ .

In the following theorem, it is stated that any random vector with components that are linear combinations of the components of an elliptical distribution is again an elliptical distribution with the same characteristic generator.

**THEOREM 2.** *For any matrix  $\mathbf{B}(m \times n)$ , any vector  $\mathbf{c}(m \times 1)$  and any random vector  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ , we have that*

$$\mathbf{B}\mathbf{Y} + \mathbf{c} \sim E_m(\mathbf{B}\boldsymbol{\mu} + \mathbf{c}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^T, \phi). \quad (14)$$

It is easily seen that Theorem 2 is a generalization of Theorem 1.

## 2.2 Moments of Elliptical Distributions

Suppose that for a random vector  $\mathbf{Y}$ , the expectation  $E(\prod_{k=1}^n Y_k^{r_k})$  exists for some set of non-negative integers  $r_1, r_2, \dots, r_n$ . Then this expectation can be found from the relation

$$E\left(\prod_{k=1}^n Y_k^{r_k}\right) = \frac{1}{i^{r_1+r_2+\dots+r_n}} \left\{ \frac{\partial^{r_1+r_2+\dots+r_n}}{\partial t_1^{r_1} \partial t_2^{r_2} \dots \partial t_n^{r_n}} E[\exp(it^T \mathbf{Y})] \right\}_{\mathbf{t}=\mathbf{0}} \quad (15)$$

where  $\mathbf{0}(n \times 1) = (0, 0, \dots, 0)^T$ .

The moments of  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  do not necessarily exist. However, from (8) and (15) we deduce that if  $E(Y_k)$  exists, then it will be given by

$$E(Y_k) = \mu_k \quad (16)$$

so that  $E(\mathbf{Y}) = \boldsymbol{\mu}$ , if the mean vector exists. Moreover, if  $Cov(Y_k, Y_l)$  and/or  $Var(Y_k)$  exist, then they will be given by

$$Cov(Y_k, Y_l) = -2\phi'(0) \sigma_{kl} \quad (17)$$

and/or

$$Var(Y_k) = -2\phi'(0) \sigma_k^2, \quad (18)$$

where  $\phi'$  denotes the first derivative of the characteristic generator. In short, if the covariance matrix of  $\mathbf{Y}$  exists, then it is given by

$$Cov(\mathbf{Y}) = -2\phi'(0) \boldsymbol{\Sigma}.$$

A necessary condition for this covariance matrix to exist is

$$|\phi'(0)| < \infty,$$

see Cambanis et al. (1981).

In the following theorem, we prove that any multivariate elliptical distribution with mutually independent components must necessarily be multivariate normal, see Kelker (1970).

**THEOREM 3.** *Let  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  with mutually independent components  $Y_k$ . Assume that the expectations and variances of the  $Y_k$  exist and that  $var(Y_k) > 0$ . Then it follows that  $\mathbf{Y}$  is multivariate normal.*

*Proof.* Independence of the random variables and existence of their expectations imply that the covariances exist and are equal to 0. Hence, we find that  $\boldsymbol{\Sigma}$  is a diagonal matrix, and that

$$\phi(\mathbf{t}^T \mathbf{t}) = \prod_{k=1}^n \phi(t_k^2)$$

holds for all  $n$ -dimensional vectors  $\mathbf{t}$ . This equation is known as Hamel's equation, and its solution has the form

$$\phi(x) = e^{-\alpha x},$$

for some positive constant  $\alpha$  satisfying  $\alpha = -\phi'(0)$ . To prove this, first note that

$$\phi(\mathbf{t}^T \mathbf{t}) = \phi\left(\sum_{k=1}^n t_k^2\right) = \prod_{k=1}^n \phi(t_k^2)$$

or equivalently,

$$\phi(u_1 + \cdots + u_n) = \phi(u_1) \cdots \phi(u_n).$$

Consider the partial derivative with respect to  $u_k$  for some  $k = 1, 2, \dots, n$ , we have

$$\begin{aligned} \frac{\partial \phi}{\partial u_k} &= \lim_{h \rightarrow 0} \frac{\phi(u_1 + \cdots + (u_k + h) + \cdots + u_n) - \phi(u_1 + \cdots + u_n)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\phi(u_1 + \cdots + u_n) \phi(h) - \phi(u_1 + \cdots + u_n)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\phi(u_1 + \cdots + u_n) [\phi(h) - \phi(0)]}{h} \\ &= \phi(u_1) \cdots \phi(u_n) \phi'(0). \end{aligned}$$

But the left-hand side is

$$\frac{\partial \phi}{\partial u_k} = \phi(u_1) \cdots \phi'(u_k) \cdots \phi(u_n) = \phi(u_1) \cdots \phi(u_n) \frac{\phi'(u_k)}{\phi(u_k)}.$$

Thus, equating the two we get

$$\frac{\phi'(u_k)}{\phi(u_k)} = \phi'(0)$$

which gives the desired solution  $\phi(x) = e^{-\alpha x}$  with  $\alpha = -\phi'(0)$ . Thus, this leads to the characteristic generator of a multivariate normal.  $\square$

### 2.3 Multivariate Densities of Elliptical Distributions

An elliptically distributed random vector  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  does not necessarily possess a multivariate density function  $f_{\mathbf{Y}}(\mathbf{y})$ . A necessary condition for  $\mathbf{Y}$  to possess a density is that  $\text{rank}(\boldsymbol{\Sigma}) = n$ . In the case of a multivariate normal random vector  $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ , the density is well-known to be

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\sqrt{(2\pi)^n |\boldsymbol{\Sigma}|}} \exp \left[ -\frac{1}{2} (\mathbf{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}) \right]. \quad (19)$$

For elliptical distributions, one can prove that if  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  has a density, then it will be of the form

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{c}{\sqrt{|\boldsymbol{\Sigma}|}} g \left[ (\mathbf{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}) \right] \quad (20)$$

for some non-negative function  $g(\cdot)$  satisfying the condition

$$0 < \int_0^\infty z^{n/2-1} g(z) dz < \infty \quad (21)$$

and a normalizing constant  $c$  given by

$$c = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[ \int_0^\infty z^{n/2-1} g(z) dz \right]^{-1}. \quad (22)$$

Also, the opposite statement holds: Any non-negative function  $g(\cdot)$  satisfying the condition (21) can be used to define an  $n$ -dimensional density  $\frac{c}{\sqrt{|\boldsymbol{\Sigma}|}} g \left[ (\mathbf{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}) \right]$  of an elliptical distribution, with  $c$  given by (22). The function  $g(\cdot)$  is called the *density generator*. One sometimes writes  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{g})$  for the  $n$ -dimensional elliptical distributions generated from the function  $g(\cdot)$ . A detailed proof of these results, using spherical transformations of rectangular coordinates, can be found in Landsman & Valdez (2002).

Note that for a given characteristic generator  $\phi$ , the density generator  $g$  and/or the normalizing constant  $c$  may depend on the dimension of the random vector  $\mathbf{Y}$ . Often one considers the class of elliptical distributions of dimensions  $1, 2, 3, \dots$ , all derived from the same characteristic generator  $\phi$ . In

case these distributions have a density, we will denote their respective density generators by  $g_n$  where the subscript  $n$  denotes the dimension of the random vector  $\mathbf{Y}$ .

From (19), one immediately finds that the density generators and the corresponding normalizing constants of the multivariate normal random vectors  $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  for  $n = 1, 2, \dots$  are given by

$$g_n(u) = \exp(-u/2) \tag{23}$$

and

$$c_n = (2\pi)^{-n/2}, \tag{24}$$

respectively.

In Table 1, we consider some well-known families of the class of multivariate elliptical distributions. Each family consists of all elliptical distributions constructed from one particular characteristic generator  $\phi(u)$ . For more details about these families of elliptical distributions, see Landsman & Valdez (2003) and the references in their paper.

Table 1  
Some Well-Known Families of Elliptical Distributions with  
their Characteristic Generator and/or Density Generators.

Family	Density $g_n(u)$ or characteristic $\phi(u)$ generators
Bessel	$g_n(u) = (u/b)^{a/2} K_a \left[ (u/b)^{1/2} \right]$ , $a > -n/2, b > 0$ where $K_a(\cdot)$ is the modified Bessel function of the 3rd kind
Cauchy	$g_n(u) = (1 + u)^{-(n+1)/2}$
Exponential Power	$g_n(u) = \exp[-r(u)^s]$ , $r, s > 0$
Laplace	$g_n(u) = \exp(- u )$
Logistic	$g_n(u) = \frac{\exp(-u)}{[1 + \exp(-u)]^2}$
Normal	$g_n(u) = \exp(-u/2)$ ; $\phi(u) = \exp(-u/2)$
Stable Laws	$\phi(u) = \exp \left[ -r(u)^{s/2} \right]$ , $0 < s \leq 2, r > 0$
Student- $t$	$g_n(u) = \left( 1 + \frac{u}{m} \right)^{-(n+m)/2}$ , $m > 0$ an integer

As an example, let us consider the elliptical Student- $t$  distribution  $E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ , with  $g_n(u) = \left( 1 + \frac{u}{m} \right)^{-(n+m)/2}$ . We will denote this multivariate distribution (with  $m$  degrees of freedom) by



$t_n^{(m)}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . Its multivariate density is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} \left( 1 + \frac{(\mathbf{y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu})}{m} \right)^{-(n+m)/2}. \quad (25)$$

In order to determine the normalizing constant, first note from (22) that

$$c_n = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[ \int_0^\infty z^{n/2-1} g(z) dz \right]^{-1} = \frac{\Gamma(n/2)}{\pi^{n/2}} \left[ \int_0^\infty z^{n/2-1} \left( 1 + \frac{z}{m} \right)^{-(n+m)/2} dz \right]^{-1}.$$

Performing the substitution  $u = 1 + (z/m)$ , we find

$$\int_0^\infty z^{n/2-1} \left( 1 + \frac{z}{m} \right)^{-(n+m)/2} dz = m^{n/2} \int_1^\infty (1 - u^{-1})^{n/2-1} u^{-m/2-1} du.$$

Making one more substitution  $v = 1 - u^{-1}$ , we get

$$\int_0^\infty z^{n/2-1} \left( 1 + \frac{z}{m} \right)^{-(n+m)/2} dz = m^{n/2} \frac{\Gamma(n/2) \Gamma(m/2)}{\Gamma((n+m)/2)},$$

from which we find

$$c_n = \frac{\Gamma((n+m)/2)}{(m\pi)^{n/2} \Gamma(m/2)}. \quad (26)$$

From Theorem (1) and (12), we have that the marginals of the multivariate elliptical Student- $t$  distribution are again Student- $t$  distributions, hence,  $Y_k \sim t_1^{(m)}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ . The results above lead to

$$f_{Y_k}(y) = \frac{\Gamma\left(\frac{m+1}{2}\right)}{(m\pi)^{1/2} \Gamma\left(\frac{m}{2}\right)} \frac{1}{\sigma_k} \left[ 1 + \frac{1}{m} \left( \frac{y - \mu_k}{\sigma_k} \right)^2 \right]^{-(m+1)/2}, \quad k = 1, 2, \dots, n, \quad (27)$$

which is indeed the well-known density of a univariate Student- $t$  random variable with  $m$  degrees of freedom. Its mean is

$$E(Y_k) = \mu_k, \quad (28)$$

whereas it can be verified that its variance is given by

$$Var(Y_k) = \frac{m}{m-2} \sigma_k^2, \quad (29)$$

provided the degrees of freedom  $m > 2$ . Note that  $\frac{m}{m-2} = -2\phi'(0)$ , where  $\phi$  is the characteristic generator of the family of Student- $t$  distributions with  $m$  degrees of freedom.

In order to derive the characteristic function of  $Y_k$ , note that

$$\begin{aligned} E(e^{itY_k}) &= \frac{\Gamma\left(\frac{m+1}{2}\right)}{(m\pi)^{1/2} \Gamma\left(\frac{m}{2}\right)} e^{i\mu_k t} \int_{-\infty}^\infty e^{i\sigma_k t z} \left( 1 + \frac{1}{m} z^2 \right)^{-(m+1)/2} dz \\ &= 2 \frac{\Gamma\left(\frac{m+1}{2}\right)}{(m\pi)^{1/2} \Gamma\left(\frac{m}{2}\right)} \frac{e^{i\mu_k t}}{\sigma_k (m\sigma_k^2)^{-(m+1)/2}} \int_0^\infty \cos(tz) (m\sigma_k^2 + z^2)^{-(m+1)/2} dz. \end{aligned} \quad (30)$$

Hence, from Gradshteyn & Ryzhik (2000, p. 907), we find that the characteristic function of  $Y_k \sim t_1^{(m)}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  is given by

$$E(e^{itY_k}) = e^{i\boldsymbol{\mu}_k t} \frac{1}{2^{m/2-1} \Gamma(\frac{m}{2})} (t\sqrt{m}\sigma_k)^{m/2} K_{m/2}(t\sqrt{m}\sigma_k), \quad (31)$$

where  $K_\nu(\cdot)$  is the Bessel function of the second kind. For a similar derivation, see Witkovsky (2001). Observe that equation (31) can then be used to find the characteristic generator for the family of Student- $t$  distributions. Furthermore, note that if the random vector  $\mathbf{Y}$  has independent components  $Y_k$  that are Student- $t$  distributed with  $m$  degrees of freedom, it follows from Theorem (3) that  $\mathbf{Y}$  cannot belong to the family of elliptical distributed random vectors.

## 2.4 Spherical Distributions

An  $n$ -dimensional random vector  $\mathbf{Z} = (Z_1, \dots, Z_n)^T$  is said to have a multivariate standard normal distribution if all the  $Z_i$ 's are mutually independent and standard normally distributed. We will write this as  $\mathbf{Z} \sim N_n(\mathbf{0}_n, \mathbf{I}_n)$ , where  $\mathbf{0}_n$  is the  $n$ -vector with  $i$ -th element  $E(Z_i) = 0$  and  $\mathbf{I}_n$  is the  $n \times n$  covariance matrix which equals the identity matrix in this case. The characteristic function of  $\mathbf{Z}$  is given by

$$E[\exp(i\mathbf{t}^T \mathbf{Z})] = \exp(-\frac{1}{2}\mathbf{t}^T \mathbf{t}), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n). \quad (32)$$

Hence from (32), we find that the characteristic generator of  $N_n(\mathbf{0}_n, \mathbf{I}_n)$  is given by  $\phi(u) = \exp(-u/2)$ . The class of multivariate spherical distributions is an extension of the class of standard multivariate normal distributions.

**DEFINITION 2.** A random vector  $\mathbf{Z} = (Z_1, \dots, Z_n)^T$  is said to have an  $n$ -dimensional spherical distribution with characteristic generator  $\phi$  if  $\mathbf{Z} \sim E_n(\mathbf{0}_n, \mathbf{I}_n, \phi)$ .

We will often use the notation  $S_n(\phi)$  for  $E_n(\mathbf{0}_n, \mathbf{I}_n, \phi)$  in the case of spherical distributions. From the definition above, we find the following corollary.

**COROLLARY 1.**  $\mathbf{Z} \sim S_n(\phi)$  if and only if

$$E[\exp(i\mathbf{t}^T \mathbf{Z})] = \phi(\mathbf{t}^T \mathbf{t}), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n).. \quad (33)$$

Consider an  $m$ -dimensional random vector  $\mathbf{Y}$  such that

$$\mathbf{Y} \stackrel{d}{=} \boldsymbol{\mu} + \mathbf{A}\mathbf{Z}, \quad (34)$$

for some vector  $\boldsymbol{\mu}(n \times 1)$ , some matrix  $\mathbf{A}(n \times m)$  and some  $m$ -dimensional elliptical random vector  $\mathbf{Z} \sim S_m(\phi)$ . Then it is straightforward to prove that  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ , where the variance-covariance matrix is given by  $\boldsymbol{\Sigma} = \mathbf{A}\mathbf{A}^T$ .

From equation (33), it immediately follows that the correlation matrices of members of the class of spherical distributions are identical. That is, if we let  $Corr(\mathbf{Z}) = (r_{ij})$  be the correlation matrix, then

$$r_{ij} = \frac{Cov(Z_i, Z_j)}{\sqrt{Var(Z_i)Var(Z_j)}} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}.$$

The factor  $-2\phi'(0)$  in the covariance matrix as shown in (17) enters into the covariance structure but cancels in the correlation matrix. Note that although the correlations between different components of a spherical distributed random variable are 0, this does not imply that these components are mutually independent. As a matter of fact, it can only be independent if they belong to the family of multivariate normal distributions.

Observe that from the characteristic functions of  $\mathbf{Z}$  and  $\mathbf{a}^T\mathbf{Z}$ , one immediately finds the following result.

LEMMA 1.  $\mathbf{Z} \sim S_n(\phi)$  if and only if for any  $n$ -dimensional vector  $\mathbf{a}$ , one has

$$\frac{\mathbf{a}^T\mathbf{Z}}{\sqrt{\mathbf{a}^T\mathbf{a}}} \sim S_1(\phi). \quad (35)$$

As a special case of this result, we find that any component  $Z_i$  of  $\mathbf{Z}$  has a  $S_1(\phi)$  distribution.

From the results concerning elliptical distributions, we find that if a spherical random vector  $\mathbf{Z} \sim S_n(\phi)$  possess a density  $f_{\mathbf{Z}}(\mathbf{z})$ , then it will have the form

$$f_{\mathbf{Z}}(\mathbf{z}) = cg(\mathbf{z}^T\mathbf{z}), \quad (36)$$

where the density generator  $g$  satisfies the condition (21) and the normalizing constant  $c$  satisfies (22). Furthermore, the opposite also holds: any non-negative function  $g(\cdot)$  satisfying the condition (21) can be used to define an  $n$ -dimensional density  $cg(\mathbf{z}^T\mathbf{z})$  of a spherical distribution with the normalizing constant  $c$  satisfying (22). One often writes  $S_n(g)$  for the  $n$ -dimensional spherical distribution generated from the density generator  $g(\cdot)$ .

## 2.5 Conditional Distributions

It is well-known that if  $(Y, \Lambda)$  has a bivariate normal distribution, then the conditional distribution of  $Y$ , given that  $\Lambda = \lambda$ , is normal with mean and variance given by

$$E(Y | \Lambda = \lambda) = E(Y) + r(Y, \Lambda) \frac{\sigma_Y}{\sigma_\Lambda} [\lambda - E(\Lambda)] \quad (37)$$

and

$$Var(Y | \Lambda = \lambda) = \left(1 - r(Y, \Lambda)^2\right) \sigma_Y^2, \quad (38)$$

where  $r(Y, \Lambda)$  is the correlation coefficient between  $Y$  and  $\Lambda$ :

$$r(Y, \Lambda) = \frac{Cov(Y, \Lambda)}{\sqrt{Var(Y) Var(\Lambda)}}. \quad (39)$$

In the following theorem, it is stated that this conditioning result can be generalized to the class of bivariate elliptical distributions. This result will be useful for developing bounds for sums of random variables in Section 4.

**THEOREM 4.** *Let the random vector  $\mathbf{Y} = (Y_1, \dots, Y_n)^T \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  with density generator denoted by  $g_n(\cdot)$ . Define  $Y$  and  $\Lambda$  to be linear combinations of the variates of  $\mathbf{Y}$ , i.e.  $Y = \boldsymbol{\alpha}^T \mathbf{Y}$  and  $\Lambda = \boldsymbol{\beta}^T \mathbf{Y}$ , for some  $\boldsymbol{\alpha}^T = (\alpha_1, \alpha_2, \dots, \alpha_n)$  and  $\boldsymbol{\beta}^T = (\beta_1, \beta_2, \dots, \beta_n)$ . Then, we have that  $(Y, \Lambda)$  has an elliptical distribution:*

$$(Y, \Lambda) \sim E_2 \left( \boldsymbol{\mu}_{(Y, \Lambda)}, \boldsymbol{\Sigma}_{(Y, \Lambda)}, \phi \right) \quad (40)$$

where the respective parameters are given by

$$\boldsymbol{\mu}_{(Y, \Lambda)} = \begin{pmatrix} \mu_Y \\ \mu_\Lambda \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}^T \boldsymbol{\mu} \\ \boldsymbol{\beta}^T \boldsymbol{\mu} \end{pmatrix}, \quad (41)$$

$$\boldsymbol{\Sigma}_{(Y, \Lambda)} = \begin{pmatrix} \sigma_Y^2 & r(Y, \Lambda) \sigma_Y \sigma_\Lambda \\ r(Y, \Lambda) \sigma_Y \sigma_\Lambda & \sigma_\Lambda^2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha} & \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\beta} \\ \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\beta} & \boldsymbol{\beta}^T \boldsymbol{\Sigma} \boldsymbol{\beta} \end{pmatrix}. \quad (42)$$

Furthermore, conditionally given  $\Lambda = \lambda$ , the random variable  $Y$  has a univariate elliptical distribution:

$$Y | \Lambda = \lambda \sim E_1 \left( \mu_Y + r(Y, \Lambda) \frac{\sigma_Y}{\sigma_\Lambda} (\lambda - \mu_\Lambda), \left(1 - r(Y, \Lambda)^2\right) \sigma_Y^2, \phi_a \right), \quad (43)$$

for some characteristic generator  $\phi_a(\cdot)$  depending on  $a = (\lambda - \mu_\Lambda)^2 / \sigma_\Lambda^2$ .

*Proof.* The joint distribution of  $(Y, \Lambda)$  immediately follows from Theorem (1). We know that if its joint density exists, it will be of the form

$$f_{Y, \Lambda}(y, \lambda) = \frac{c_2}{\sqrt{1 - r^2} \sigma_Y \sigma_\Lambda} \times g_2 \left[ \frac{1}{1 - r^2} \left( \left( \frac{y - \mu_Y}{\sigma_Y} \right)^2 - 2r \left( \frac{y - \mu_Y}{\sigma_Y} \right) \left( \frac{\lambda - \mu_\Lambda}{\sigma_\Lambda} \right) + \left( \frac{\lambda - \mu_\Lambda}{\sigma_\Lambda} \right)^2 \right) \right], \quad (44)$$

where we simply wrote  $r = r(Y, \Lambda)$ . Consider next the characteristic function of the conditional random variable  $Y | \Lambda = \lambda$ :

$$\begin{aligned} \varphi_{Y|\Lambda}(t) &= E(e^{itY} | \Lambda = \lambda) = \int_{-\infty}^{\infty} e^{ity} f_{Y|\Lambda}(y | \lambda) dy \\ &= \frac{1}{f_\Lambda(\lambda)} \int_{-\infty}^{\infty} e^{ity} f_{Y, \Lambda}(y, \lambda) dy, \end{aligned}$$

where  $f_{Y, \Lambda}(y, \lambda)$  is given in (44). Using the transformation  $w_1 = (y - \mu_Y) / \sigma_Y$  and  $w_2 = (\lambda - \mu_\Lambda) / \sigma_\Lambda$ , we have

$$\varphi_{Y|\Lambda}(t) = e^{it\mu_Y} \int_{-\infty}^{\infty} e^{i(t\sigma_Y)w_1} \frac{c_2/c_1}{\sqrt{1 - r^2}} \frac{g_2 \left[ \frac{1}{1 - r^2} (w_1^2 - 2rw_1w_2 + w_2^2) \right]}{g_1(w_2^2)} dw_1.$$

Since we can write  $\frac{1}{1-r^2} (w_1^2 - 2rw_1w_2 + w_2^2) = \left(\frac{w_1-rw_2}{\sqrt{1-r^2}}\right)^2 + w_2^2$  and using the transformation  $w_1^* = (w_1 - rw_2) / \sqrt{1-r^2}$ , we have

$$\varphi_{Y|\Lambda}(t) = e^{it(\mu_Y + \sigma_Y r w_2)} \int_{-\infty}^{\infty} \exp \left[ i \left( \sigma_Y \sqrt{1-r^2} t \right) w_1^* \right] \frac{c_2}{c_1} \frac{g_2(w_1^{*2} + w_2^2)}{g_1(w_2^2)} dw_1^*. \quad (45)$$

Note that

$$\int_{-\infty}^{\infty} \frac{c_2}{c_1} \frac{g_2(w_1^{*2} + w_2^2)}{g_1(w_2^2)} dw_1^* = 1$$

which clearly indicates that

$$\frac{c_2}{c_1} \frac{g_2(w_1^{*2} + w_2^2)}{g_1(w_2^2)}$$

is a legitimate density. Furthermore, it can be shown that it is the density of a spherical random variable, denoted by  $W$ , with density generator

$$g_a(w) = \frac{g_2(w+a)}{\int_{-\infty}^{\infty} u^{-1/2} g_2(u+a) du}$$

where  $a = w_2^2$ , see Fang et al. (1990), pages 39 to 41. Thus, the characteristic function of this spherical random variable  $W$  can be expressed as

$$\varphi_W(t) = \phi_a(t^2)$$

for some characteristic generator  $\phi_a(\cdot)$  which depends on  $a$ . Therefore, the characteristic function in (45) simplifies to

$$\begin{aligned} \varphi_{Y|\Lambda}(t) &= e^{it(\mu_Y + \sigma_Y r w_2)} \psi_W \left( \sigma_Y \sqrt{1-r^2} t \right) \\ &= e^{it(\mu_Y + \sigma_Y r w_2)} \phi_a \left( \sigma_Y^2 (1-r^2) t^2 \right). \end{aligned}$$

We see it to be the characteristic function of an elliptical random variable with parameters in (37) and (38).  $\square$

From this theorem, it follows that the characteristic function of  $Y|\Lambda = \lambda$  is given by

$$E(e^{iYt} | \Lambda = \lambda) = \exp \left( i \mu_{Y|\Lambda=\lambda} t \right) \phi_a \left( \sigma_{Y|\Lambda=\lambda}^2 t^2 \right)$$

where

$$\mu_{Y|\Lambda=\lambda} = \mu_Y + r(Y, \Lambda) \frac{\sigma_Y}{\sigma_\Lambda} (\lambda - \mu_\Lambda)$$

and

$$\sigma_{Y|\Lambda=\lambda}^2 = \left( 1 - r(Y, \Lambda)^2 \right) \sigma_Y^2.$$

### 3 Log-Elliptical Distributions

Multivariate log-elliptical distributions are natural generalizations of multivariate log-normal distributions. For any  $n$ -dimensional vector  $\mathbf{x} = (x_1, \dots, x_n)^T$  with positive components  $x_i$ , we define

$$\log \mathbf{x} = (\log x_1, \log x_2, \dots, \log x_n)^T.$$

Recall that an  $n$ -dimensional random vector has a multivariate log-normal distribution if  $\log \mathbf{X}$  has a multivariate normal distribution. In this case we have that  $\log \mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ .

**DEFINITION 3.** *The random vector  $\mathbf{X}$  is said to have a multivariate log-elliptical distribution with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  if  $\log \mathbf{X}$  has an elliptical distribution:*

$$\log \mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi). \quad (46)$$

In the sequel, we shall denote  $\log \mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  as  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ . When  $\boldsymbol{\mu} = \mathbf{0}_n$  and  $\boldsymbol{\Sigma} = \mathbf{I}_n$ , we shall write  $\mathbf{X} \sim LS_n(\phi)$ . Clearly, if  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  and  $\mathbf{X} = \exp(\mathbf{Y})$ , then  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ .

If the density of  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  exists, then the density of  $\mathbf{Y} = \log \mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$  also exists. From (20), it follows that the density of  $\mathbf{X}$  must be equal to

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c}{\sqrt{|\boldsymbol{\Sigma}|}} \left( \prod_{k=1}^n x_k^{-1} \right) g \left[ (\log \mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\log \mathbf{x} - \boldsymbol{\mu}) \right], \quad (47)$$

see Fang et al. (1990). The density of the multivariate log-normal distribution with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  follows from (19) and (47). Furthermore, any marginal distribution of a log-elliptical distribution is again log-elliptical. This immediately follows from the properties of elliptical distributions.

**THEOREM 5.** *Let  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ . If the mean of  $X_k$  exists, then it is given by*

$$E(X_k) = e^{\mu_k} \phi(-\sigma_k^2).$$

*Provided the covariances exist, they are given by*

$$\text{Cov}(X_k, X_l) = \exp(\mu_k + \mu_l) \left\{ \phi \left[ -(\sigma_k + \sigma_l)^2 \right] - \phi(-\sigma_k^2) \phi(-\sigma_l^2) \right\}.$$

*Proof.* Define the vector  $\mathbf{a}_k = (0, 0, \dots, 0, 1, 0, \dots, 0)^T$  to consist of all zero entries, except for the  $k$ -th entry which is 1. Thus, for  $k = 1, 2, \dots, n$ , we have

$$\begin{aligned} E(X_k) &= E(e^{Y_k}) = E\left(e^{\mathbf{a}_k^T \mathbf{Y}}\right) = \varphi_{\mathbf{a}_k^T \mathbf{Y}}(-i) \\ &= \exp(\mathbf{a}_k^T \boldsymbol{\mu}) \phi(-\mathbf{a}_k^T \boldsymbol{\Sigma} \mathbf{a}_k) \\ &= e^{\mu_k} \phi(-\sigma_k^2) \end{aligned}$$

and the result for the mean immediately follows. For the covariance, first define the vector  $\mathbf{b}_{kl} = (0, 0, \dots, 1, 0, \dots, 0, 1, 0, \dots, 0)^T$  to consist of all zero entries, except for the  $k$ -th and  $l$ -th entries which are each 1. Note that  $\nu_{kl} = E(X_k X_l) - E(X_k)E(X_l)$  where

$$\begin{aligned} E(X_k X_l) &= E\left(e^{\mathbf{b}_{kl}^T \mathbf{Y}}\right) = \varphi_{\mathbf{b}_{kl}^T \mathbf{Y}}(-i) \\ &= \exp\left(\mathbf{b}_{kl}^T \boldsymbol{\mu}\right) \phi\left(-\mathbf{b}_{kl}^T \boldsymbol{\Sigma} \mathbf{b}_{kl}\right) \\ &= \exp\left(\mu_k + \mu_l\right) \phi\left[-\left(\sigma_k + \sigma_l\right)^2\right] \end{aligned}$$

and the result should now be obvious.  $\square$

Some risk measures for log-elliptical distributions are derived in the next theorem.

**THEOREM 6.** *Let  $X \sim LE_1(\mu, \sigma^2, \phi)$  and  $Z \sim S_1(\phi)$  with density  $f_Z(x)$ , then*

$$\begin{aligned} F_X^{-1}(p) &= \exp\left(\mu + \sigma F_Z^{-1}(p)\right), \quad 0 < p < 1, \\ E[(X - d)_+] &= e^\mu \phi(-\sigma^2) F_{Z^*}\left(\frac{\mu - \log d}{\sigma}\right) - d F_Z\left(\frac{\mu - \log d}{\sigma}\right), \quad d > 0, \\ E[X | X > F_X^{-1}(p)] &= \frac{e^\mu}{1-p} \phi(-\sigma^2) F_{Z^*}\left(F_Z^{-1}(1-p)\right), \quad 0 < p < 1, \end{aligned}$$

where the density of  $Z^*$  is given by

$$f_{Z^*}(x) = \frac{f_Z(x) e^{\sigma x}}{\phi(-\sigma^2)}.$$

*Proof.* (1) The quantiles of  $X$  follow immediately from  $\log X \stackrel{d}{=} \mu + \sigma Z$ .

(2) Using  $f_X(x) = \frac{1}{\sigma x} f_Z\left(\frac{\log x - \mu}{\sigma}\right)$ , and substituting  $x$  by  $t = \frac{\log x - \mu}{\sigma}$ , we find that

$$\int_d^\infty x f_X(x) dx = e^\mu \phi(-\sigma^2) F_{Z^*}\left(\frac{\mu - \log d}{\sigma}\right).$$

The stated result then follows from

$$E[(X - d)_+] = \int_d^\infty x f_X(x) dx - d F_Z\left(\frac{\mu - \log d}{\sigma}\right).$$

(3) The expression for the tail conditional expectation follows from

$$E[X | X > F_X^{-1}(p)] = F_X^{-1}(p) + \frac{1}{1-p} E\left[(X - F_X^{-1}(p))_+\right].$$

$\square$

Note that the density of  $Z^*$  in the theorem above can be interpreted as the Esscher transform with parameter  $\sigma$  of  $Z$ . Further, note that the expression for the quantiles holds for any one-dimensional elliptical distribution, whereas the expressions for the stop-loss premiums and tail conditional expectations were only derived for continuous elliptical distributions. We also have that if  $g$  is the normalized density generator of  $Z \sim S_1(\phi)$ , then

$$F_Z(x) = \int_{-\infty}^x g(z^2) dz.$$

## 4 Convex Order Bounds for Sums of Random Variables

This section describes the bounds for (the distribution function of) sums of random variables as presented in Dhaene et al. (2002a, 2002b). We first define the notions of convex order and comonotonicity, which are essential ingredients for developing the bounds.

### 4.1 Convex Order

In the sequel, all random variables are assumed to have a finite mean. In this case, we find

$$E(X) = \int_0^{\infty} \bar{F}_X(x) dx - \int_{-\infty}^0 F_X(x) dx. \quad (48)$$

In actuarial science, it is common to replace a random variable with another which is “less attractive” but with a simpler structure so that the distribution function can at least be easier to determine. In this paper, the notion of “less attractive” will be translated in terms of convex order, as defined below.

**DEFINITION 4.** *A random variable  $X$  is said to precede another random variable  $Y$  in convex order, written as  $X \preceq_{cx} Y$ , if*

$$E[v(X)] \leq E[v(Y)]$$

*holds for all convex functions  $v$  for which the expectations exist.*

Alternatively, it can be shown that  $X \preceq_{cx} Y$  if and only if  $E(X) = E(Y)$  and

$$E[(X - d)_+] \leq E[(Y - d)_+] \text{ for all } d, \quad (49)$$

see e.g. Shaked & Shanthikumar (1994). As stop-loss premiums can be seen as measures for the upper tail of the distribution function,  $X \preceq_{cx} Y$  means that observing large outcomes is more likely for  $Y$  than for  $X$ . Other characterizations of convex ordering can be found in Dhaene et al. (2002a).

### 4.2 Comonotonicity

Consider an  $n$ -dimensional random vector  $\mathbf{X} = (X_1, \dots, X_n)^T$  with multivariate distribution function given by  $F_{\mathbf{X}}(\mathbf{x}) = \Pr(X_1 < x_1, \dots, X_n < x_n)$ , for any  $\mathbf{x} = (x_1, \dots, x_n)^T$ . It is well-known that this



multivariate distribution function satisfies the so called Fréchet bounds:

$$\max \left( \sum_{k=1}^n F_{X_k}(x_k) - (n-1), 0 \right) \leq F_{\mathbf{X}}(\mathbf{x}) \leq \min \left( F_{X_1}(x_1), \dots, F_{X_n}(x_n) \right),$$

see Hoeffding (1940) or Fréchet (1951).

DEFINITION 5. A random vector  $\mathbf{X}$  is said to be comonotonic if its joint distribution is given by the Fréchet upper bound, i.e.,

$$F_{\mathbf{X}}(\mathbf{x}) = \min \left( F_{X_1}(x_1), \dots, F_{X_n}(x_n) \right).$$

Alternative characterizations of comonotonicity of a random vector are given in the following theorem, the proof of which can be found in Dhaene et al. (2002a).

THEOREM 7. Suppose  $\mathbf{X}$  is an  $n$ -dimensional random vector. Then the following statements are equivalent:

1.  $\mathbf{X}$  is comonotonic.
2.  $\mathbf{X} \stackrel{d}{=} \left( F_{X_1}^{-1}(U), \dots, F_{X_n}^{-1}(U) \right)$  for  $U \sim \text{Uniform}(0, 1)$  where  $F_{X_k}^{-1}(\cdot)$  denotes the quantile function defined by

$$F_{X_k}^{-1}(q) = \inf \{ x \in \mathbf{R} \mid F_X(x) \geq q \}, \text{ for } 0 \leq q \leq 1.$$

3. There exists a random variable  $Z$  and non-decreasing functions  $h_1, \dots, h_n$  such that

$$\mathbf{X} \stackrel{d}{=} (h_1(Z), \dots, h_n(Z)).$$

In the sequel, we shall use the superscript  $c$  to denote comonotonicity of a random vector. Hence, the vector  $(X_1^c, \dots, X_n^c)$  is a comonotonic random vector with the same marginals as the vector  $(X_1, \dots, X_n)$ . The former vector is called the comonotonic counterpart of the latter.

Consider the comonotonic random sum

$$S^c = X_1^c + \dots + X_n^c. \tag{50}$$

In Dhaene et al. (2002a), it is proven that each quantile of  $S^c$  is equal to the sum of the corresponding quantiles of the marginals involved:

$$F_{S^c}^{-1}(q) = \sum_{k=1}^n F_{X_k}^{-1}(q), \quad 0 \leq q \leq 1. \tag{51}$$

Furthermore, they showed that in case all marginal distributions  $F_{X_k}$  are strictly increasing, the stop-loss premiums of a comonotonic sum  $S^c$  can easily be computed from the stop-loss premiums of the marginals:

$$E[(S^c - d)_+] = \sum_{k=1}^n E[(X_k - d_k)_+], \tag{52}$$

where the  $d_k$ 's are determined by

$$d_k = F_{X_k}^{-1}(F_{S^c}(d)). \quad (53)$$

This result can be extended to the case of marginal distributions that are not necessarily strictly increasing, see Dhaene et al. (2002a).

### 4.3 Convex Bounds

In this subsection, we shall provide lower and upper for the sum

$$S = X_1 + \cdots + X_n. \quad (54)$$

Proofs for these bounds can be found in Dhaene et al. (2002).

The comonotonic counterpart of  $\mathbf{X} = (X_1, \dots, X_n)^T$  is denoted by  $\mathbf{X}^c = (X_1^c, \dots, X_n^c)^T$ , where  $X_k^c$  is given by

$$X_k^c = F_{X_k}^{-1}(U), \quad (55)$$

with  $U \sim U(0, 1)$ . The comonotonic convex upper bound for the sum  $S^c$  is defined by

$$S^c = X_1^c + \cdots + X_n^c. \quad (56)$$

It can be shown that

$$S \leq_{cx} S^c, \quad (57)$$

which implies that  $S^c$  is indeed a convex order upper bound for  $S$ .

Let us now suppose that we have additional information available about the dependency structure of  $\mathbf{X}$ , in the sense that there is some random variable  $\Lambda$  with a known distribution function and such that we also know the distributions of the random variables  $X_k | \Lambda = \lambda$  for all outcomes  $\lambda$  of  $\Lambda$  and for all  $k = 1, \dots, n$ . Let  $F_{X_k | \Lambda}^{-1}(U)$  be a notation for the random variable  $f_k(U, \Lambda)$ , where the function  $f_k$  is defined by  $f_k(u, \lambda) = F_{X_k | \Lambda = \lambda}^{-1}(u)$ . Now, consider the random vector  $\mathbf{X}^u = (X_1^u, \dots, X_n^u)^T$ , where  $X_k^u$  is given by

$$X_k^u = F_{X_k | \Lambda}^{-1}(U). \quad (58)$$

The improved upper bound (corresponding to  $\Lambda$ ) of the sum  $S$  is then defined as

$$S^u = X_1^u + \cdots + X_n^u. \quad (59)$$

Notice that the random vectors  $\mathbf{X}^c$  and  $\mathbf{X}^u$  have the same marginals. It can be proven that

$$S \leq_{cx} S^u \leq_{cx} S^c, \quad (60)$$

which means that the sum  $S^u$  is indeed an improved upper bound (in the sense of convex order) for the original sum  $S$ .

Finally, consider the random vector  $\mathbf{X}^l = (X_1^l, \dots, X_n^l)^T$ , where  $X_k^l$  is given by

$$X_k^l = E(X_k | \Lambda). \quad (61)$$

Using Jensen's inequality, it is straightforward to prove that the sum

$$S^l = X_1^l + \dots + X_n^l \quad (62)$$

is a convex order lower bound for  $S$ :

$$S^l \leq_{cx} S. \quad (63)$$

## 5 Non-Independent Log-Elliptical Risks

In this section we develop convex lower and upper bounds for sums involving log-elliptical random variables. It generalizes the results for the log-normal case as obtained in Kaas, Dhaene & Goovaerts (2000). Consider a series of deterministic non-negative payments  $\alpha_1, \dots, \alpha_n$ , that are due at times  $1, \dots, n$  respectively. The present value random variable  $S$  is defined by

$$S = \sum_{i=1}^n \alpha_i \exp[-(Y_1 + \dots + Y_i)], \quad (64)$$

where the random variable  $Y_i$  represents the continuously compounded rate of return over the period  $(i-1, i)$ ,  $i = 1, 2, \dots, n$ . Furthermore, define  $Y(i) = Y_1 + \dots + Y_i$ , the sum of the first  $i$  elements of the random vector  $\mathbf{Y} = (Y_1, \dots, Y_n)^T$ , and  $X_i = \exp[-Y(i)]$ . Using these notations, we can write the present value random variable in (64) as

$$S = \sum_{i=1}^n \alpha_i e^{-Y(i)} = \sum_{i=1}^n X_i. \quad (65)$$

We will assume that the return vector  $\mathbf{Y} = (Y_1, \dots, Y_n)^T$  belongs to the class of multivariate elliptical distributions, i.e.  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ , with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  given by

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T, \quad \boldsymbol{\Sigma} = (\sigma_{kl}) \text{ for } k, l = 1, 2, \dots, n. \quad (66)$$

Thus, the random vector  $\mathbf{X} = (X_1, \dots, X_n)^T$  is a log-elliptical random vector. From (13), we find that  $Y(i) \sim E_1(\mu(i), \sigma^2(i), \phi)$  with

$$\mu(i) = \sum_{k=1}^i \mu_k, \quad (67)$$

$$\sigma^2(i) = \sum_{k=1}^i \sum_{l=1}^i \sigma_{kl}. \quad (68)$$

In order to develop lower and improved upper bounds for  $S$ , we define the conditioning random variable  $\Lambda$  as a linear combination of the random variables  $Y_i$ ,  $i = 1, 2, \dots, n$ :

$$\Lambda = \sum_{i=1}^n \beta_i Y_i.$$

Using the property of elliptical distributions described in (14), we know that  $\Lambda \sim E_1(\mu_\Lambda, \sigma_\Lambda^2, \phi)$  where

$$\mu_\Lambda = \sum_{i=1}^n \beta_i \mu_i \tag{69}$$

and

$$\sigma_\Lambda^2 = \sum_{i,j=1}^n \beta_i \beta_j \sigma_{ij}. \tag{70}$$

Note that if the mean and variance of  $\Lambda$  exist, then they are given by  $E(\Lambda) = \mu_\Lambda$  and  $Var(\Lambda) = -2\phi'(0) \sigma_\Lambda^2$ , respectively.

**PROPOSITION 1.** *Let  $S$  be the present value sum as defined in (64), with  $\alpha_i$ ,  $i = 1, \dots, n$ , non-negative real numbers and  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ . Let the conditioning random variable  $\Lambda$  be defined by*

$$\Lambda = \sum_{i=1}^n \beta_i Y_i. \tag{71}$$

*Then the comonotonic upper bound  $S^c$ , the improved upper bound  $S^u$  and the lower bound  $S^l$  are given by*

$$S^c = \sum_{i=1}^n \alpha_i \exp[-\mu(i) + \sigma(i) F_Z^{-1}(U)], \tag{72}$$

$$S^u = \sum_{i=1}^n \alpha_i \exp\left[-\mu(i) - r_i \sigma(i) F_Z^{-1}(U) + \sqrt{1 - r_i^2} \sigma(i) F_Z^{-1}(V)\right], \tag{73}$$

$$S^l = \sum_{i=1}^n \alpha_i \exp[-\mu(i) - r_i \sigma(i) F_Z^{-1}(U)] \phi_a(-\sigma^2(i) (1 - r_i^2)), \tag{74}$$

where  $U$  and  $V$  are mutually independent uniform(0,1) random variables,  $Z \sim S_1(\phi)$ , and  $r_i$  is defined by

$$r_i = \frac{\sum_{k=1}^i \sum_{l=1}^n \beta_l \sigma_{kl}}{\sigma(i) \sigma_\Lambda}. \tag{75}$$

*Proof.* (a) From

$$X(i) \stackrel{d}{=} \alpha_i e^{\mu(i) + \sigma(i)Z},$$

we find that

$$F_{S^c}^{-1}(p) = \sum_{i=1}^n F_{X(i)}^{-1}(p) = \sum_{i=1}^n \alpha_i e^{\mu(i) + \sigma(i) F_Z^{-1}(p)}, \quad 0 < p < 1,$$

Hence, the comonotonic upper bound  $S^c$  of  $S$  is given by (72).

(b) In order to derive the lower bound  $S^l$ , we first determine the characteristic function of the bivariate random vector  $(Y(i), \Lambda)$  for  $i = 1, \dots, n$ . For any 2-vector  $(t_1, t_2)^T$ , we find

$$E[\exp(i(t_1 Y(i) + t_2 \Lambda))] = E[\exp(is^T \mathbf{Y})]$$

with

$$s_k = \begin{cases} t_1 + t_2 \beta_k & : k = 1, \dots, i, \\ t_2 \beta_k & : k = i + 1, \dots, n. \end{cases}$$

As  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ , this leads to

$$\begin{aligned} E[\exp(i(t_1 Y(i) + t_2 \Lambda))] &= \exp(is^T \boldsymbol{\mu}) \cdot \phi(\mathbf{s}^T \boldsymbol{\Sigma} \mathbf{s}) \\ &= \exp(i\mathbf{t}^T \boldsymbol{\mu}^*) \cdot \phi(\mathbf{t}^T \boldsymbol{\Sigma}^* \mathbf{t}) \end{aligned}$$

with

$$\begin{aligned} \boldsymbol{\mu}^* &= (\mu(i), \mu_\Lambda)^T, \\ \boldsymbol{\Sigma}^* &= \begin{pmatrix} \sigma^2(i) & \sigma_{1,2}^* \\ \sigma_{2,1}^* & \sigma_\Lambda^2 \end{pmatrix} \end{aligned}$$

and

$$\sigma_{1,2}^* = \sigma_{2,1}^* = r_i \sigma(i) \sigma_\Lambda.$$

From Theorem (1), we can conclude that the bivariate random vector  $(Y(i), \Lambda)$  is elliptical:

$$(Y(i), \Lambda)^T \sim E_2(\boldsymbol{\mu}^*, \boldsymbol{\Sigma}^*, \phi).$$

Thus, from Theorem (4), we know that  $(Y(i) | \Lambda = \lambda) \sim E_1(\mu_{Y(i)|\Lambda}, \sigma_{Y(i)|\Lambda}^2, \phi_a)$  with

$$\mu_{Y(i)|\Lambda} = E(Y(i) | \Lambda = \lambda) = \mu(i) + r_i \frac{\sigma(i)}{\sigma_\Lambda} (\lambda - \mu_\Lambda) \quad (76)$$

and

$$\sigma_{Y(i)|\Lambda}^2 = \text{Var}(Y(i) | \Lambda = \lambda) = \sigma^2(i) (1 - r_i^2). \quad (77)$$

Note that

$$\begin{aligned} r_i &= \text{Corr}(Y(i), \Lambda) = \text{Corr}\left(\sum_{k=1}^i Y_k, \sum_{l=1}^n \beta_l Y_l\right) \\ &= \frac{\text{Corr}\left(\sum_{k=1}^i Y_k, \sum_{l=1}^n \beta_l Y_l\right)}{\sigma(i) \sigma_\Lambda} = \frac{\sum_{k=1}^i \sum_{l=1}^n \beta_l \sigma_{kl}}{\sigma(i) \sigma_\Lambda}. \end{aligned}$$

Consequently, using the moment generating function of an elliptical, we have

$$\begin{aligned} E\left(\alpha_k e^{-Y(i)} \mid \Lambda = \lambda\right) &= \alpha_i E\left(e^{-Y(i)} \mid \Lambda = \lambda\right) = \alpha_i M_{Y(i)|\Lambda}(-1) \\ &= \alpha_i \exp\left(-\mu_{Y(i)|\Lambda}\right) \cdot \phi_a\left(-\sigma_{Y(i)|\Lambda}^2\right). \end{aligned}$$

Thus, if  $V \sim U(0, 1)$ , then a lower bound for the present value random sum in (64) can be written as  $S^l = X_1^l + \dots + X_n^l$  with

$$\begin{aligned} X_i^l &= E\left(\alpha_i e^{-Y(i)} \mid \Lambda\right) \\ &= \alpha_i \exp\left(-\mu(i) - r_r \frac{\sigma(i)}{\sigma_\Lambda} (\mu_\Lambda + \sigma_\Lambda F_Z^{-1}(V) - \mu_\Lambda)\right) \cdot \phi_a\left(-\sigma^2(i) (1 - r_i^2)\right) \\ &= \alpha_i \exp\left(-\mu(i) - r_i \sigma(i) F_Z^{-1}(V)\right) \cdot \phi_a\left(-\sigma^2(i) (1 - r_i^2)\right). \end{aligned} \quad (78)$$

(c) As suggested in Section 4, an improved upper bound can be developed using the additional information as provided by  $\Lambda = \sum_{i=1}^n \beta_i Y_{i..}$ . Now, define  $X_i^u = F_{X_i|\Lambda}^{-1}(U)$  as in (59). Because  $(Y(i) \mid \Lambda = \lambda) \sim E_1\left(\mu_{Y(i)|\Lambda}, \sigma_{Y(i)|\Lambda}^2, g_1\right)$ , it follows that

$$F_{X_i|\Lambda=\lambda}^{-1}(u) = \alpha_i \exp\left[F_Z^{-1}(u) \sigma_{Y(i)|\Lambda} - \mu_{Y(i)|\Lambda}\right].$$

Thus, if  $U \sim U(0, 1)$ , we have

$$\begin{aligned} F_{X_i|\Lambda}^{-1}(U) &= \alpha_i \exp\left[F_Z^{-1}(U) \sigma_{Y(i)|\Lambda} - \mu_{Y(i)|\Lambda}\right] \\ &= \alpha_i \exp\left[F_Z^{-1}(U) \sqrt{\sigma^2(i) (1 - r_i^2)} - \mu(i) - r_i \sigma(i) F_Z^{-1}(V)\right] \\ &= \alpha_i \exp\left[F_Z^{-1}(U) \sigma(i) \sqrt{1 - r_i^2} - \mu(i) - r_i \sigma(i) F_Z^{-1}(V)\right] \\ &= \alpha_i \exp\left\{\sigma(i) \left[F_Z^{-1}(U) \sqrt{1 - r_i^2} - r_i F_Z^{-1}(V)\right] - \mu(i)\right\}. \end{aligned}$$

Thus, an improved upper bound is expressed as in (73).  $\square$

## 6 Concluding Remarks

In this paper we derived upper and lower bounds for the distribution function of a sum of non-independent log-elliptical random variables. The results presented in this paper extend the results of Dhaene, Denuit, Kaas, Goovaerts & Vyncke (2002a, 2002b) who constructed bounds for sums of log-normal random variables. The extension to the class of elliptical distributions makes sense because

it makes the ideas developed in Kaas, Dhaene & Goovaerts (2000) also applicable in situations where the shape of log-normal distributions is not fitted, but heavier tailed distributions such as Student- $t$  are required. As multivariate log-elliptical distributions share many of the tractable properties of multivariate log-normal distributions, it is not surprising to find that the bounds developed for the log-elliptical case are very similar in form to those developed for the log-normal case. The upper bound is based on the sum of comonotonic random variables while lower and improved upper bounds can be constructed from conditioning on some additional available random variable.

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