Ruin Probabilities with Dependent Claims

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Abstract

In classical risk theory, the surplus process is a very important model for understanding how the capital or surplus of an insurance company evolve over time. By adding to the previous surplus the current premium flow and deducting the claims made during the period, the process gives the value of the capital that is available to the insurer at each point in time. Each period is tracked so that the surplus never gets below zero because if it does, it provides an indication of ruin, that is, the company is in a position of negative cash flow. However, the first time that ruin occurs is important and the company must ensure this does not happen because it can leave the company inoperable. This time to ruin is so much a function of the initial capital and the pricing structure of the insurer's book of business, although how claims evolve over time can also directly impact the level of surplus. Claims generally are out of the company's control, but it can manage its surplus so that it can predictably estimate the level of claims that will emerge over time. One distinguishing feature of a typical mathematical structure of the surplus process is the assumption that individual claims do arise independently. We know that this assumption of independence is no longer realistic and reasonable because individual risks, are usually homogeneous and share common characteristics that claims from one can induce claims of another. In other words, the risks do not exhibit independence. In this paper, we investigate the effects of dependent claims on the probability of ruin and the time-to-ruin, given ruin occurs. However, unlike the case of independence where there may be a more tractable solution, it is not straightforward to get closed form solutions to the probability of ruin. Instead, we apply simulation procedures to provide us insight into the statistical distribution of the time to ruin when claims are dependent. We find that in the presence of dependent claims, the time to ruin occurs faster.

1 Introduction

Insurance companies are in the business of risks. They exist to pool together risks faced by individuals or companies who in the event of a loss are compensated by the insurer to reduce the financial burden. In its simplest form, when certain events occur, an insurance contract will provide the policyholder the right to claim all or a portion of the loss. In exchange for this entitlement, the policyholder pays a specified amount called the premium and the insurer is obligated to honor its promises when they come due.

In order to ensure that it will be able to pay its promised obligations, the insurance company sets aside amount called the reserve or surplus from which it can draw from when claims are due. The company generally does not accumulate surplus overnight but it does so over long periods of time from possible excesses of premiums collected over claim amounts paid. Additional sources of surplus accumulation is possible such as investment income but the traditional approach of risk theory has been to ignore the effect of interest although there is increasing literature on this subject. See, for example, Asmussen(2000).

The surplus process studied in classical risk theory is a very important stochastic framework for understanding how the company's capital or surplus evolves over time. Beginning with an initial surplus u_0, the company's surplus at time t is given by

$$U(t) = u_0 + \Pi(t) - S(t), \tag{1}$$

where $\Pi(t)$ is the total premiums received up until time t and $S(t)$ is the aggregate claims paid up to time t. A realization of this surplus process is depicted in Figure 1. A quantity of interest here is usually the so-called probability of ruin which provides a measure of how certain will the company be able to support its book of business. The time to ruin is defined to be the first time that the surplus in (1) becomes negative and is therefore

$$T = \inf\{t : U(t) < 0\}.$$

If $U(t) \geq 0$ for all $t > 0$, that is, the surplus never reaches zero, then we define $T = \infty$. This enables us to define ruin probabilities. The finite-horizon probability of ruin is

$$\Psi(u_0, t) = \text{Prob}(T < t \,|U(0) = u_0) \tag{2}$$

and the infinite-horizon probability of ruin is

$$\Psi(u_0) = \Psi(u_0, \infty) = \text{Prob}(T < \infty \,|U(0) = u_0). \tag{3}$$

This is more often called the ultimate probability of ruin. The surplus process together with probabilities of ruin have been extensively examined in the actuarial literature. See ASMUSSEN(2000), BOWERS, ET AL.(1997), BUHLMANN(1970), and ROLSKI, ET AL.(1999). The classical collective risk theory has originated from LUNDBERG(1909).

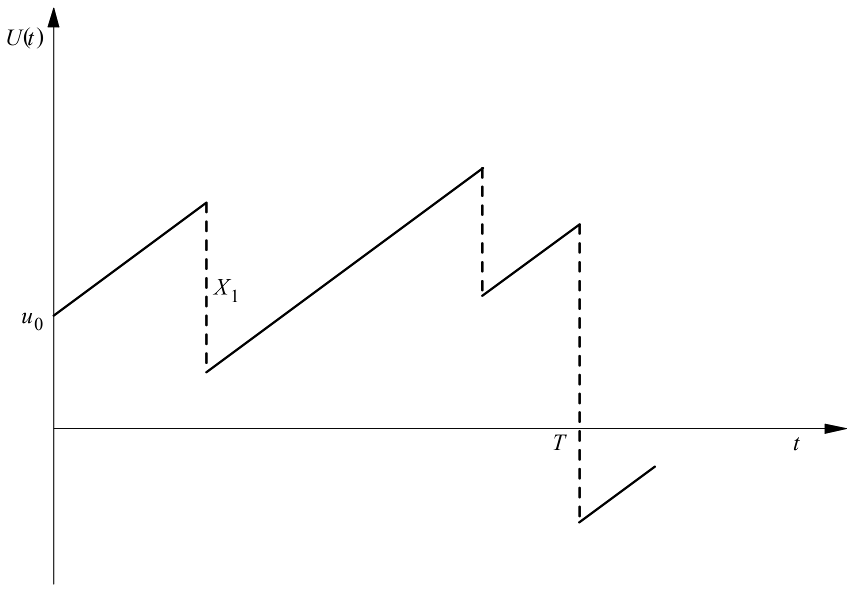

Figure 1: A Sample Path of the Insurance Company Surplus Process

In the classical set-up, the aggregate claims process $\{S(t) : t \geq 0\}$ is typically represented as

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)} = \sum_{k=1}^{N(t)} X_k \tag{4}$$

and is the total claims amount paid over the period from time 0 to t. Here, $\{N(t) : t \geq 0\}$ represents the claims number process and is independent of the claim amount process $\{X_k\}$. Because of mathematical tractability, the claim amounts X_k are commonly assumed to be independent and identically random variables. This paper examines departure from this assumption and its effect on the insurer's probability of ruin. There has been a recent surge of interest in studying classical actuarial results when the assumption of independence no longer holds. This increase in interest is understandable because the assumption of independent risks no longer seems realistic. Dependence of risks exists in practice in several situations. For example, the risks of an entire insurance portfolio may be influenced by a singular catastrophic event such as earthquake, storm, fire, or epidemic. As yet another example, an insurance portfolio may cover the lives of persons who may have common characteristics such as a family (husband, wife, children) or a group (employees of a corporation, members of a professional organization), whose mortality will be dependent to a certain extent. See Dhaene and Goovaerts(1997).

In this paper, we analyze the company's probability of ruin when we remove the assumption of independence in claim occurrence. Assuming that within a period the company has fixed n policyholders, we re-express the sum in (4) as

$$S(t) = Y_1 + Y_2 + \cdots + Y_n = \sum_{k=1}^{n} Y_k = \sum_{k=1}^{n} (I_k B_k) \tag{5}$$

where each Y is written as the product of a Bernoulli claim indicator I and the size of claim B should a claim occur. We will assume that the incidence of claims among the policyholders are no longer independent so that this induces the dependence in our framework. We continue with the usual assumption of independence of the claim sizes, when claims occur. This approach of dependence between claims occurrences has recently been proposed by Denuit, et al.(2002) where they analyzed the impact of this dependence in the individual risk model. They also distinguish between two types of dependence: the global dependence induced by a common environment and local dependence leading to subdivision of risks into independent classes. However, we do not need to distinguish between such types of dependence here because we choose to use "copulas" to express the dependencies. Copulas are functions that link the marginal distributions to their joint multivariate distribution and contain parameter(s) that describes the dependency. In particular, we express the joint distribution function of $I_1, I_2, ..., I_n$ as a copula function

$$F_{I_1 \cdots I_n}(i_1, ..., i_n) = C(F_{I_1}(i_1), ..., F_{I_n}(i_n)) \tag{6}$$

where $F_{I_1 \cdots I_n}(\cdot, ..., \cdot)$ is the multivariate distribution function and $F_{I_k}(\cdot)$ for $k = 1, 2, ..., n$ are the marginals. Note that if we assume the Bernoulli claim indicator has distribution

$$\text{Prob}(I_k = 0) = p_k \quad \text{and} \quad \text{Prob}(I_k = 1) = q_k = 1 - p_k,$$

then the marginals will be

$$F_{I_k}(0) = p_k = 1 - q_k \quad \text{and} \quad F_{I_k}(1) = 1.$$

Because the marginals are discrete, the copula function expressed in (6) may not be uniquely determined. See, for example, Nelsen(1999) and Sklar(1959).

Although it is only recently that there has been a number of research in the area of dependence in the collective risk theory, as early as the 1980's, many researchers have recognized the limitations of assuming independence. Gerber(1982) investigated the probability of ruin when the claim amounts follow a linear (ARMA) model. His work was later extended by Promislow(1991) who relaxed the boundedness assumption of the claim amounts made by Gerber. Using large deviations theory, Nyrhinen(1998) derived Lundberg-like bounds on $\Psi(u)$ while similar results were derived by Muller and Pflug(2001) when Markov inequalities are used.

More related to our work is that by Cossette and Marceau(2000) who studied ruin probabilities within a discrete time setting. The dependence structure was made within the claims number process and they showed that under dependence, the probability of ruin is increased and the adjustment coefficient is decreased. Furthermore, Marceau, et al.(1999) examined various forms of dependence within the individual risk model including expressing the joint distribution of the claims occurrence as a copula, while Genest, et al.(2000) expressed this copula in the Archimedean form:

$$C(F_{I_1}(i_1), ..., F_{I_n}(i_n)) = \psi^{-1}[\psi(F_{I_1}(i_1)) + \cdots + \psi(F_{I_1}(i_1))], \tag{7}$$

where ψ, the Archimedean generator, is decreasing and convex and satisfies $\psi(1) = 0$.

Using simulation, this paper extends some of the work of the authors mentioned above by examining what happens to the ruin probability for various levels of dependence. The dependence structure will be expressed in copula form which as stated earlier links the multivariate distribution to their joint marginals. We induce the dependency on the probability of claims. Then to estimate probabilities of ruin, we produced a number of sample paths of the surplus process defined in (1) by simulation, followed each path until either ruin or a fixed period in the future to terminate the process. Recently, Albrecher and Kantor (2002) also employed simulation to investigate the effect of dependent claims on the probability of ruin. Their approach was different from ours in the sense that they examined dependence of consecutive claims according to a Markov-type risk process and later investigated the effect of the Lundberg exponent (or more commonly called adjustment coefficient). Nyrhinen (1998) suggested a Lundberg coefficient appropriate for general dependency structure and this was investigated by Albrecher and Kantor (2002) using simulation.

The remainder of this paper is organized as follows. In Section 2, we discuss alternative representation of the aggregate claims process. This representation is exactly the individual risk model and is convenient for specifying the dependency of claims structure. In Section 3, we briefly discuss about copulas and how they are to be used to express the joint distribution of the claim occurrences. This section reviews some of the fundamental results and properties of copulas, and gives some examples of copulas. Section 4 summarizes some useful classical results. In Section 5, we describe the simulation procedure used to estimate the probabilities of ruin and the time-to-ruin, given ruin occurs. In particular, what we did was simulated a large number of trajectories of the surplus process, counted the number of times ruined occurred, and later recorded the time-to-ruin for those that experienced ruin. This section also describes assumptions regarding the copula used, the various levels of dependencies, the loading, premium rate calculation, and marginal distribution of size of claims. Section 6 summarizes and provides a discussion of the simulation results. Section 7 concludes the paper.

2 The Aggregate Claims Process

An alternative representation of the aggregate claims process is made in this section. Similar to the individual risk model, this representation is to express each claim as the product of a claims occurrence and the amount associated with a claim. It then allows us to impose the dependency structure on the claims occurrence. Consider a portfolio of $n(t)$ insurance risks Y_1, Y_2,..., $Y_{n(t)}$ for the period $[0,t]$. The aggregate claims is defined to be the sum of these risks:

$$S(t) = Y_1 + Y_2 + \cdots + Y_{n(t)} = \sum_{k=1}^{n(t)} Y_k, \tag{8}$$

where generally the risks are non-negative random variables, i.e. $X_k \geq 0$. It is clear that $n(t)$ represents the total number of exposures to claim and that $n(1) \leq n(2) \leq \cdots \leq n(t)$ for any t, with strict inequality mainly because of new policies coming in during a period. In a typical set-up of the individual risk model, each insurance risk can therefore be expressed as the product of the indicator

$$I_k = \begin{cases} 1, & \text{if claim occurs} \\ 0, & \text{otherwise} \end{cases}.$$

and the amount of benefit if claim occurs, denoted by B_k. For simplicity and for reasons we think it is realistic, we shall assume that I and B are independent. The indicator random variable I_k has a Bernoulli distribution with

$$\text{Prob}(I_k = 0) = p_k \quad \text{and} \quad \text{Prob}(I_k = 1) = q_k = 1 - p_k.$$

The benefit amount B_k we shall assume has a distribution function

$$F_{B_k}(b) = \text{Prob}(B_k \leq b).$$

Furthermore, we shall assume its moment generating function exists and is

$$M_{B_k}(t) = E\left(e^{B_k t}\right),$$

and its mean and variance are

$$\mu_k = E(B_k) \quad \text{and} \quad Var(B_k) = \sigma_k^2,$$

respectively. It is straightforward to find the mean and variance of the aggregate claims:

$$\begin{aligned} E[S(t)] &= \sum_{k=1}^{n(t)} E(Y_k) = \sum_{k=1}^{n(t)} E[E(Y_k | I_k)] \\ &= \sum_{k=1}^{n(t)} E(Y_k | I_k = 1) q_k = \sum_{k=1}^{n(t)} q_k \mu_k \end{aligned} \tag{9}$$

and

$$Var[S(t)] = \sum_{k=1}^{n(t)} Var(Y_k) + 2\sum_{i<j}\sum Cov(Y_i, Y_j) \tag{10}$$

where

$$\begin{aligned} Var(Y_k) &= E\left[E\left(Y_k^2 | I_k\right)\right] - [E(Y_k)]^2 \\ &= E\left[q_k B_k^2\right] - (q_k \mu_k)^2 \\ &= q_k E\left(B_k^2\right) - q_k^2 \mu_k^2 \\ &= q_k \sigma_k^2 - q_k (1 - q_k) \mu_k^2 \end{aligned} \tag{11}$$

and

$$\begin{aligned} Cov(Y_i, Y_j) &= E(I_i B_i I_j B_j) - E(Y_i) E(Y_j) \\ &= E(I_i I_j) \mu_i \mu_j - q_i \mu_i q_j \mu_j \\ &= [Cov(I_i, I_j)] \mu_i \mu_j \text{ for } i \neq j. \end{aligned} \tag{12}$$

Similarly, the moment generating function of Y_k can be derived as follows:

$$\begin{aligned} M_{Y_k}(t) &= E\left(e^{Y_k t}\right) = E\left[E\left(e^{Y_k t} | I_k\right)\right] \\ &= E\left[p_k + q_k e^{B_k t}\right] \\ &= p_k + q_k M_{B_k}(t) \\ &= M_{I_k}\left[\log\left(M_{B_k}(t)\right)\right], \end{aligned} \tag{13}$$

where $M_{I_k}(\cdot)$ is the moment generating function of the indicator I_k. The moment generating function of the sum in (8) can also immediately be evaluated using

$$M_{S(t)}(u) = E\left(e^{S(t)u}\right) = E\left[\exp\left(\sum_{k=1}^{n(t)} Y_k u\right)\right]. \tag{14}$$

Equations (9) to (14) applies even if the individual risks are not independent. In the case of independence, the results are already well-known. See, for example, BOWERS, ET AL.(1997) and KLUGMAN, ET AL.(1998). It is well-known that the sum of independent Bernoulli trials has a binomial distribution. For illustrative purpose, we provide Figures 2 and 3 which displays the resulting distribution of the total number of claims when the Bernoulli trials are no longer independent. In each figure, there are 4 different distributions corresponding to various levels of correlations. Figure 2 is the case where we have $n = 10$ Bernoulli trials while Figure 3 corresponds to the case where $n = 50$ Bernoulli trials. These figures apparently indicate that departure from independence can lead to a whole different shape of the sum of the distribution particularly for higher correlations and for larger number of trials.

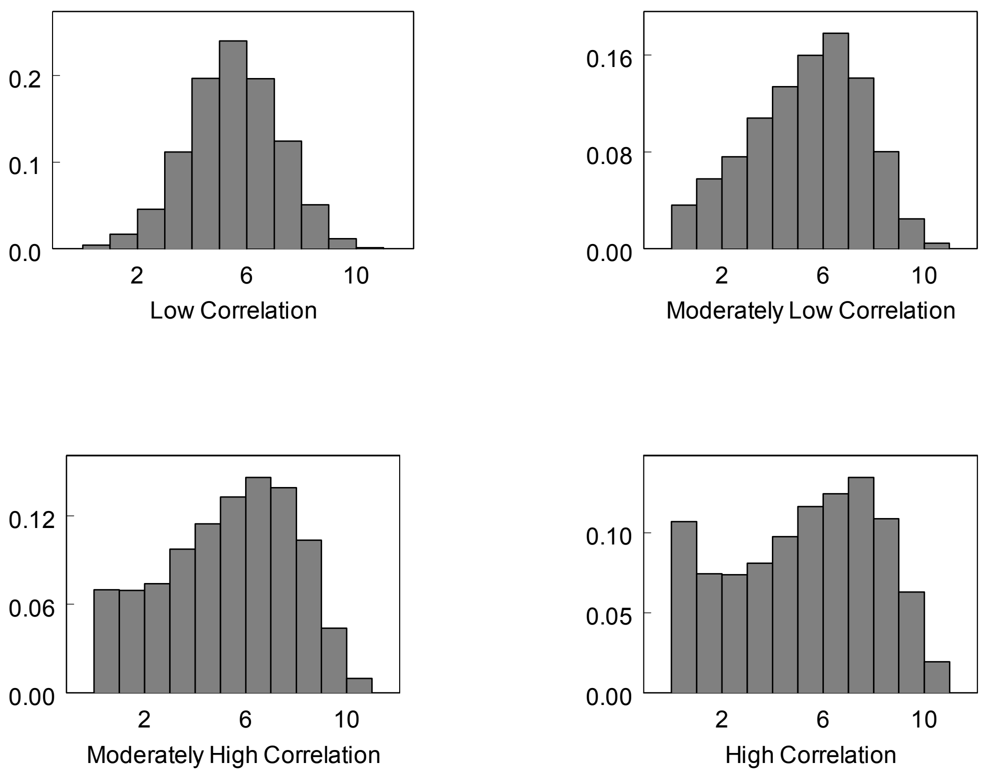

Figure 2: Distributions of Sums of $n = 10$ Bernoulli trials

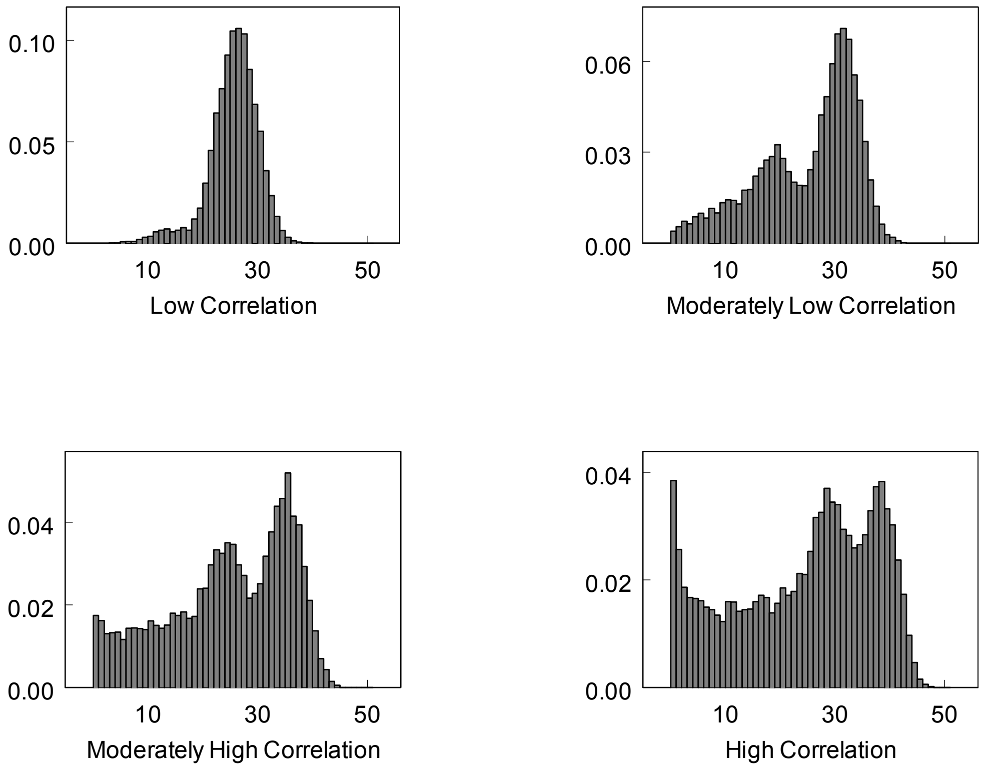

Figure 3: Distributions of Sums of $n = 50$ Bernoulli trials

3 The Joint Distribution of Claim Occurrences

Suppose that an n-dimensional random vector $\mathbf{X} = (X_1, X_2, ..., X_n)$ has the cumulative distribution function

$$F(x_1, x_2, ..., x_n) = \text{Prob}(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n). \tag{15}$$

We can decompose this c.d.f. F into the univariate marginals of X_k for $k = 1, 2, ..., n$ and another distribution function called a copula. Before formally defining a copula function, let us examine the properties of a multivariate distribution function. Following Joe (1997), a function F with support $\mathbf{R}^n$ and range $[0, 1]$ is a multivariate cumulative distribution function if it satisfies the following:

1. it is right-continuous;

2. $\lim_{x_k \to \infty} F(x_1, x_2, ..., x_n) = 0$, for $k = 1, 2, ..., n$;

3. $\lim_{x_k \to \infty, \, \forall k} F(x_1, x_2, ..., x_n) = 1$; and

4. the following rectangle inequality holds: for all $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ with $a_k \leq b_k$ for $k = 1, 2, ..., n$,we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} F(x_{1i_1}, ..., x_{ni_n}) \geq 0, \tag{16}$$

where $x_{k1} = a_k$ and $x_{k2} = b_k$.

Suppose $\mathbf{u} = (u_1, ..., u_n)$ belong to the n-cube $[0,1]^n$. A copula, $C(\mathbf{u})$, is a function, with support $[0,1]^n$ and range $[0,1]$, that is a multivariate cumulative distribution function whose univariate marginals are $U(0,1)$. As a consequence of this definition, we see that

$$C(u_1, ..., u_{k-1}, 0, u_{k+1}, ..., u_n) = 0 \tag{17}$$

and

$$C(1, ..., 1, u_k, 1, ..., 1) = u_k \tag{18}$$

for all $k = 1, 2, ..., n$. Any copula function C is therefore the distribution of a multivariate uniform random vector. From the definition of a multivariate distribution function, the rectangle inequality leads us to

$$\begin{aligned} & \text{Prob}(a_1 \leq U_1 \leq b_1, ..., a_n \leq U_n \leq b_n) \\ = & \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1i_1}, ..., u_{ni_n}) \geq 0, \end{aligned}$$

for all $u_k \in [0,1]$, $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ with $a_k \leq b_k$ for $k = 1, 2, ..., n$, and $u_{k1} = a_k$ and $u_{k2} = b_k$.

The significance of copulas in examining the dependence structure of $X_1, X_2, ..., X_n$ comes from a result which first appeared in Sklar (1959). Known as Sklar's theorem, it relates the marginal distribution functions to copulas. Suppose $\mathbf{X}$ is a random vector with joint distribution function F as expressed in (15). According to Sklar (1959), there exists a copula function C such that

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)) \tag{19}$$

where F_k is the kth univariate marginal, for $k = 1, 2, ..., n$. The function C need not be unique, but it is unique if the univariate marginals are absolutely continuous. For absolutely continuous univariate marginals, the unique copula function is clearly

$$C(u_1, ..., u_n) = F\left(F_1^{-1}(x_1), ..., F_n^{-1}(x_n)\right) \tag{20}$$

where $F_1^{-1}, ..., F_n^{-1}$ denote the quantile functions of the univariate marginals $F_1, ..., F_n$. From equation (19), it becomes apparent that the copula is a function which "couples,", "links," or "connects" the joint distribution to its marginals.

In the purely discrete case, denote the kth univariate distribution function by $F_k(i_k) = \text{Prob}(X_k \le i_k)$ together with its probability mass function

$$\begin{aligned} p_k(i_k) &= \text{Prob}(X_k = i_k) \\ &= \text{Prob}(X_k \le i_k) - \text{Prob}\left(X_k \le i_k^-\right) \\ &= F_k(i_k) - F_k\left(i_k^-\right), \end{aligned} \tag{21}$$

where i_k belongs to its set of support, say D_k. A copula function C then that is associated with the joint distribution of $X_1, ..., X_n$ will satisfy

$$P(i_1, ..., i_n) = C(F_1(i_1), ..., F_n(i_n))$$

for all $i_k \in D_k$, where $P(\cdot, ..., \cdot)$ denotes the cumulative distribution of the discrete random vector. Here the copula C although it exists, need not be unique.

An example of a copula is the independence copula which is given by

$$C(u_1, ..., u_n) = u_1 \cdots u_n$$

and is the copula associated with the joint distribution of independent random variables $X_1, X_2, ..., X_n$. This copula is often denoted simply by $\mathbf{\Pi}(u_1, ..., u_n)$. Another very important copula is the normal copula. Denote the density and cumulative distribution functions of a univariate standard normal by $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, so that

$$\Phi(z) = \int_{-\infty}^{z} \phi(w)\, dw, \text{ where } \phi(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}.$$

Consider an n-variate normal random vector $\mathbf{Z} = (Z_1, Z_2, ..., Z_n)$ with standard normal marginals, i.e. $Z_k \sim N(0,1)$ for $k = 1, 2, ..., n$ and positive-definite, symmetric variance-covariance matrix $\mathbf{V} = (v_{ij})$. Clearly, the elements of $\mathbf{V}$ satisfy

$$v_{ij} = \begin{cases} 1, & \text{if } i = j \\ \text{corr}(Z_i, Z_j), & \text{if } i \neq j \end{cases}.$$

The joint density of $\mathbf{Z}$ can be expressed as

$$f(z_1, ..., z_n) = \frac{1}{\sqrt{(2\pi)^n |\mathbf{V}|}} \exp\left(-\frac{1}{2}\mathbf{z}^T \mathbf{V} \mathbf{z}\right), \tag{22}$$

with $\mathbf{z} = (z_1, ..., z_n)$. Now denote the joint distribution function by

$$H(z_1, ..., z_n) = \int_{-\infty}^{z_n} \int_{-\infty}^{z_{n-1}} \cdots \int_{-\infty}^{z_1} f(z_1, ..., z_n)\, dz_1 \cdots dz_n. \tag{23}$$

The copula defined by

$$C(u_1, ..., u_n) = H\left(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)\right) \tag{24}$$

is called the normal copula and is easily seen to define a multivariate uniform cumulative distribution function. Although the copula in (24) does not appear to be simple in form, it generally leads to simple simulation procedures. For those interested, we refer them to the paper by WANG (1998).

4 Classical Results

Because we want to be able to compare our results with some already well-known results in the subject of ruin probabilities, we briefly summarize some of these classical results. In classical risk theory, the company's surplus process as earlier defined in equation (1) is re-written in the following form

$$U(t) = u_0 + \Pi(t) - S(t) = u_0 + \Pi(t) - \sum_{k=1}^{N(t)} X_k$$

where the aggregate claims process $\{S(t)\}$ consists of the sum of individual claim amounts assumed to be independent and identically distributed random variables, together with a random number of claims $\{N(t)\}$ also assumed to be independent of the claim sizes. The premium process typically has the form $\Pi(t) = ct$, where c, the rate of premium per unit of time, is expressed as

$$c = E(N)\,E(X)\,(1+\theta)\,, \tag{25}$$

with θ denoting the relative security loading. There is usually the additional positive requirement for this loading because otherwise ruin becomes certain. See ASMUSSEN (2000) for justification. The most common assumption for $\{N(t)\}$ is that it follows a Poisson process at a rate of λ. The aggregate claims process then is said to have a compound Poisson distribution, a family of compound distributions which may include those where the number of claims is not Poisson. See PANJER AND WILLMOT (1984). A useful result for computing ruin probability is

$$\Psi(u_0) = \frac{e^{-Ru_0}}{E\left[e^{-RU(T)}\,|T<\infty\right]}. \tag{26}$$

A proof for this is available in standard textbooks like BOWERS, ET AL. (1997) and KLUGMAN, ET AL. (1998) and is usually based on the compound Poisson assumption. However, ASMUSSEN (2000) proved it using martingales, without assuming compound Poisson distribution, but instead imposing a martingale requirement on the process $\left\{e^{-RS(t)}\right\}$. Here, R refers to the adjustment coefficient and is the smallest positive solution to

$$E\left\{e^{-r[S(t)-ct]}\right\} = 1. \tag{27}$$

In the case of compound Poisson distribution for the aggregate claims, the adjustment coefficient formula in (27) becomes

$$\lambda\left[M_X(r) - 1\right] = cr. \tag{28}$$

In addition, in the compound Poisson model, the probability of ruin satisfies the following integro-differential equation

$$\Psi'(u_0) = \frac{\lambda}{c}\left\{\Psi(u_0) - \int_0^{u_0}\Psi(u_0 - x)\,dF_X(x) - \left[1 - F_X(u_0)\right]\right\} \tag{29}$$

which can be alternatively used to evaluate ruin probabilities. The proof for (29) can be found in KLUGMAN, ET AL. (1998) and ROLSKI, ET AL. (1999). Another less familiar representation of the probability of ruin is the so-called *Pollaczek-Khinchine formula* and is given by

$$\Psi(u_0) = \left[1 - \frac{\lambda}{c} E(X)\right] \sum_{n=1}^{\infty} \left[\frac{\lambda}{c} E(X)\right]^n \left[\overline{F_X^{S}}^{*n}(u_0)\right] \tag{30}$$

where $\overline{F}$ denotes right-tail probability and $F_X^S(u) = \frac{1}{E(X)} \int_0^{u_0} \overline{F}_X(u)\, du$.

In the compound Poisson model where individual claims are assumed to be exponential with mean parameter α^{-1}, the formula for the probability of ruin is given by

$$\Psi(u_0) = \frac{\lambda}{\alpha c} e^{-(\alpha - \lambda/c)u_0}.$$

One may use either of the methods described above to show this. Furthermore, one may also derive explicit solutions for the probability of ruin for other claim distributions such as mixture of exponentials, gamma, Erlang, or the more broad family of distributions called the *Phase-Type family.*

Because it is difficult to derive explicit solutions for the probability of ruin for many classes of distributions particularly those with heavy tails, approximations and bounds have been developed over the years as alternatives. For example, one can use the integro-differential equation above to derive the Cramer-Lundberg approximation:

$$\Psi(u_0) \approx Ce^{-Ru_0} \tag{31}$$

where

$$C = \frac{\theta E(X)}{E(Xe^{RX}) - (1+\theta) E(X)}.$$

The Lundberg inequality

$$\Psi(u_0) \leq Ce^{-Ru_0} \tag{32}$$

provides an upper bound. Both these results first appeared respectively in Cramer (1930) and Lundberg (1909).

There has been some recent developments and attempts in evaluating probability of ruin when there is departure from the assumption of independence. Besides some of those work already mentioned in the introduction, one interesting result is attributed to Glynn and White (1994). A proof is also outlined in Asmussen (2000).

Consider a sequence of random variables $X_1, X_2, ...$ and denote by $S_n = \sum_{k=1}^{n} X_k$, $T = \inf\{n : S_n > u_0\}$ and $\Psi(u_0) = \text{Prob}(T < \infty)$. Then Glynn & White proved the following large deviation result for the probability of ruin:

$$\lim_{u_0 \to \infty} \frac{\log[\Psi(u_0)]}{\log[e^{-Ru_0}]} = 1, \tag{33}$$

subject to some regularity condition on the cumulant generating function

$$\kappa_n(t) = \log E\left(e^{tS_n}\right).$$

Here, the resulting adjustment coefficient R is defined by the solution to

$$\lim_{n \to \infty} \frac{1}{n} \kappa_n(R) = 0.$$

In effect, the adjustment coefficient then gives an approximation similar to a Lundberg exponential bound:

$$\Psi(u_0) \approx e^{-Ru_0}.$$

The proof of this exercise is based on the idea of "large deviations". Similar idea has been used by NYRHINEN (1998) to develop the suitable Lundberg adjustment coefficient for a general dependency structure. This adjustment coefficient, as again stated earlier, has been investigated by ALBRECHER AND KANTOR (2002) and they found that the coefficient actually decreases with increasing positive dependence. Here, dependence of claims is represented by consecutive claims with a Markov-type structure.

5 The Simulation Procedure

We now describe the simulation process employed to generate estimates of probabilities of ruin and the distributions for time-to-ruin resulting from those that experienced ruin. We simulated a large number of trajectories or sample paths of the surplus process as defined in (1) by first initializing a surplus at time 0 of u_0, and then generating claims and receiving premiums. Figure 4 provides a few sample path realizations from the simulation procedure, with the bottom half providing sample trajectories that led to ruin. We followed the process until ruin occurs. It was possible that some of these sample paths will never lead to ruin and therefore necessary to terminate the process at some finite time, say t^+. Ruin therefore occurs when the sample path led to a surplus below zero prior to the right-censor time t^+. Our unit of time period is practically "months", which means that corresponding to a right-censor time of $t^+ = 1,200$, this is equivalent to 100 "years". We counted the number of sample paths then where ruin has occurred and divided that number by the number M of simulated paths to derive the estimated probability of ruin. Denoting by $I_k(\cdot)$ the indicator of ruin for the kth path and by T the time to ruin, the estimate for a finite time ruin probability can be expressed as

$$\widehat{\Psi}\left(u_0, t^+\right) = \frac{1}{M}\sum_{k=1}^{M} I_k\left(T < t^+\right). \tag{34}$$

It is clear that this estimate is unbiased, that is,

$$E\left[\widehat{\Psi}\left(u_0, t^+\right)\right] = \frac{1}{M}\sum_{k=1}^{M} E\left[I_k\left(T < t^+\right)\right] = \frac{1}{M}\sum_{k=1}^{M} \Psi\left(u_0, t^+\right) = \Psi\left(u_0, t^+\right),$$

and that its variance can be evaluated using

$$Var\left[\widehat{\Psi}\left(u_0, t^+\right)\right] = \frac{1}{M^2}\sum_{k=1}^{M} Var\left[I_k\left(T < t^+\right)\right] = \frac{1}{M}\left\{\Psi\left(u_0, t^+\right)\left[1 - \Psi\left(u_0, t^+\right)\right]\right\}. \tag{35}$$

Using large-sample arguments, a $100(1-\alpha)\%$ confidence interval can therefore be developed using

$$\widehat{\Psi}\left(u_0, t^+\right) \pm z_{1-\alpha/2} \cdot \sqrt{\frac{1}{M}\left\{\widehat{\Psi}\left(u_0, t^+\right)\left[1 - \widehat{\Psi}\left(u_0, t^+\right)\right]\right\}}. \tag{36}$$

We generate each trajectory by first subdividing the whole time period of t^+ into unit time periods which for convenience corresponds to "month" and generated components in the surplus stochastic process. We assumed a total of $n = 10,000$ static policies, that is, there will always be this much exposure in the insurance portfolio for each time period. We denoted by q the probability of a claim per time period so that by assuming a probability of a claim of 0.01 in 12 months, we have $q = 1 - 0.99^{1/12}$.

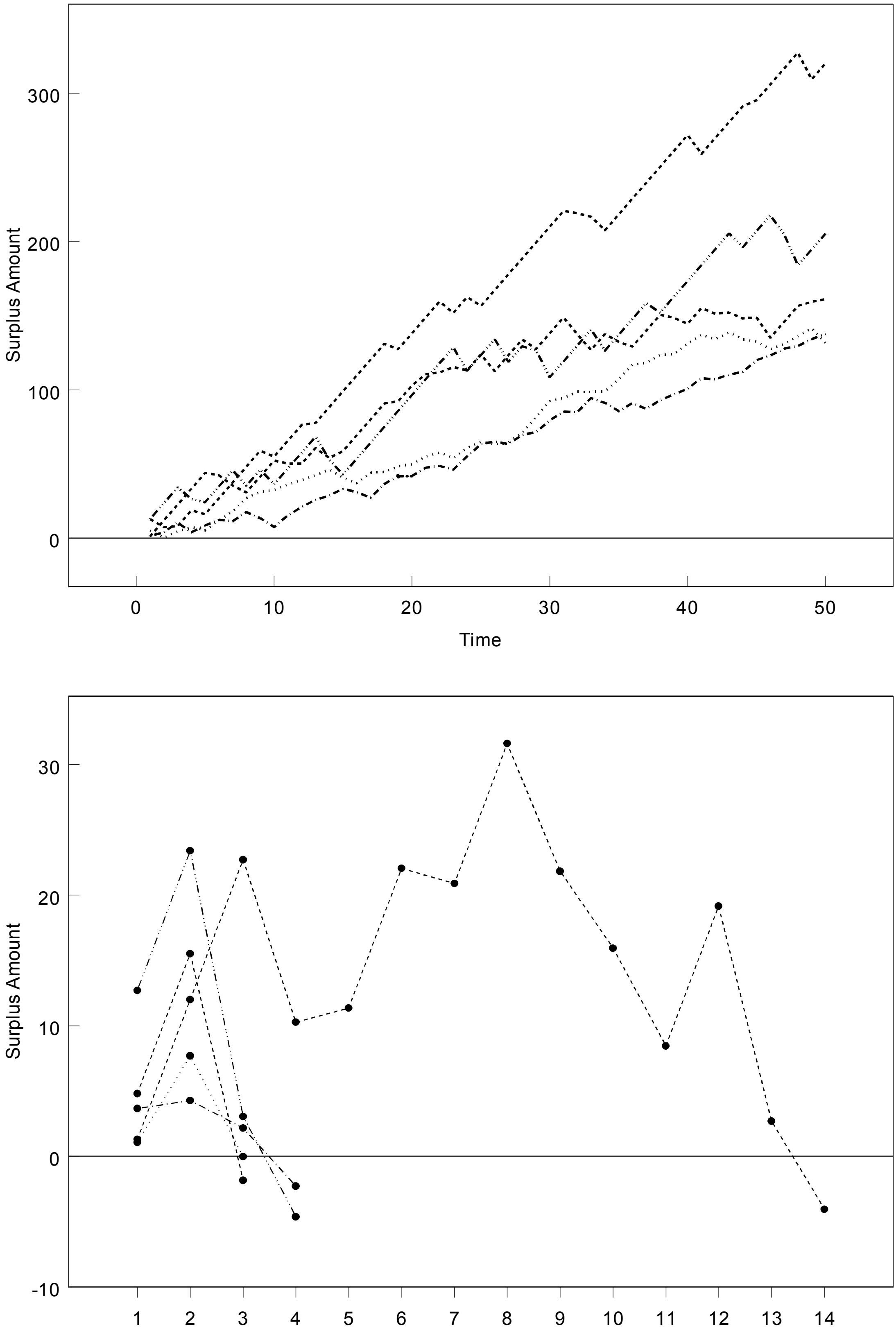

Figure 4: Some Sample Path Realization of the Surplus Process

Individual claims are assumed to follow a specified distribution with cumulative distribution function denoted by F_B and mean $E(B)$. For the premium rate process, we assumed a loading factor of θ so that in each time period, the premium received was equal to $nqE(B)(1+\theta)$ and in effect,

$$\Pi(t) = nqE(B)(1+\theta)t$$

where none of these variables are stochastic. The key component in the simulation is generating the claims. For each time period $[t, t+1)$ for each individual policy, we generated a value of 0 to indicate no claim and 1 to indiciate a claim. This is accomplished by simulating an $n \times 1$ vector $(u_1, ..., u_n)$ with uniform marginals whereby the dependence structure for claims incidence is specified. This vector is then converted to a vector consisting of 1's and 0's, with 1 indicating occurrence of claims when $u_k < q$. We then counted the number of claims in the time period by setting

$$N = \sum_{k=1}^{n} I(u_k < q)$$

where $I(\cdot)$ is an indicator function. For these N policies that went on claim, we generated a vector of independent claim amounts $(b_1, b_2, ..., b_N)$ with marginals from F_B, using the standard method of inverting the cumulative distribution function. For our purposes, we assumed that in each time period, there is always $n = 10,000$ policies, which accordingly gives a stationary number of policyholders. In effect, we are assuming that terminated policies are being replaced by new policies.

For claim incidence where the dependence structure was injected, we generated the vector according to the specified copula. For presentation in this paper, we specifically chose the Frank's copula which in the multivariate dimension, the form is given by

$$C(u_1, ..., u_n) = -\frac{1}{\log \eta} \log \left[1 + \frac{\prod_{k=1}^{n} (\eta^{u_k} - 1)}{(\eta - 1)^{n-1}} \right]. \tag{37}$$

In can be shown that this satisfies the definition of a copula. Note that the Frank's family of copulas belong to the class of Archimedean copula which are of the form

$$C(u_1, ..., u_n) = \psi^{-1}[\psi(u_1) + \cdots + \psi(u_n)]$$

where the generator for this family is

$$\psi(t) = \log\left(\frac{\eta^t - 1}{\eta - 1}\right).$$

This generator is a convex function and therefore from NELSEN (1998), the multivariate version satisfies the definition of a copula. Now, to generate random vectors from the Frank's family of copulas, we employ the algorithm suggested by MARSHALL AND OLKIN (1988) which showed that the Frank copula can be constructed from a frailty framework with the frailty random variable being a discrete logarithmic random variable Z with parameter $1 - \eta$. Thus, to generate from a multivariate Frank copula, we:

1. Generate a z from a discrete logarithmic variable with parameter $1 - \eta$. DEVROYE (1986) provides for algorithm to do this.

2. Generate n independent and identically distributed Uniform$[0, 1]$ random variables; denote this by $\mathbf{u}^* = (u_1^*, ..., u_n^*)$.

3. Set $\mathbf{u} = M_Z(z^{-1} \log \mathbf{u}^*)$ where $M_Z(\cdot)$ is the moment generating function for Z and is equal to $M_Z(t) = \frac{\log(1-(1-\eta)e^t)}{\log(\eta)}$, and $\log \mathbf{u}^* = (\log u_1^*, ..., \log u_n^*)$.

6 Results and Discussion

Our initial round of simulation results are summarized and discussed in this section. For simplicity, we set the loading factor to be $\theta = 28\%$ and individual claim amounts follow an exponential distribution with mean 1. We examined various level of dependence structure: $\eta = 0.1$, 0.2, 0.4, 0.9, 1.0 where $\eta = 1$ corresponds to the case of independence. We also examined various levels of initial surplus and the total number of sample paths simulated were $10,000$.

Figure 5 provides the ruin probability estimates $\widehat{\Psi}(u_0, t^+)$, together with 95% confidence interval, as a function of intial capital or surplus u_0 in the case where $\eta = 0.1$ corresponding to a high correlation, $\eta = 0.2$ corresponding to a moderately high correlation, $\eta = 0.4$ corresponding to a moderate correlation, $\eta = 0.9$ corresponding to a low correlation, and $\eta = 1$ the case of independence. We only examined positive levels of correlation because we believe in reality the claims occurrence will exhibit positive correlation.

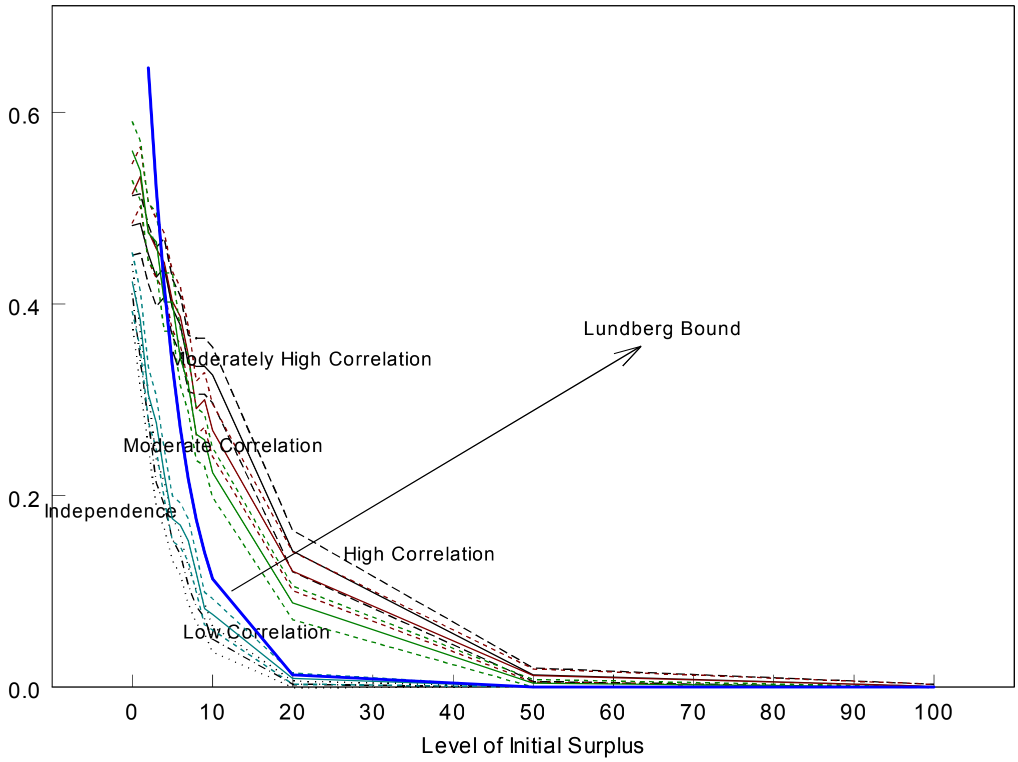

Figure 5: Probabilities of Ruin $\widehat{\Psi}\,(u_0, t^+)$ for Various Levels of Dependence

The Lundberg upper bound that appear in Figure 5 using equation (32) with the adjustment coefficient calculated assuming the Compound Poisson approximation to the individual model as described in Gooaverts and Dhaene (1996). There are a couple of observations that can be made from this figure. The Lundberg bound always stays above the ruin probability in the case of independence, however, such is not always true particularly when there is a stronger level of dependence. The higher this level of dependence, as measured by the Frank's copula parameter η, the further outward it departs from this bound. This means that in the presence of correlations between claims, there is a higher chance the company will ruin than that indicated by the case of independence, which is what is often used in practice, and much more so than that indicated by the Lundberg approximation. However, another interesting observation can be made though with low level of initial surplus. While it can be observed that larger dependence in claims lead to larger probability of ruin in general, the Lundberg approximation appears to provide an upper bound even for higher correlations in claims, but only for certain low levels of initial surplus.

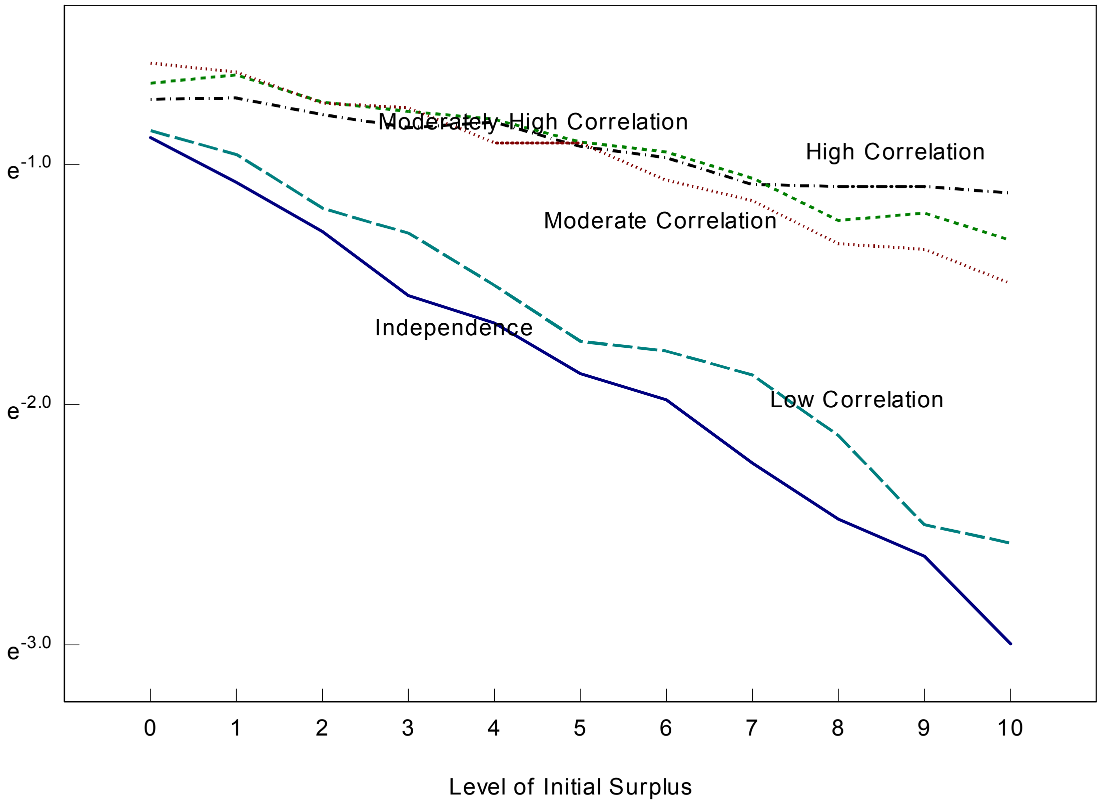

Figure 6: Probabilities of Ruin $\widehat{\Psi}\left(u_0, t^+\right)$ on a Logarithmic Scale

In Figure 7, we display probabilities of ruin as a function of η, the dependence parameter in the Frank's copula. Recall that $\eta = 1$ corresponds to the case of independence. First, we note that even in the presence of dependence of claims, generally smaller initial surplus lead to higher levels of probabilities of ruin. This is generally true in the case of independence and it is just being carried over in the case of non-independence. When viewed as a function of the dependence parameter, it appears that probabilities of ruin increases with levels of correlation. Note that the dependence parameter η in the Frank's copula is inversely proportional to correlation measures. For larger surplus, the distinction for various levels of dependence becomes immaterial because the level of ruin probabilities are already so small to be able to make a distinction.

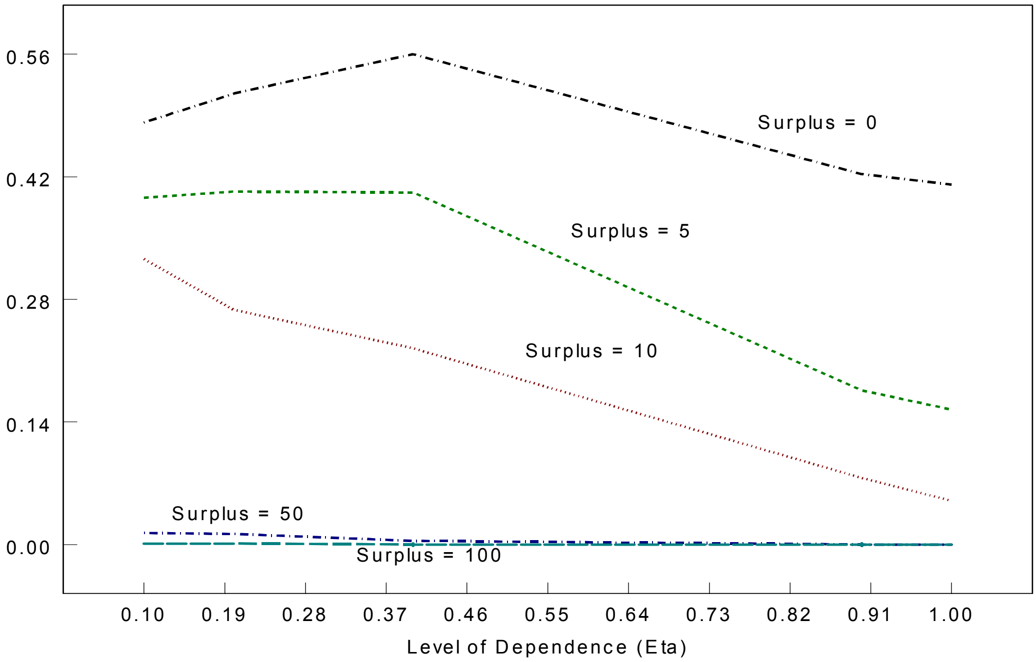

Figure 7: Probabilities of Ruin $\widehat{\Psi}(u_0, t^+)$ as a Function of Dependence η

A common important question often asked is now given an insurance company ruins, what is the most likely time of ruin? When does ruin actually occur, given that it occurs? Our simulation results can easily produce the time-to-ruin for those simulated paths where ruin has occurred. We recorded the time-of-ruin and Figure 8 provides the distribution of the time-to-ruin, given ruin occurs, in the case of independence. In contrast, we provide Figure 9 which displays the distribution of the time-to-ruin, given ruin occurs, for the case where $\eta = 0.1$ corresponding to a high correlation. Not to overwhelm the reader, we display this comparison here only between the case of independence and the case of strong dependence. For the other levels of η, the results are graphically displayed in the appendix.

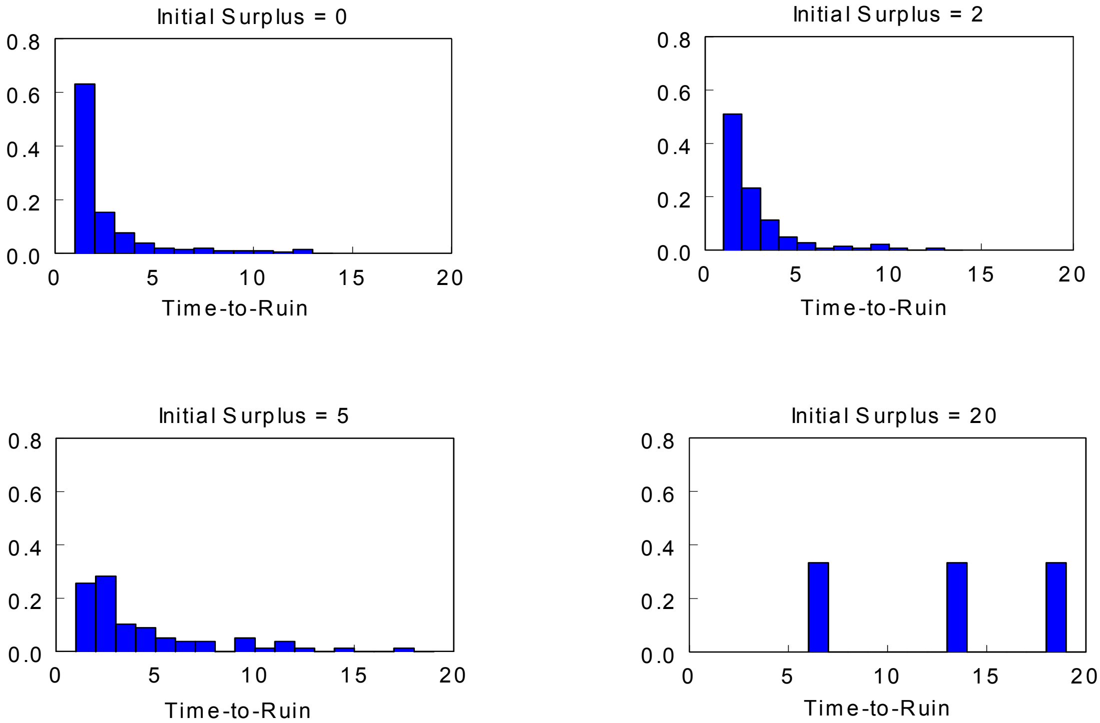

Figure 8: Distribution of the Time-to-Ruin in the Case of Independence

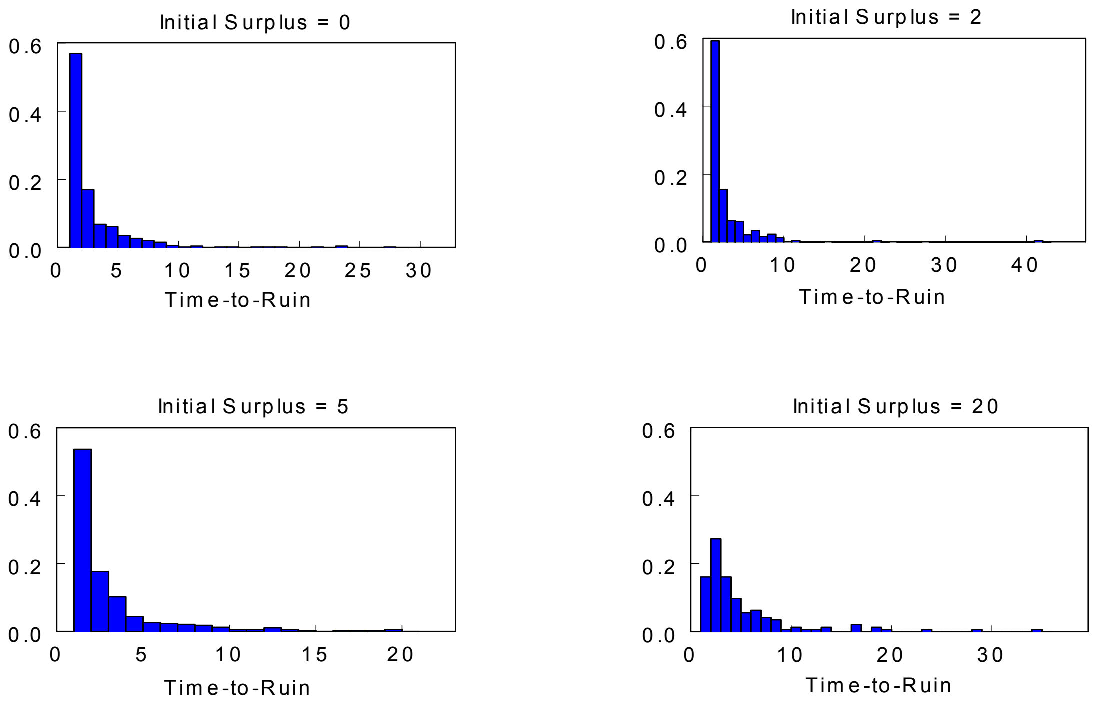

Figure 9: Distribution of the Time-to-Ruin in the Case of High Correlation ($\eta = 0.1$)

Table 1

Some Summary Statistics on the Time-to-Ruin

	Initial Surplus $u_0 = 0$					Initial Surplus $u_0 = 2$				
	Dependence (η)					Dependence (η)				
Statistics	1.0	0.9	0.4	0.2	0.1	1.0	0.9	0.4	0.2	0.1
Number	207	332	560	541	479	140	233	474	476	473
Mean	2.1	2.1	2.0	2.5	2.5	2.2	2.3	2.5	2.2	2.5
Std Dev	2.2	2.6	2.6	3.8	3.0	2.0	2.3	3.5	3.2	3.8
Minimum	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
Maximum	12.0	29.0	25.0	42.0	27.0	12.0	18.0	34.0	32.0	41.0
	Initial Surplus $u_0 = 5$					Initial Surplus $u_0 = 20$				
	Dependence (η)					Dependence (η)				
Statistics	1.0	0.9	0.4	0.2	0.1	1.0	0.9	0.4	0.2	0.1
Number	78	138	402	404	388	3	7	88	121	142
Mean	3.8	3.5	3.3	2.8	2.6	12.3	9.4	8.3	6.0	4.7
Std Dev	3.5	3.2	3.9	3.6	3.0	6.0	5.9	8.5	5.2	5.2
Minimum	1.0	1.0	1.0	1.0	1.0	6.0	2.0	1.0	1.0	1.0
Maximum	17.00	19.0	38.0	26.0	19.0	18.0	17.0	52.0	32.0	34.0

Alternatively, we provide some summary statistics (number, mean, standard deviation, minimum and maximum) in Table 1 of the distributions of the time-to-ruin. This table provides the values for all levels of dependence examined in this paper and for various levels of surplus. We display only surplus level of up to $u_0 = 20$, because beyond this level, ruin becomes rare so that the distribution of the time-to-ruin becomes meaningless.

According to these results, the average time to ruin, given ruin occurs, is about 2 to 3 time periods in the case where there is zero initial surplus. This seems to be not at all surprising given the main purpose of an initial capital is to be able to absorb adverse and unexpected deviations. If there is no such surplus available in the beginning to act as a buffer, the company is expected to ruin early and in fact, the probability of ruin is so much higher than with those companies having more than zero initial surplus. For higher levels of surplus, this leads to a longer average time-to-ruin, apparently the initial surplus provides some capacity to absorb shocks from expectations particularly in claims. This observation is true across various levels of dependence in claims. For example, when surplus is 5, the average time-to-ruin ranges from 2.6 to about 3.8 time periods. There is shorter time-to-ruin for larger level of correlations in claims. Again, this is probably intuitively right as the positive correlations generally imply that claims induce other claims, therefore, whenever a claim occurs, this increases the probability that other policyholders will also claim.

7 Conclusion

In this paper, we have attempted to analyze ruin probabilities in the presence of dependent claims. It is important to recognize the possibility that claims within an insurance portfolio exhibit some form of dependence. Just consider the following:

- A policyholder may have multiple insurance policies and thereby creating a portfolio with duplicate policies;
- Insurance coverage may be for members of a family or employees of an organization; and
- There may be other common characteristics of a group of policyholders in the portfolio that possibly create dependencies such as common location for which they may be exposed to natural catastrophes like floods or earthquakes.

This paper advocates the use of a copula function to specify the dependence structure possible within the insurance portfolio. In particular, we assumed that the occurrence of claims are dependent and this is where we specified the copula structure, on the incidence of claims. However, we continued with the usual assumption that amounts of claims are independent, and that within a policyholder, its incidence and amount are also independent. One of the primary advantage of using the copula to specify the dependence structure is its versatility. It is easy to incorporate into the model and we believe that all the information about the dependency within the claims is embedded into this single copula function. It was then easy to examine the effects of various dependency structures on the probability of ruin. However, as in the case of independence, it is often difficult to get closed-form solutions for the probabilitities of ruin. This paper uses simulation to perform the analysis.

Our simulation results reveal that in the presence of dependency in claim occurrence:

1. Ruin probabilities are much higher than that indicated by the case of independence; the difference is much larger for higher level of initial surplus.

2. Ruin probabilities can violate the Lundberg upper bound particularly for large initial surpluses, not so with low level of initial surplus.

3. Ruin probabilities is an increasing function of the level of correlation among claims.

4. Given ruin occurs, there does not appear to be significant differences in the distribution of time-to-ruin, except that there is slightly higher frequencies of ruining early for stronger dependence.

We ask the reader to interpret these results with caution. The results revealed are based on a set of assumptions for which the reader must understand. Sensitivity of some of these assumptions may well have to be examined. Another possible limitation of our model specification is applying the copula structure on a set of Bernoulli random variables. It is well known that the copula representation of dependent discrete random variables is not unique. There is nothing wrong with specifying a copula on dependent discrete random variables like the way we did in this paper. But this is part of our assumption, that the inputted copula function is correct. However, in practice, this copula representation may well have to be estimated from data, in which case, the non-uniqueness of the copula may pose some estimation problems. Perhaps a different dependence representation may be imposed, but still the procedures employed here in this paper would still be useful.

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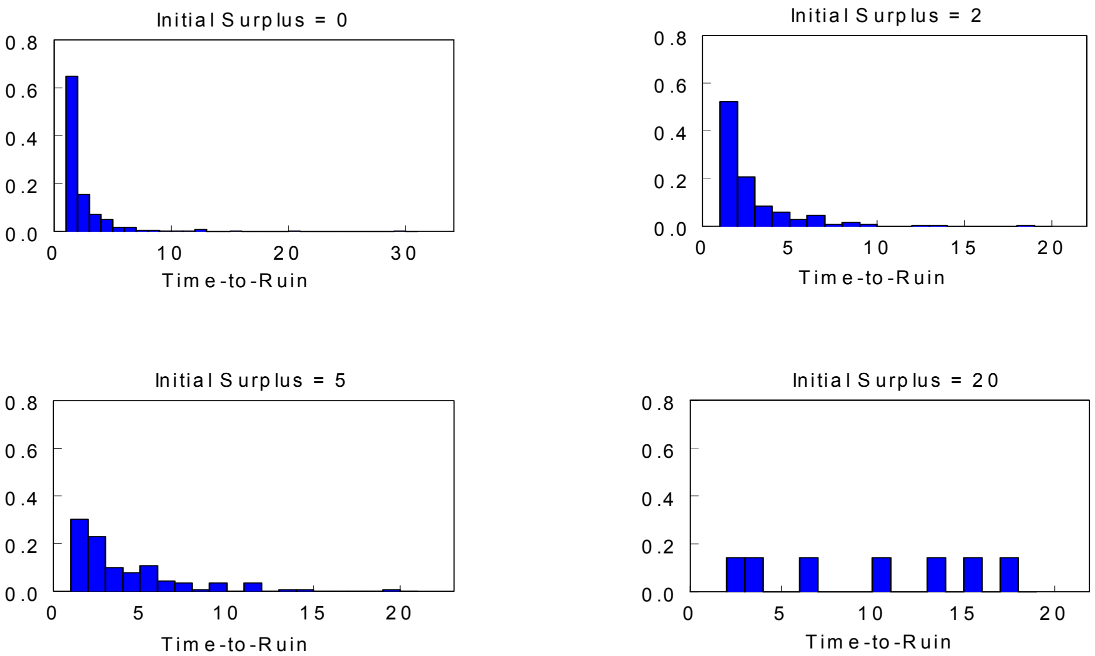
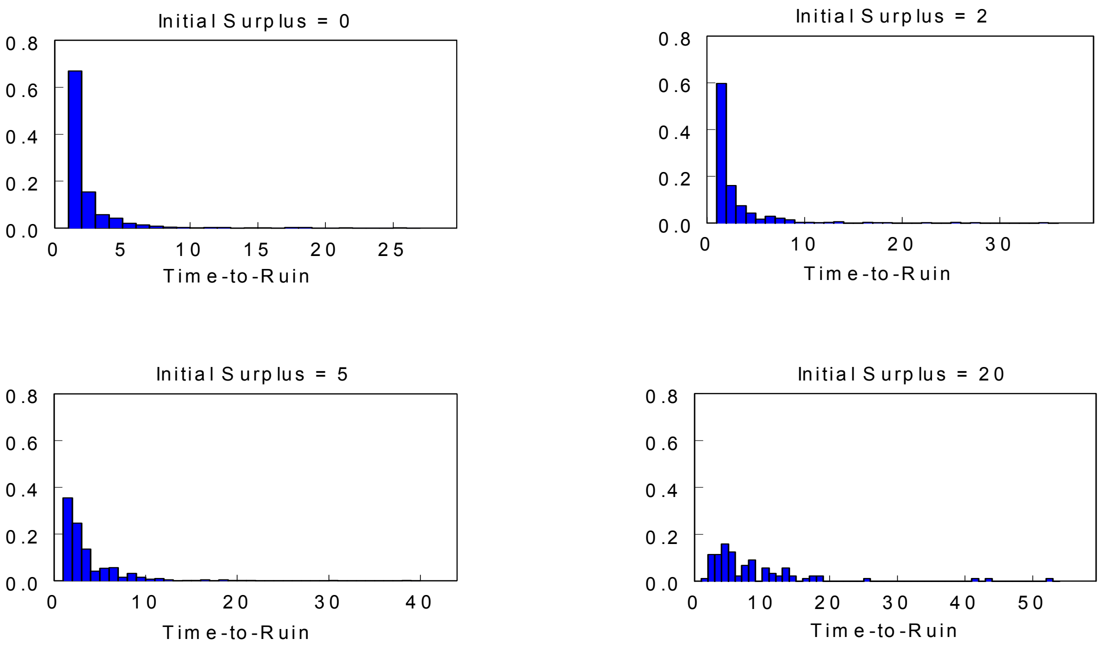
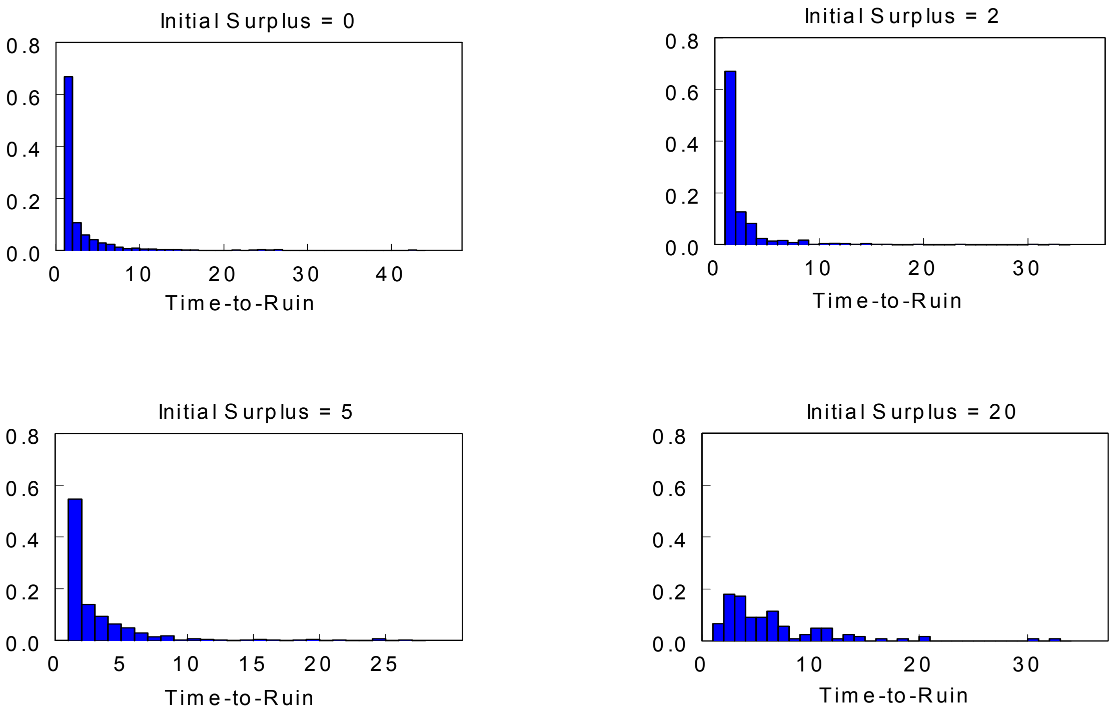
Appendix



Appendix 1a: Distribution of Time-to-Ruin in the Case where $\eta = 0.9$



Appendix 1b: Distribution of Time-to-Ruin in the Case where $\eta = 0.4$



Appendix 1c: Distribution of Time-to-Ruin in the Case where $\eta = 0.2$