Simulating Exchangeable Multivariate Archimedean Copulas and its Applications*

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February 2006

Abstract

Multivariate exchangeable Archimedean copulas are one of the most popular classes of copulas that are used in actuarial science and finance for modelling risk dependencies and for using them to quantify the magnitude of tail dependence. Owing to the increase in popularity of copulas to measure dependent risks, generating multivariate copulas has become a very crucial exercise. Current methods for generating multivariate Archimedean copulas could become a very difficult task as the number of dimension increases. The resulting analytical procedures suggested in the existing literature do not offer much guidance for practical implementation. This paper presents an algorithm for generating multivariate exchangeable Archimedean copulas based on a multivariate extension of a bivariate result. A procedure for generating bivariate Archimedean copulas has been originally proposed in Genest and Rivest (1993) and again later described in Nelsen (1999) and Embrechts, et al. (2002a, 2002b). Using a proof that is simply based on fundamental Jacobian techniques for deriving distributions of transformed random variables, we are able to extend the bivariate result into the multivariate case allowing us to develop an interesting algorithm to generate exchangeable Archimedean copulas. As auxiliary results, we are able to derive the distribution function of an n-dimensional Archimedean copula, a result already known in Genest and Rivest (2001) but our approach of proving this result is based on a different perspective. This paper focuses on this class of copulas that has one generating function and one parameter that characterizes the dependence structure of the joint distribution function.

*The authors thank the Australian Research Council through the Discovery Grant DP0345036 and the UNSW Actuarial Foundation of the Institute of Actuaries of Australia for financial support. Corresponding author.

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1 Preliminaries and motivation

This paper develops practical algorithms for simulating from a class of exchangeable multivariate Archimedean copulas. Archimedean copulas are increasingly popular in actuarial science and financial risk management for reasons that they are easily constructed, practically implementable, and possess many interesting properties making them suitable for modelling various dependence structures. Recent literature on simulations of multivariate Archimedean copulas can be seen in Whelan (2004). His work requires evaluation of contour Cauchy integrals. Archimedean copulas are often characterized by a generator, which is a single-valued function, thereby reducing the search for a high dimensional distribution function. Let φ be a mapping from $[0,1]$ to $[0,1]$ and we shall call this function an Archimedean generator if it satisfies the following three conditions (see Chapter 4 of Nelsen (1999)):

1. $\varphi(1) = 0$;
2. φ is monotonically decreasing; and
3. φ is convex.

Let $\mathbf{u} = (u_1, ..., u_n)'$ be an n-dimensional unit vector with $u_k \in [0,1]$ for all $k = 1, ..., n$. A copula C is termed Archimedean if there exists a generator function φ such that C has the form

$$C(\mathbf{u}) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_n)), \tag{1}$$

where φ^{-1} denotes the inverse function of the generator. Notice that if the first derivative, φ', exists, then by condition (2), we must have $\varphi' \leq 0$, and if the second derivative, φ'', exists, then by condition (3), we must have $\varphi'' \leq 0$. For our purposes, we consider only Archimedean generators which are continuous and whose higher derivatives exist. For C to be an n-dimensional copula, φ^{-1} must be completely monotone so that if all derivatives exist, we must have

$$(-1)^k \frac{d^k \varphi^{-1}(u)}{du^k} \geq 0 \text{ for } k = 1, 2, ..., n.$$

The definition of "completely monotonic" is defined in page 122 of Nelsen (1999). Archimedean generators associated with a particular Archimedean copula are not necessarily unique, but they are up to a constant. If φ is a generator of an Archimedean copula, then $a\varphi$ for some positive constant a also generates the same Archimedean copula.

Section 3 gives examples of Archimedean generators and their corresponding copulas. For example, the Frank copula is generated by $\varphi(u) = -\log\left[\left(e^{\theta u} - 1\right) / \left(e^{\theta} - 1\right)\right]$ for $\theta \neq 1$ but has the independence copula as a limiting case when $\theta \to 1$. The Gumbel copula has generator

$$\varphi(u) = (-\log u)^{1/\theta} \tag{2}$$

for some $\theta > 1$ and is therefore useful for describing positive dependencies. Our numerical illustration later in the paper focuses on the applications of this copula. In addition, the independence copula is indeed Archimedean in the sense that the generator is $\varphi(u) = -\log(u)$. Archimedean copulas first appeared in Genest and McKay

(1986) and their statistical properties for inference procedures are well established in Genest and Rivest (1993).

Our objective in this paper is to develop an algorithm to generate an n-dimensional random vector $\mathbf{X} = (X_1, ..., X_n)'$ whose distribution is

$$F_{\mathbf{X}}(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n))$$

where F_k denotes the marginal distribution function for $k = 1, ..., n$ and C has the copula of the form (1). There has been a number of simulation algorithms offered in the literature some of which are discussed in Frees and Valdez (1998), but the most popular has been the bivariate case constructed using the distribution function of the bivariate copula. The resulting algorithm is based on the following Theorem.

Theorem 1 *Let $(U_1, U_2)'$ be a bivariate random vector with uniform marginals and let its bivariate distribution function be defined by the Archimedean copula of the form $C(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$ for some generator φ. Define the random variables $S = \varphi(U_1) / [\varphi(U_1) + \varphi(U_2)]$ and $T = C(U_1, U_2)$. The joint distribution function of (S, T) is characterized by*

$$H(s, t) = P(S \leq s, T \leq t) = s \times K_C(t) \tag{3}$$

where $K_C(t) = t - \varphi(t) / \varphi'(t)$.

The proof of this theorem are detailed in Nelsen (1999), Genest and Rivest (1993), and Embrechts, et al. (2001).

We see from this Theorem that S and T are independent and that S is uniformly distributed on $(0, 1)$. This result provides a mechanism for then generating from a bivariate Archimedean copula as follows. Suppose (X_1, X_2) has a bivariate distribution function based on the two-dimensional Archimedean copula with generator φ and marginal distributions F_1 and F_2. The resulting simulation procedure is then:

1. Simulate two independent $U(0, 1)$ random variables, say s and w.
2. Set $t = K_C^{-1}(w)$ where $K_C(t) = t - \varphi(t) / \varphi'(t)$.
3. Set $u_1 = \varphi^{-1}(s\varphi(t))$ and $u_2 = \varphi^{-1}((1 - s)\varphi(t))$.
4. The desired simulated values are $x_1 = F_1^{-1}(u_1)$ and $x_2 = F_2^{-1}(u_2)$.

This paper provides the multi-dimensional extension of the result in the above theorem so that a corresponding simulation algorithm can be developed. To accomplish this, we begin with an n-dimensional random vector $\mathbf{U} = (U_1, ..., U_n)'$ with uniform $U(0, 1)$ marginals and with distribution function generated by an Archimedean generator φ so that

$$C(u_1, ..., u_n) = P(U_1 \leq u_1, ..., U_n \leq u_n) = \varphi^{-1}\left(\sum\nolimits_{k=1}^{n} \varphi(u_k)\right).$$

Define the transformed random variables

$$S_k = \sum\nolimits_{j=1}^{k} \varphi(u_j) / \sum\nolimits_{j=1}^{k+1} \varphi(u_j) \text{ for } k = 1, ..., n-1 \tag{4}$$

and together with the copula function

$$T = C(U_1, ..., U_n) = \varphi^{-1}\left(\sum_{k=1}^{n} \varphi(U_k)\right), \tag{5}$$

we are able to characterize the resulting joint distribution of the transformed random vector $(S_1, ..., S_{n-1}, T)$ using the method of Jacobian transformation. Furthermore, as in the two dimensional case, we find that indeed $S_1, ..., S_{n-1}$ and T are independent. The details of the sketch of these results are discussed in the subsequent section which provide for the main result of this paper.

2 Main proposition

In this section of the paper, we discuss the main result and provide a sketch of the detailed proof. First, notice that the inverse of the transformations in (4) and (5) satisfy the following set of equations:

$$\begin{aligned}
\varphi(U_1) &= S_1 \cdots S_{n-1}\varphi(T) \\
\varphi(U_2) &= (1-S_1) S_2 \cdots S_{n-1}\varphi(T) \\
\varphi(U_3) &= (1-S_2) S_3 \cdots S_{n-1}\varphi(T) \\
& \quad \ldots\ldots \\
\varphi(U_{n-1}) &= (1-S_{n-2}) S_{n-1}\varphi(T) \\
\varphi(U_n) &= (1-S_{n-1})\varphi(T)
\end{aligned} \tag{6}$$

and we shall denote by J the Jacobian $n \times n$ matrix of the transformation as defined by

$$J = \left(\frac{\partial(u_1, ..., u_n)}{\partial(s_1, ..., s_{n-1}, t)}\right) = \begin{pmatrix} \partial u_1/\partial s_1 & \partial u_1/\partial s_2 & . \; . \; . & \partial u_1/\partial t \\ \partial u_2/\partial s_1 & \partial u_2/\partial s_2 & . \; . \; . & \partial u_2/\partial t \\ . & . & . \; . \; . & . \\ . & . & . \; . \; . & . \\ . & . & . \; . \; . & . \\ \partial u_n/\partial s_1 & \partial u_n/\partial s_2 & . \; . \; . & \partial u_n/\partial t \end{pmatrix}.$$

In the appendix, we show that the determinant of this Jacobian matrix has the following representation

$$|J| = s_1^0 s_2^1 s_3^2 \cdots s_{n-1}^{n-2} \frac{[\varphi(t)]^{n-1}\varphi'(t)}{\varphi'(u_1)\cdots\varphi'(u_n)}. \tag{7}$$

We now state and prove the main proposition in this paper.

Theorem 2 *Let $(U_1, ..., U_n)'$ be an n-dimensional random vector with uniform marginals and joint distribution function defined by the Archimedean copula*

$$C(u_1, ..., u_n) = \varphi^{-1}(\varphi(u_1) + \cdots + \varphi(u_n))$$

for some generator φ. Define the n transformed random variables $S_1, S_2, ..., S_{n-1}$ and T as in (4) and (5). The joint distribution function of $(S_1, ..., S_{n-1}, T)$ is characterized by

$$\begin{aligned}
h(s_1, ..., s_{n-1}, t) &= \frac{\partial^n P(S_1 \le s_1, ..., S_{n-1} \le s_{n-1}, T \le t)}{\partial s_1 \cdots \partial s_{n-1}\partial t} \\
&= s_1^0 s_2^1 s_3^2 \cdots s_{n-1}^{n-2} \times \varphi^{-1(n)}(t)[\varphi(t)]^{n-1}\varphi'(t)
\end{aligned} \tag{8}$$

where $\varphi^{-1(n)}$ denotes the n-th derivative of φ^{-1}, the inverse of the generator.

Proof. By noting that the density of the Archimedean copula can be expressed as

$$c(u_1,...,u_n) = \frac{\partial^n C(u_1,...,u_n)}{\partial u_1 \cdots \partial u_n} = \varphi^{-1(n)}(C(u_1,...,u_n)) \prod_{k=1}^{n} \varphi'(u_k)$$

and using the determinant of the Jacobian in (7), we find that we can then write the joint density of $S_1, S_2, ..., S_{n-1}$ and T as

$$\begin{aligned} h(s_1,...,s_{n-1},t) &= |J| \times \varphi^{-1(n)}(t) \prod_{k=1}^{n} \varphi'(u_k) \\ &= s_1^0 s_2^1 s_3^2 \cdots s_{n-1}^{n-2} \frac{[\varphi(t)]^{n-1} \varphi'(t)}{\varphi'(u_1) \cdots \varphi'(u_n)} \times \varphi^{-1(n)}(t) \prod_{k=1}^{n} \varphi'(u_k) \\ &= s_1^0 s_2^1 s_3^2 \cdots s_{n-1}^{n-2} \times \varphi^{-1(n)}(t) [\varphi(t)]^{n-1} \varphi'(t) \end{aligned}$$

and so the result immediately follows. ■

The independence of $S_1, S_2, ..., S_{n-1}$ and T becomes obvious. Furthermore, although $S_1, S_2, ...$ and S_{n-1} each have support on $(0,1)$, their marginal distribution are not uniform, except for S_1.

Corollary 1 *The transformed variables $S_1, S_2, ..., S_{n-1}$ and T are indeed independent.*

Proof. Trivial, following from their joint density as expressed in Theorem 2. ■

Corollary 2 *The marginal densities for S_k, for $k = 1, 2, ..., n-1$ are given by*

$$f_{S_k}(s) = ks^{k-1}, \text{ for } s \in (0,1). \tag{9}$$

Proof. Trivial also, following immediately from joint density as expressed in Theorem 2. ■

It follows therefore that the marginal distribution function can be written as

$$F_{S_k}(s) = s^k, \text{ for } s \in (0,1), \ k = 1, 2, ..., n-1. \tag{10}$$

We also can immediately derive the distribution of the copula $T = C(U_1, ..., U_n)$ and we state this as a corollary.

Corollary 3 *The marginal density for T is given by*

$$f_T(t) = \frac{1}{(n-1)!} \times \varphi^{-1(n)}(t) [\varphi(t)]^{n-1} \varphi'(t) \tag{11}$$

for $t \in (0,1)$.

Proof. Each of the density of S_k for $k = 1, 2, ..., n-1$ needs a factor of k and this results in the joint density

$$h(s_1,...,s_{n-1},t) = \left(\prod_{k=1}^{n-1} ks_k^{k-1}\right) \times \frac{1}{(n-1)!} \varphi^{-1(n)}(t) [\varphi(t)]^{n-1} \varphi'(t).$$

Integrating out all the s_k therefore gives the marginal density for T. The result immediately follows. ■

Using the above corollary, the marginal distribution function for the copula T can be expressed as

$$F_T(t) = \frac{1}{(n-1)!} \times \int_0^t \varphi^{-1(n)}(w) [\varphi(w)]^{n-1} \varphi'(w) \, dw$$

and by repeated application of integration by parts, one can also show that this cumulative distribution function has the representation

$$\begin{aligned} F_T(t) &= \sum\nolimits_{k=0}^{n-1} \frac{1}{k!} \times (-1)^k \varphi^{-1(k)}(\varphi(t)) [\varphi(t)]^k \\ &= t + \sum\nolimits_{k=1}^{n-1} \frac{1}{k!} \times (-1)^k \varphi^{-1(k)}(\varphi(t)) [\varphi(t)]^k, \end{aligned} \tag{12}$$

a result that was also shown in Genest & Rivest (2001). For example, in the bivariate case, we would have

$$F_T(t) = t - \varphi(t) / \varphi'(t),$$

a result that was stated in Theorem 1.

3 The simulation algorithm

It is well-known that the distribution function of any continuous random variable has a Uniform $U(0,1)$ distribution so that all the F_{S_k} and F_T in (10) and (12) have this distribution. Thus, in order to generate an n-tuple vector $(X_1, ..., X_n)$ with an Archimedean copula, we follow the following procedure:

1. Generate n independent $U(0,1)$ random variables. Denote them by $w_1, ..., w_n$.
2. For $k = 1, 2, ..., n-1$, set $s_k = w_k^{1/k}$.
3. Set $t = F_T^{-1}(w_n)$.
4. Set $u_1 = \varphi^{-1}(s_1 \cdots s_{n-1} \varphi(t))$, $u_n = \varphi^{-1}((1 - s_{n-1}) \varphi(t))$ and for $k = 2, ..., n$, set $u_k = \varphi^{-1}\left((1 - s_{k-1}) \prod_{j=k}^{n-1} s_j \cdot \varphi(t)\right)$.
5. The desired values are $x_k = F_k^{-1}(u_k)$ for $k = 1, 2, ..., n$.

In terms of practical implementation, the biggest challenge is to be able to explicitly express (12) and then find its inverse function. Indeed this task requires finding the k-th derivative $\varphi^{-1(k)}$ of the inverse of the generator. For some Archimedean types, this can be a daunting task. We consider some Archimedean generators and derive the corresponding k-th derivative of its inverse. Notice that most of these examples contain a single parameter capturing all the dependencies among the random variables. This single parameter may be viewed as a disadvantage, but at the same, it provides for simplicity and tractability.

For the development about each copula, we suggest the books by Nelsen (1999) and Joe (1997). For discussion of the applications of these copulas, we refer the reader Frees and Valdez (1998).

Example 3.1: *The Clayton copula*
The Clayton copula is constructed based on the generator

$$\varphi(u) = \left(u^{-\theta} - 1\right)/\theta \tag{13}$$

for some $\theta > 0$. It can be shown that its inverse function is

$$\varphi^{-1}(u) = (1 + \theta u)^{-1/\theta},$$

and its corresponding k-th derivative has the form

$$\varphi^{-1(k)}(u) = (-1)^k (1 + \theta u)^{-(1+k\theta)/\theta} \prod\nolimits_{j=0}^{k-1} (1 + j\theta). \blacksquare$$

Example 3.2: *The Gumbel-Hougaard copula*
The Gumbel-Hougaard copula is constructed based on the generator

$$\varphi(u) = (-\log u)^{1/\theta} \tag{14}$$

for some $\theta > 1$. It can be shown that its inverse function is

$$\varphi^{-1}(u) = \exp\left(-u^{\theta}\right),$$

and its corresponding k-th derivative can be expressed in the form

$$\varphi^{-1(k)}(u) = (-1)^k \theta \exp\left(-u^{\theta}\right) u^{-k+1/\theta} \times \Psi_{k-1}\left(u^{\theta}\right)$$

where $\Psi_k(x)$ is a function of x that can be recursively determined, beginning with $\Psi_0(x) = 1$, as follows

$$\Psi_k(x) = [\theta(x-1) + k] \Psi_{k-1}(x) - \theta x \Psi'_{k-1}(x). \blacksquare$$

Example 3.3: *The Frank copula*
The Frank copula is constructed based on the generator

$$\varphi(u) = -\log\left(\frac{e^{-\theta u} - 1}{e^{-\theta} - 1}\right) \tag{15}$$

for some $\theta \neq 1$. It can be shown that its inverse function is

$$\varphi^{-1}(u) = -\log\left(1 + e^{-u}\left(e^{-\theta} + 1\right)\right)/\theta,$$

and its corresponding k-th derivative can be expressed in the form

$$\varphi^{-1(k)}(u) = -\Psi_{k-1}\left(\left(1 + e^{-u}\left(e^{-\theta} + 1\right)\right)^{-1}\right)/\theta$$

where $\Psi_k(x)$ is a function of x that can be recursively determined, beginning with $\Psi_0(x) = x - 1$, as follows

$$\Psi_k(x) = x(1-x)\Psi'_{k-1}(x). \blacksquare$$

4 Simulation illustrations

To illustrate the applicability of the simulation algorithm proposed in this paper, we consider the case of the aggregation of risks and the case of the expansion of risks. In the aggregation of risks, we evaluate the total capital required for an insurance company with multiple lines of business. In the expansion of risks, we evaluate the required amount of increase in capital required when the same insurance company decides to expand by adding a new line of business.

4.1 Aggregation of risks

Consider an insurance company with four different lines of business, with line k facing a loss of X_k for $k = 1, 2, 3, 4$. For simplicity, we assume that loss for each line of business has the same log-normal marginal distribution specified by the density of the form

$$f_k(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2\right]. \tag{16}$$

The log-normal parameters μ and σ are chosen so that its mean and its variance are both equal to 1. The aggregate loss for the insurance company is the random variable

$$S = X_1 + \cdots + X_4 \tag{17}$$

consisting of the sum of the losses arising from each line of business. To evaluate the capital required for this portfolio, we are interested in the distribution of this aggregate loss. Suppose that the lines of business are "dependent" in some sense. The distribution function of aggregate loss is the joint distribution of the random variables $X_1, \cdot, X_4$. To obtain the distribution of the aggregate loss, we simulate m samples of the four tuples $(x_{1i}, ..., x_{4i})$ for $i = 1, 2, ..., m$, and then we sum the components to get the distribution of the aggregate loss.

Sums of random variables are well studied in classical risk theory, but with the individual risks being assumed to be mutually independent because computation of the aggregate claims becomes more tractable in this case. One can determine the exact form of the distribution for the aggregate claims for special families assuming independence. Several exact and approximate recursive methods have been proposed for computing the aggregate claims in the case of discrete marginal distributions, see e.g. Dhaene & De Pril (1994) and Dhaene & Vandebroek (1995). A classical approach to approximating the aggregate claims distribution is through a Normal distribution based on Central Limit Theorem, but other approximations based on a translated Gamma distribution or the Normal power approximation have been proposed as improvements, see e.g. Kaas, Goovaerts, Dhaene & Denuit (2000). For sums of dependent random variables, finding the explicit form of the distribution is less well-known but approximations using convex order bounds and comonotonicity have been proposed by Dhaene, et al. (2002a, 2002b).

The common approaches to assessing the economic capital are Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) risk measures. The VaR risk measure is a quantile measure, for any p between 0 and 1. For CTE risk measure, it is the expected loss beyond VaR, it allows for the severity of the potential loss beyond the VaR.

The p-quantile risk measure for the aggregate loss random variable S, which we denote by $\text{VaR}_p[S]$, is defined by

$$\text{VaR}_p[S] = \inf\{s \in \mathbb{R} \mid F_S(s) \geq p\}, \tag{18}$$

where $F_S(s) = P(S \leq s)$, the cumulative distribution function of S. The Tail Value-at-Risk at level p for which we denote by $\text{CTE}_p[S]$, is defined by

$$\text{CTE}_p[S] = E[S \,|S > \text{VaR}_p[S]]. \tag{19}$$

In general, p is a large number between 95% and 100%, e.g. 95% or 99%. The VaR of the aggregate loss can be interpreted as the amount for which there is a probability of $(1-p)$ of losing beyond this amount whereas, the CTE can be interpreted as the average of the top $(1-p)$ losses. Details of the risk measures and the properties of coherent risk measures can be seen in Artzner et al (1999). The nonparametric estimates of the VaR for this risk measure can be determined by inverting the empirical cumulative distribution function of the aggregate loss. The CTE is the average of the aggregate loss beyond the corresponding VaR.

For the choice of copulas, we consider two popular Archimedean copulas: the Gumbel-Hougaard copula and the Frank copula. The form of the Gumbel-Hougaard copula is as follows

$$C(u_1, u_2, u_3, u_4) = \exp\left\{-\left[\sum_{k=1}^{4}(-\log u_k)^{1/\theta}\right]^{\theta}\right\}, \tag{20}$$

while that of the Frank copula, we have

$$C(u_1, u_2, u_3, u_4) = -\frac{1}{\theta}\log\left\{1 + \frac{\prod_{k=1}^{4}\left(e^{-\theta u_k} - 1\right)}{\left(e^{-\theta} - 1\right)^4}\right\}. \tag{21}$$

The Archimedean generators for each respective copula are given in (14) and (15).

4.1.1 Results of the simulation

For purposes of illustration, we generated a total of $m = 1,000$ samples of the random vectors (X_1, X_2, X_3, X_4) whose copula structure has either the Gumbel-Hougaard or the Frank copula. For the Gumbel-Hougaard copula, we decided to choose the dependence parameter value of $\theta = 2$, and for the Frank copula, we chose $\theta = 5.75$. Both dependence parameters translate to a Kendall's tau correlation coefficient of 50%. These can be derived from the Kendall's tau formulas

$$\tau(\theta) = 1 - \frac{1}{\theta}$$

for the Gumbel-Hougaard and

$$\tau(\theta) = 1 + \frac{4}{\theta}\left[\int_0^{\theta}\frac{z}{\theta(e^z - 1)}dz - 1\right]$$

for the Frank copula. See Frees and Valdez (1998) for discussion of these Kendall's tau coefficients for these copula forms.

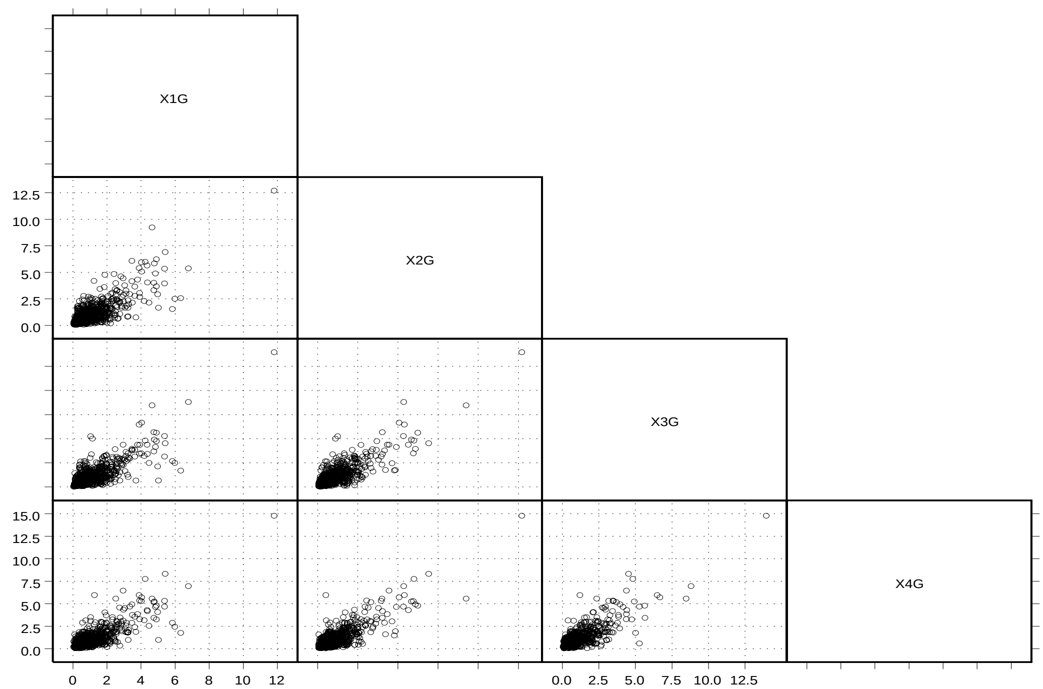

Figure 1: Matrix plots of the 1,000 simulated observations from the 4-dimensional Gumbel-Hougaard copula with Kendall's tau of 0.5 and with Log-Normal Marginals

Figures 1 and 2 display the pairwise scatter plots of the 1,000 simulated observations from the 4-dimensional Gumbel-Hougaard and the Frank copulas, respectively. These simulated observations are produced using the algorithm presented in Section 3 of this paper. It is apparent from these figures that despite equal tau correlation, the resulting dependence structures are different for both copulas. From these visual representations, it appears that there is implied positive dependencies on the tails for the Gumbel-Hougaard copula, but negative dependencies on the tails for the Frank copula.

A copula has a distribution function as expressed in (12). This appears difficult to evaluate but we can visualize the implied distribution function of the copula based on the simulated observations. Figure 3 compares these distribution functions for both the Gumbel-Hougaard and the Frank copulas.

4.1.2 The distribution of the aggregate loss

From the 1,000 simulated observations of 4-tuples $(x_{1i}, ..., x_{4i})$, $i = 1, ..., 1000$, we can deduce the implied empirical distribution of the aggregate loss as described in (17). Table 1 provides some summary statistics for the resulting aggregate loss both for the Gumbel-Hougaard and the Frank copulas.

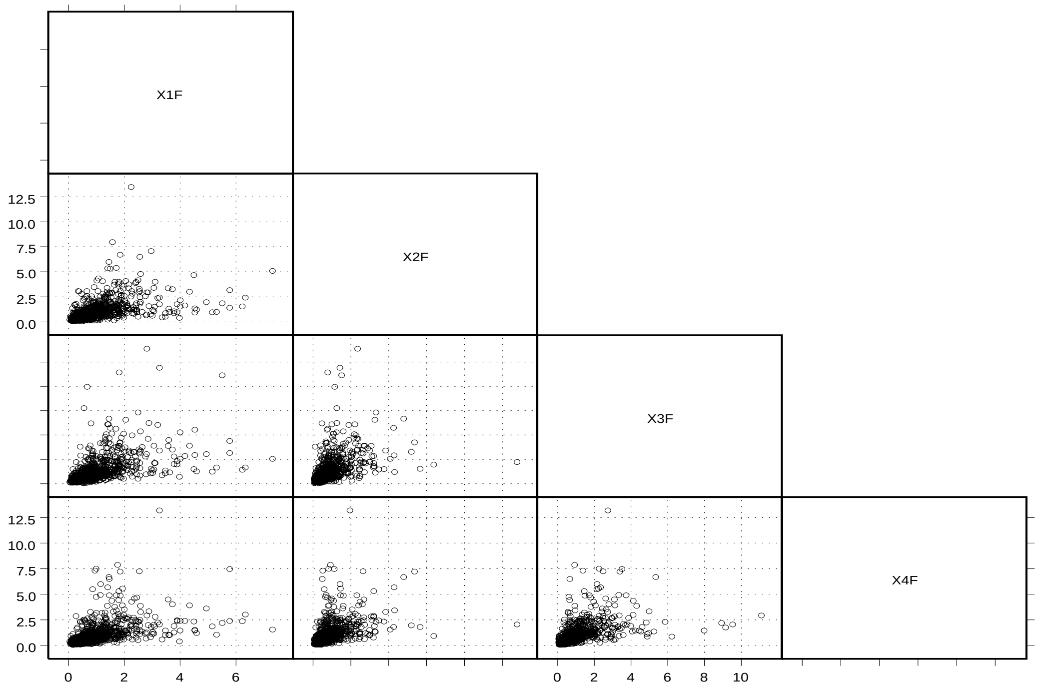

Figure 2: Matrix plots of the 1,000 simulated observations from the 4-dimensional Frank copula with Kendall's tau of 0.5 and with Log-Normal Marginals

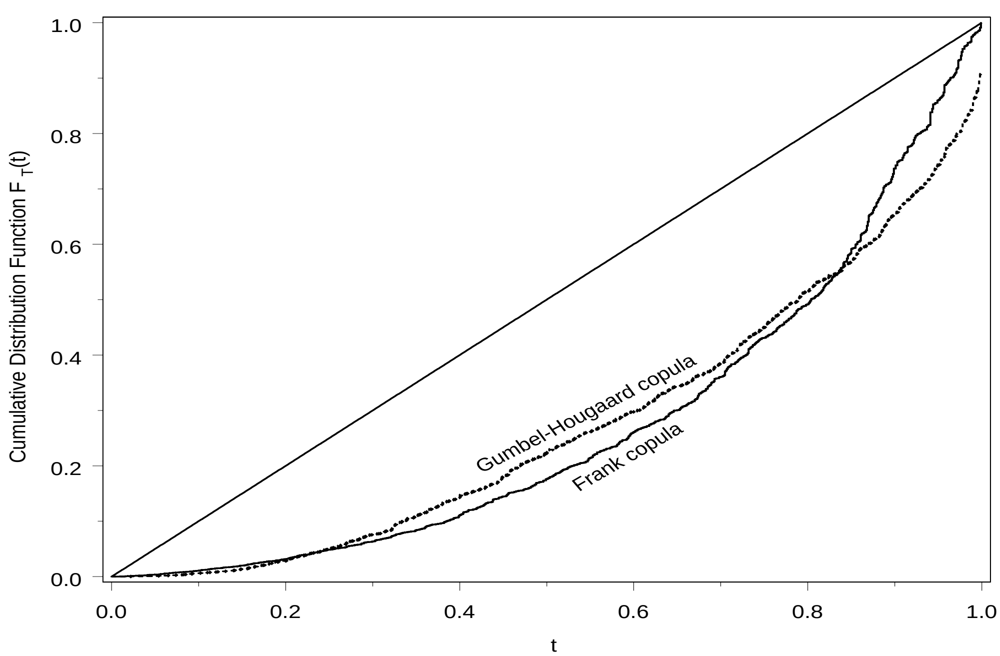

Figure 3: The cumulative distribution function of the copula - Gumbel-Hougaard versus Frank copulas. The 45-degree straight line is the implied CDF of a Uniform distribution.

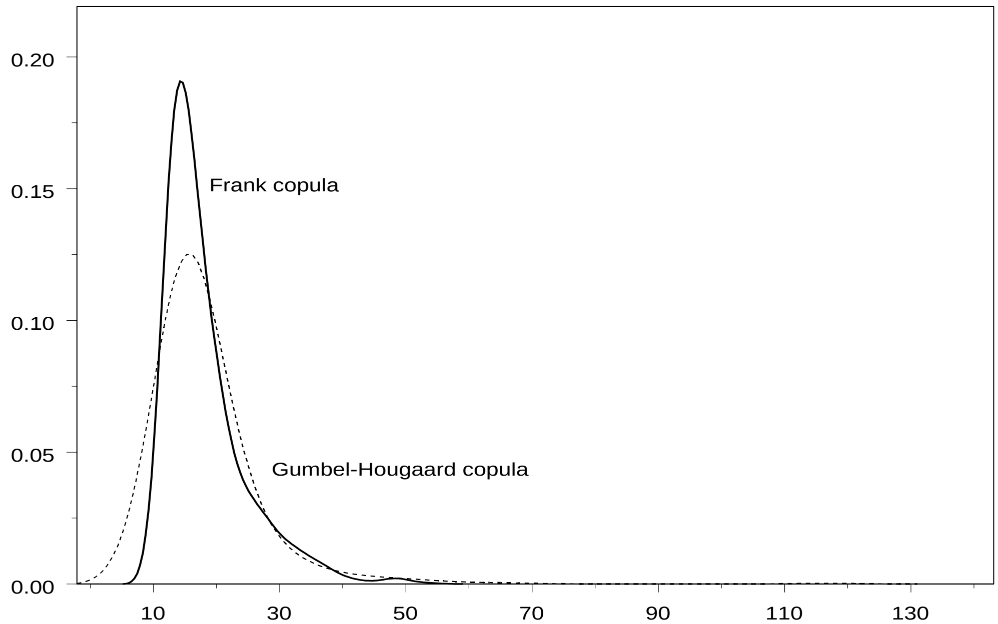

Figure 4: The empirical distribution of the aggregate loss: Gumbel-Hougaard copula versus the Frank copula

	Gumbel-Hougaard copula	Frank copula
Number	1,000	1,000
Mean	4.04	4.02
Median	2.97	3.00
Standard deviation	3.82	3.19
Minimum	0.39	0.48
Maximum	53.28	21.63
$\text{VaR}_{0.95}[S]$	10.59	10.76
$\text{VaR}_{0.99}[S]$	18.75	14.79
$\text{CTE}_{0.95}[S]$	16.40	13.55
$\text{CTE}_{0.99}[S]$	25.63	18.11

Table 1: Summary statistics of the aggregate loss: Gumbel-Hougaard vs Frank copulas

The results show that the mean of the aggregate loss is 4.04 for the Gumbel-Hougaard copula and 4.02 for the Frank copula, with respective standard deviations of 3.82 and 3.19. The respective medians are 2.97 and 3.00, both of which are smaller than their means. This indicates that both distributions are highly positively skewed. This is because the marginals are all log-normal which is positively skewed. Figure 4 provides the empirical distribution for the aggregate loss for both copulas.

Table 1 also provides a summary of the VaR and CTE for both families of copulas. These values indicate the capital requirements for holding the risk of the aggregate loss. The range of these values are between approximately 10 and 26. Both Table 1 and Figure 4 show that the VaR and the CTE tend to be higher for the Gumbel-Hougaard copula than for the Frank copula.

4.2 Business expansion

It is a common practice for an insurance company to expand their business by adding a line of business. For example, a general insurance company may be thinking of expanding its existing product lines by the adding a new product, such as cargo insurance, if this is not currently in its existing lines of business. The typical question to ask usually is how the addition of this new product line will impact its capital requirements. Modelling the marginal loss distribution of the additional line is typically straightforward, and if no existing data is available, the company actuary may wish to draw empirical results from the industry experience. The difficult part may have to do with modelling the dependency structure of its entire portfolio with the addition of the new line of business.

For the purpose of illustration, we continue with our illustrations resulting from the previous sub-section, where the insurer has existing 4 lines of business and with simulations, we were able to deduce the distribution of the aggregate loss. In the case of a business expansion, we denote the loss arising from the 5-th line of business as X_5 and we assume that it again has a log-Normal distribution with density of the form given in (16) but with the parameters chosen to be such that it has a mean and a variance of 2. That is, we assume that this additional line of business is more risky than the existing 4 lines of business. To simplify the procedure, we focus on the Gumbel-Hougaard copula. The parameter chosen is still $\theta = 2$ and has therefore resulting tau correlation of 50%.

Figure 5 provides a comparison of the resulting distribution of the new aggregate loss expressed as

$$S^* = S + X_5 \tag{22}$$

and the distribution of the aggregate loss from the existing portfolio, i.e. S. As is expected, there appears to be a longer tail of the aggregate distribution of the new portfolio than that of the existing portfolio.

For the capital requirements, we have summarized the VaR and CTE risk measures before and after the addition of the new line of business. The extra capital required then arising out of the extra addition of a new line of business is the difference between the risk measures before and after the addition. Table 2 provides a numerical summary of the extra capital required. We also provide the extra capital required, if the new line of business is treated as a stand-alone business, that is, we evaluated its capital requirements as if it is operating as a separate company. The loss arising from this new line is treated independently from the losses of the existing portfolio. The results show that the amount of extra capital required (CTE at 99% CI) is about 36% higher if the new line of business is operating on a stand-alone basis. This shows that the impact of the dependency risk to the capital requirements can be very significant.

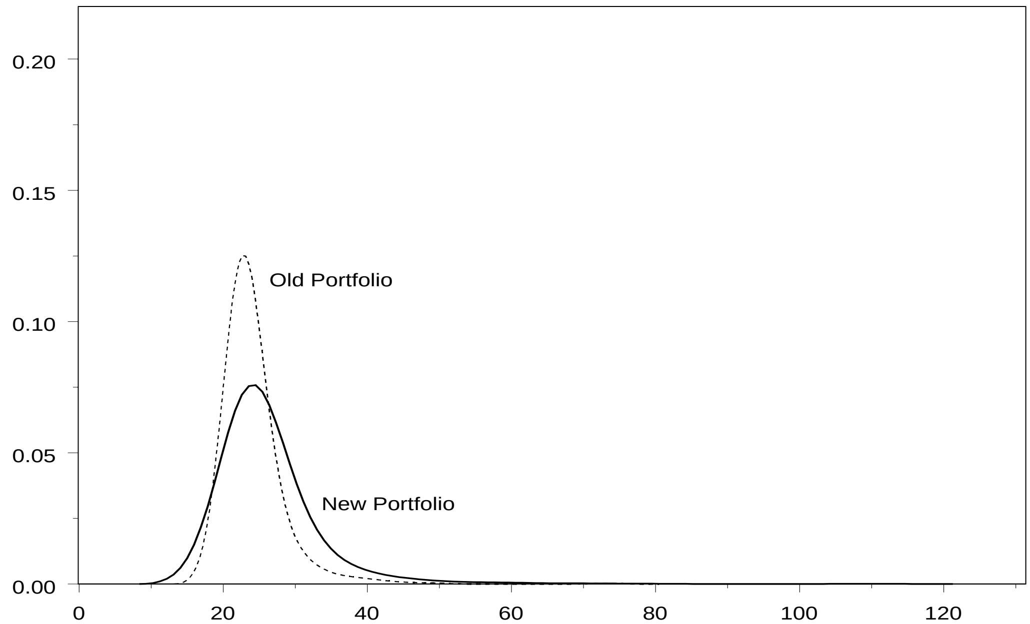

Figure 5: Comparison of the aggregate loss distribution arising from the new portfolio and that of the old portfolio

	Old Portfolio	New Portfolio	Extra Capital	Stand-alone Capital
$\text{VaR}_{0.95}[S]$	10.59 (0.62)	17.70 (0.61)	7.10	6.14
$\text{VaR}_{0.99}[S]$	18.75 (1.77)	28.75 (2.28)	10.00	14.25
$\text{CTE}_{0.95}[S]$	16.40 (1.98)	25.14 (2.52)	8.74	10.99
$\text{CTE}_{0.99}[S]$	25.63 (5.38)	37.45 (6.85)	11.81	18.70

* estimated standard errors are in parenthesis.

Table 2: Capital requirements before and after the addition of the new line of business*

Figure 6 compares the sensitivity of the CTE to the Kendall's correlation coefficient between the old and the new portfolio. The graph shows that, in both cases, the amount of capital required increases with the Kendall's correlation coefficient. This shows that the impact of the correlation coefficient to the amount of capital required can be significant.

5 Final remarks

In this paper, we presented an algorithm for generating multivariate random vectors whose copula structure has the Archimedean form. The case for generating from a one-parameter bivariate exchangeable Archimedean copula is already well-known.

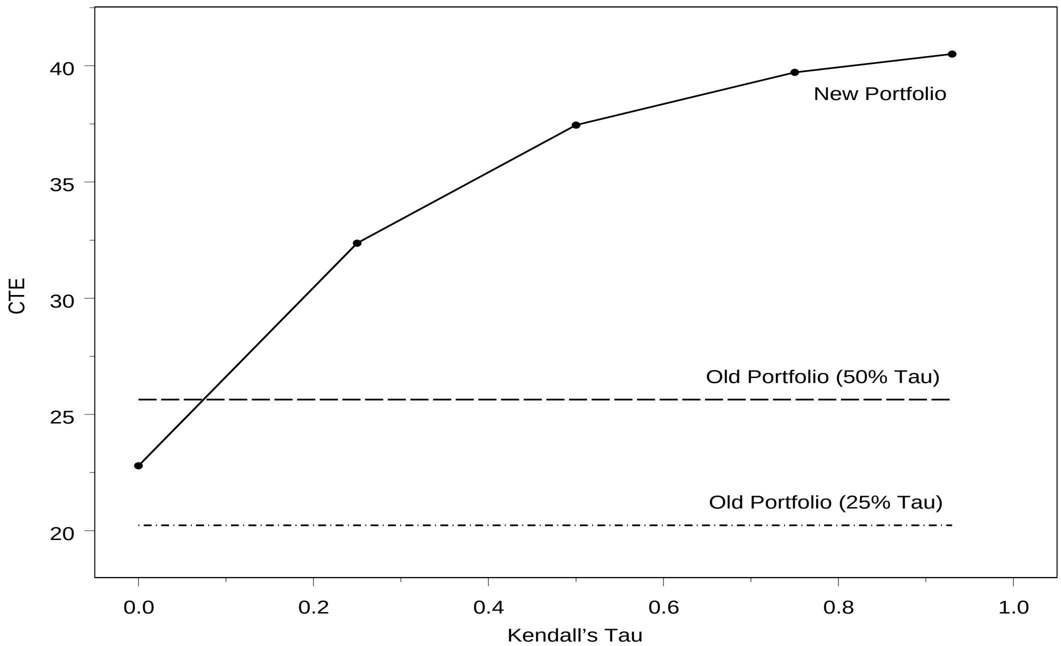

Figure 6: Examining the sensitivity of the dependence parameter (tau correlation) on the capital requirement - new versus old portfolio

See for example the work of Embrechts, et al. (2001) and their new textbook on quantitative risk management (McNeil, et al., 2005). We have extended the bivariate algorithm into the case of more than two dimension and we also provided a detailed proof of this extension. Archimedean copulas, because of some of the many useful properties they possess making them tractable for inference purposes, have become a widely used practical tool for modelling dependent risks in finance and insurance. Generating from this copula has therefore become an important exercise when evaluating dependencies, and as a result, there is increasing need to have an analytical procedure and algorithm that is mathematically tractable and practically implementable. It has been the purpose of this paper to present such an algorithm. Furthermore, in order to demonstrate the usefulness and the reasonableness of the results, we have considered illustrative examples of generating from the Gumbel-Hougaard and the Frank family of Archimedean copulas. We find that our simulation results appear to be reasonably as expected. In terms of practicality, we also provided illustration of evaluating the extra capital required for the addition of a new line of business.

Our illustrations show that both the dependency risk and the choice of the correlation coefficient will have significant impact on the amount of capital required for an insurance company running multiple lines of business.

A Appendix: Derivation of the Determinant of the Jacobian Matrix

In this appendix, we present details of the derivation of the determinant of the Jacobian matrix as defined in Section 2, and that this is indeed equal to (7). We now use the set of equations in (6). The first row of the Jacobian matrix has elements equal to

$$\partial u_1/\partial s_k = \prod_{j=1,j\neq k}^{n-1} s_j \times \varphi\left(t\right)/\varphi'\left(u_1\right),$$

for $k = 1, ..., n-1$ and

$$\partial u_1/\partial t = \prod_{j=1}^{n-1} s_j \times \varphi'\left(t\right)/\varphi'\left(u_1\right).$$

Similarly, for the second row, we would have

$$\partial u_2/\partial s_1 = -\prod_{j=2}^{n-1} s_j \times \varphi\left(t\right)/\varphi'\left(u_2\right),$$

$$\partial u_2/\partial s_k = \left(1-s_1\right)\prod_{j=2,j\neq k}^{n-1} s_j \times \varphi\left(t\right)/\varphi'\left(u_2\right),$$

for $k = 2, ..., n-1$ and

$$\partial u_2/\partial t = \left(1-s_1\right)\prod_{j=2}^{n-1} s_j \times \varphi'\left(t\right)/\varphi'\left(u_2\right).$$

Generalizing to the m-th, we find the elements have the form

$$\partial u_m/\partial s_k = 0$$

for $k \leq m-2$,

$$\partial u_m/\partial s_k = \left(1-s_{m-1}\right)\prod_{j>m,j\neq k}^{n-1} s_j \times \varphi\left(t\right)/\varphi'\left(u_m\right),$$

for $m-2 < k \leq n-1$, and

$$\partial u_m/\partial t = \left(1-s_{m-1}\right)\prod_{j=m}^{n-1} s_j \times \varphi'\left(t\right)/\varphi'\left(u_m\right).$$

The last row of the Jacobian matrix have zero elements for all columns except for the last two columns. The last two columns, in this case, have entries

$$\partial u_n/\partial s_{n-1} = \varphi\left(t\right)/\varphi'\left(u_n\right)$$

and

$$\partial u_n/\partial t = \left(1-s_{n-1}\right) \times \varphi'\left(t\right)/\varphi'\left(u_n\right).$$

Now, to evaluate the determinant in the general form, we prove the result by induction. The result can be easily verified for lower dimensions such as the case where $n = 1, 2, 3$. Suppose the determinant for the case where $n = l$ holds as follows:

$$|J_l| = s_1^0 s_2^1 s_3^2 \cdots s_{l-1}^{l-2} \frac{\left[\varphi\left(t\right)\right]^{l-1}\varphi'\left(t\right)}{\varphi'\left(u_1\right)\cdots\varphi'\left(u_l\right)}. \tag{23}$$

Then we simply have to show it is true for $n = l + 1$. This result can be proven using determinant for partitioned matrix. This is because the $(l+1)$-th truncated Jacobian can be partitioned as

$$J_{l+1} = \begin{pmatrix} J_l & A_{12} \\ A_{21} & B \end{pmatrix}$$

where A_{12} is an $(l \times 1)$ column vector consisting of

$$A_{12} = (\partial u_1/\partial t, ..., \partial u_l/\partial t)',$$

A_{21} is the $(1 \times l)$ row matrix of zeros except the last entry

$$A_{12} = (0, ..., 0, \partial u_{l+1}/\partial s_l)',$$

and

$$B = \partial u_{l+1}/\partial t.$$

The determinant of the partitioned matrix is thus equal to

$$|J_{l+1}| = |J_l| \times \left|B - A_{21} J_l^{-1} A_{12}\right|$$

and the determinant evaluation should be straightforward to evaluate and results in

$$|J_{l+1}| = s_1^0 s_2^1 s_3^2 \cdots s_l^{l-1} \frac{[\varphi(t)]^l \varphi'(t)}{\varphi'(u_1) \cdots \varphi'(u_{l+1})}.$$

References

[1] Artzner, P., F. Delbaen, J.-M. Eber, D. Heath, 1999. Coherent Measures of Risk. *Mathematical Finance* 9: 203-228.

[2] Barbe, P., C. Genest, K. Ghoudi, B. Remillard, 1996. On Kendall's Process. *Journal of Multivariate Analysis* 58: 197-229.

[3] Dhaene, J., N. De Pril, 1994. On a Class of Approximate Computation Methods in the Individual Risk Model. *Insurance: Mathematics & Economics* 14: 181-196.

[4] Dhaene, J., M. Denuit, M.J. Goovaerts, R. Kaas, D. Vyncke, 2002a. The Concept of Comonotonicity in Actuarial Science and Finance: Theory. *Insurance: Mathematics & Economics* 31(1), 3-33.

[5] Dhaene, J., M. Denuit, M.J. Goovaerts, R. Kaas, D. Vyncke, 2002b. The Concept of Comonotonicity in Actuarial Science and Finance: Applications. *Insurance: Mathematics & Economics* 31(2), 133-161.

[6] Dhaene, J., M. Vandebroek, 1995. Recursions for the Individual Model. *Insurance: Mathematics & Economics* 16, 31-38.

[7] Embrechts, P., F. Lindskog, A. McNeil, 2001a. Modelling Dependence with Copulas and Applications to Risk Management, in *Handbook of Heavy Tailed Distributions in Finance*. Edited by Rachev ST, Published by Elsevier/North-Holland, Amserdam.

[8] Embrechts, P., A. McNeil, D. Straumann, 2001b. Correlations and Dependence in Risk Management: Properties and Pitfalls. *Risk Management: Value-at-Risk and Beyond* ed M. Dempster and H. Moffat, Cambridge: Cambridge University Press.

[9] Frees, E.W., E.A. Valdez, 1998. Understanding Relationships Using Copulas. *North American Actuarial Journal* 2: 1-25.

[10] Genest, C., 1998. Frank's Family of Bivariate Distributions. *Biometrika 74: 549-555.*

[11] Genest, C., R.J. McKay, 1986. Copulas archimediennes et familles de lois bidimensionnelles dont les marges sont donnees. *Canadian Journal of Statistics* 26: 187-197.

[12] Genest, C., L. Rivest, 1993. Statistical Inference Procedures for bivariate Archimedean Copulas. *Journal of the American Statistical Association* 88: 1034-1043.

[13] Genest, C., L. Rivest, 2001. On the Multivariate Probability Integral Transformation. *Statistics & Probability Letters* 53: 391-399.

[14] Gumbel, E.J., 1958. Distributions à plusieures variables don't les marges sont donné. *Comptes Rendus de l' Academie des Sciences, Paris*, 246, 2717.

[15] Joe, Harry, 1997. *Multivariate Models and Dependence Concepts.* London: Chapman and Hall.

[16] Kaas, R., J. Dhaene, M.J. Goovaerts, 2000. Upper and lower bounds for sums of random variables. *Insurance: Mathematics & Economics* 27, 151-168.

[17] Mari, D.D., S. Kotz, 2001. *Correlation and Dependence.* London: Imperial College Press.

[18] McNeil, A.J., R. Frey, P. Embrechts, 2005. *Quantitative Risk Management.* Princeton, New Jersey: Princeton University Press.

[19] Nelsen, R.B. 1999. *An Introduction to Copulas.* New York: Springer.

[20] Sklar, A., 1959. Fonctions de repartition a n dimensions et leurs marges. *Publications de l'Institut de Statistique de L'Universite de Paris* 8: 229-231.

[21] Wang, S., 1998. Aggregation of Correlated Risk Portfolios. *Proceedings of the Casualty Actuarial Society* 85: 848-939.

[22] Whelan, N., 2004. Sampling from Archimedean Copulas. *Quantitative Finance* 4: 339-352.