ON MAXIMIZING DIVIDENDS WITH INVESTMENT AND REINSURANCE

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ABSTRACT

A simple model of the surplus of an insurance business is considered with reinsurance, investments in risky and non-risky assets and dividend distribution. The uncontrolled surplus process follows a brownian motion. The business surplus is controlled by the proportion of business retained when cheap reinsurance is used to transfer risk, by the proportion of the surplus invested in risky and non-risky assets, and by the rate dividends are paid out.

The strategy that maximizes the expected present value of future dividends is computed using the Hamilton-Jacobi-Bellman (HJB) equation of the controlled process.
OUTLINE OF THE PRESENTATION

- Introduction and literature review
- Model formulation
- The control problem and the HJB equation
- Solution to the control problem
- Analysis of results
INTRODUCTION AND LITERATURE REVIEW

- The problem of optimizing the rate of dividend pay-outs of an insurance company is widely discussed in the actuarial literature.


- The emergence of papers which use tools from stochastic control theory in dividend optimization in the last decade.
LITERATURE REVIEW

- Seminal work by Martin-Lof (1994)

- Asmussen and Taksar (1997) considered maximizing the expected value of discounted dividends paid until the time of ruin. Reserve process follows a Brownian motion with drift.

- Paulsen and Gjessing (1997) determined the optimal dividend pay-outs while allowing for stochastic return of investments.

- Hojgaard and Taksar (1997, 1999) extended the results of Asmussen and Taksar where the control of risk exposure by proportional reinsurance was included.
LITERATURE REVIEW: RECENT RESULTS

- Hojgaard and Taksar (2004) follow-up paper included control of investment strategies. The objective is to find a policy, consisting of investment strategy, risk control and dividend distribution scheme which maximizes the expected total discounted dividends paid out until time of bankruptcy.

- Irgens and Paulsen (2004) finds the controls that maximize expected utility of assets of an insurance company at a terminal date. Investment and reinsurance are controlled dynamically.
THIS RESEARCH

• Differs from Hojgaard and Taksar (2004) in the objective function used in the control problem. Here the objective is to maximize the expected value of discounted utility of future dividends up to a fixed time $T$ where the shareholders are assumed to have fractional power utility.

• Dividend distribution is controlled. Irgens and Paulsen (2004) used fractional power utility in maximizing the expected utility of assets at a given terminal date. Control of dividends was not considered.
THIS RESEARCH (cont’d)

- Considers an insurance business whose uncontrolled surplus follows a Brownian motion with drift.

- There is an option to reinsure part of its business using proportional reinsurance.

- Surplus may be allocated for investment in a risky asset, whose price follows a geometric brownian motion, and in a non-risky asset.

- The company also pays out dividends to its shareholders.
RESEARCH OBJECTIVE

- The management of the company wants a dynamic reinsurance, investment and dividend distribution scheme that maximizes the expected value of discounted power utility of future dividends up to time $T$.

- Application of tools from stochastic control theory.

- The choice of the objective function, in particular the presence of the utility function in the dividend payout and a finite time horizon, was primarily because it leads to a classical stochastic control problem.
MODEL FORMULATION

- We start with a probability space \((\Omega, F, P)\), a filtration \(\{\mathcal{F}_t\}\), and independent Brownian motions \(\left\{B^{(1)}_t\right\}\) and \(\left\{B^{(2)}_t\right\}\) with respect to \(\{\mathcal{F}_t\}\). The filtration represents the information available at time \(t\) and any decision made is based on this information.

- Let \(\{X_t\}\) be the surplus process of the company at time \(t\). In the absence of control, the process evolves according to

  \[dX_t = \mu \, dt + \sigma dB^{(1)}_t\]

  with \(\mu, \sigma > 0\).
MODEL FORMULATION (cont’d)

- Reinsurance: cheap proportional insurance. Reinsurance companies have the same safety loading as the ceding company.

- Investment:
  Risky asset: price process \( \{P_t\} \), a geometric Brownian motion

\[
dP_t = P_t \left( r_p \, dt + \sigma_p dB_t^{(2)} \right)
\]

Non-risky asset: price process

\[
 dA_t = A_t r_F \, dt
\]

- Shareholders’ utility: \( G(x) = x^\eta, \) \( 0 < \eta < 1. \)
  Note that \( G(\cdot) \) represents the preferences of a risk averse decision maker.
MODEL FORMULATION (cont’d)

- control policy \( u : (a_u(t), b_u(t), l_u(t)) \)
  - \( a_u(\cdot) \): risk exposure of the company
  - \( b_u(\cdot) \): proportion of the surplus invested in the risky asset
  - \( l_u(\cdot) \): rate of dividend payout

- The controlled surplus process: \( \{X_t^u\} \)

\[
\begin{align*}
\frac{dX_t^u}{ds} &= a_u(s) \left( \mu ds + \sigma dB_s^{(1)} \right) + b_u(s) X_t^u \left( r_p ds + \sigma_p dB_s^{(2)} \right) \\
&\quad + (1 - b_u(s)) X_t^u r_F ds - l_u(s) ds \\
X_t^u &= x.
\end{align*}
\]
THE CONTROL PROBLEM

- Admissible policy $u : \{a_u(t), b_u(t), l_u(t)\}$ is adapted to the filtration $\{\mathcal{F}_t\}, 0 \leq a_u(t) \leq 1, 0 \leq b_u(t) \leq 1, 0 \leq l_u(t) \leq X^u_t$.

- Reward function for each $u$

  $$V^u(t, x) = E_{t,x} \int_t^{T^\wedge \tau} e^{-\delta s} [l_u(s)]^\eta ds$$

  Optimal return function

  $$V(t, x) = \sup_{u \in U} V^u(t, x),$$

CONTROL PROBLEM: Find $u^*$ s.t. $V^{u^*}(t, x) = V(t, x)$
THE HJB EQUATION

\[
\sup_{u \in U} \left\{ V_t + \frac{1}{2} \left( a^2 \sigma^2 + b^2 x^2 \sigma_p^2 \right) V_{xx} + \left( a \mu - l + x [r_F + b(r_p - r_F)] \right) V_x \right. \\
\left. + e^{-\delta t} l(t) \eta \right\} = 0
\]
Maximizers of the HJB equation

\[
b(t, x) = \frac{-(r_p - r_F) V_x}{x \sigma_p^2 V_{xx}}
\]

\[
a(t, x) = \frac{-\mu V_x}{\sigma^2 V_{xx}}
\]

\[
l(t, x) = \left( \frac{V_x e^{\delta t}}{\eta} \right)^{\frac{1}{\eta-1}}
\]
THE HJB EQUATION (cont’d)

Conjecture: $V(t, x) = g(t) x^\eta$

Maximizers of the HJB equation

$$b(t, x) = \frac{r_P - r_F}{\sigma_p^2 (1 - \eta)}$$

$$a(t, x) = \frac{\mu x}{\sigma^2 (1 - \eta)}$$

$$l(t, x) = x \left[ e^{\delta t} g(t) \right]^{\frac{1}{\eta - 1}}$$
THE HJB EQUATION: Solution

\[ 0 = x^\eta \left\{ g'(t) + \eta \left( r_F + \frac{1}{2(1-\eta)} \left[ \frac{(r_p - r_F)^2}{2\sigma_p^2} + \frac{\mu^2}{2\sigma^2} \right] \right) g(t) \right\} \]

\[ + (1 - \eta) \left[ e^{\delta t} g(t) \right]^\frac{1}{\eta-1} g(t) \]

Let \( \phi = \frac{(r_p-r_F)^2}{\sigma_p^2} + \frac{\mu^2}{\sigma^2} \). Then \( g(t) \) satisfies the SDE

\[ \left\{ \begin{array}{l}
0 = g'(t) + \eta \left( r_F + \frac{1}{2(1-\eta)} \phi \right) g(t) + (1 - \eta) \left[ e^{\delta t} g(t) \right]^\frac{1}{\eta-1} g(t) \\
0 = g(T) \end{array} \right. \]
THE HJB EQUATION: Solution

\[ g(t) = e^{-\delta t} \left\{ \left( \frac{1 - \eta}{\delta - \eta \left( r_F + \frac{1}{2(1-\eta)}\phi \right)} \right) \left[ 1 - e^{-\left( \frac{\delta-\eta \left( r_F + \frac{1}{2(1-\eta)}\phi \right)}{1-\eta} \right) (T-t)} \right] \right\}^{1-\eta} \]

- It is evident that \( V(t, x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}) \).

- By the Verification Theorem \( V(t, x) = g(t) x^\eta \) is the solution that we are looking for.
The optimal control is determined from:

\[ b(t, x) = \frac{r_p - r_F}{\frac{\sigma_p^2}{\sigma_p^2} (1 - \eta)} \]

\[ a(t, x) = \frac{\mu}{\sigma^2 (1 - \eta)} x \]

\[ l(x, t) = \frac{x}{\left( \frac{1 - \eta}{\delta - \eta \left( r_F + \frac{1}{2(1 - \eta)} \phi \right)} \right) \cdot \left( 1 - e^{-\left( \delta - \eta \left( r_F + \frac{1}{2(1 - \eta)} \phi \right) \right) \cdot (T - t)} \right)} \]
ANALYSIS OF THE RESULTS: The Reinsurance Proportion

\[ a(t, x) = \frac{\mu}{\sigma^2 (1 - \eta)} x \]

• \( a(t, x) \) is an increasing function of the surplus.

• This proportion increases as the risk premium \( \frac{\mu}{\sigma^2} \) of the risk process increases.

• The company buys proportional reinsurance if the current surplus \( x \) is below a surplus threshold \( A = \frac{\sigma^2 (1 - \eta)}{\mu} \). Otherwise, the company retains all business.
ANALYSIS OF THE RESULTS: The Reinsurance Proportion

\[ a^*(t, x) = \begin{cases} \frac{\mu}{\sigma^2 \cdot (1 - \eta)} x, & x < \frac{\sigma^2 (1 - \eta)}{\mu} \\ 1 & \text{otherwise.} \end{cases} \]

- Threshold amount increases as the volatility \( \sigma \) from premium income and claims increases and also as the premium rate decreases.

- Threshold is seen to be inversely proportional to the risk premium \( \frac{\mu}{\sigma^2} \).

- Threshold amount decreases as \( \eta \) approaches 1 and increases as \( \eta \) approaches 0. This is consistent with risk averse behaviour. If the shareholder is more risk averse, i.e. \( \eta \) is lower, then the threshold is higher and vice versa.
Figure 1: $a^*(t, x)$ for different values of $\sigma$.

Suppose $\mu = 4000$ and $\eta = 0.99$. In Figure 1, the dotted lines correspond to $a^*(t, x)$ with $\sigma = 1500$. The threshold for the surplus is 5.625. The solid line corresponds to $a^*(t, x)$ with $\sigma = 2500$ with threshold 15.625.
ANALYSIS OF THE RESULTS: The Investment Proportion

\[ b(t, x) = \frac{r_p - r_F}{\sigma^2_p (1 - \eta)} \]

- The optimal investment strategy is to invest a constant proportion of the surplus to the risky asset. This could be viewed as the company using a safe investment strategy.

- The proportion invested depends on the ratio \( M_p = \frac{r_p - r_F}{\sigma^2_p} \). This ratio is often called the risk premium and it represents the compensation for exposure to risk in the investment portfolio. If \( r_p < r_F \) then \( M_p < 0 \) and the required proportion is negative. This implies that the entire surplus should be invested in the non-risky asset. If \( 0 < M_p < 1 - \eta \) then a constant proportion of the surplus is invested in the risky asset. If \( M_p \geq 1 - \eta \) then the entire surplus is invested in the risky asset.
ANALYSIS OF THE RESULTS: The Investment Proportion

\[ b^*(t, x) = \begin{cases} 
0, & M_p \leq 0 \\
\frac{r_p - r_F}{\sigma_p^2 (1 - \eta)}, & 0 < M_p < 1 - \eta \\
1, & M_p \geq 1 - \eta. 
\end{cases} \]

- The proportion increases as the difference between the risk-free rate, \( r_F \), and the expected return of the risky asset, \( r_p \), increases.

- If \( \sigma_p^2 \) is small this means that the risky asset would not fluctuate much and a larger proportion results.

- As \( \eta \) is near 1 we invest a larger proportion to the risky asset. On the other hand if \( \eta \) is near zero then a larger proportion would be invested in the risk-free asset. These observations related to \( \eta \) are consistent with the behaviour of a risk averse decision maker.
Figure 2: $b^*(t, x)$ for different values of $\sigma_p^2$.

Figure 2 shows simulations of $b^*$. Suppose $r_p = .08$, $r_F = .05$, $\delta = .10$, $\eta = 0.99$ and $T = 5$. The dashed lines correspond to $b^*$ with $\sigma_p^2 = 0.10$. $M_p = 0.3 > 1 - \eta = .01$. Hence we set $b^*(t, x) = 1$. The solid line correspond to $b(x, t)$ with $\sigma_p^2 = 4$. $M_p = 0.0075 < .01$ and so $b^*(t, x) = 0.75.$
ANALYSIS OF THE RESULTS: The Dividend Payout

Define the constant $C$ such that

$$C = \left( \frac{1 - \eta}{\delta - \eta \left( r_F + \frac{1}{2(1-\eta)\phi} \right)} \right) \cdot \left( 1 - e^{-\left( \delta - \eta \left( r_F + \frac{1}{2(1-\eta)\phi} \right) \right) \cdot (T-t)} \right).$$

where $\phi = \frac{(r_p - r_F)^2}{\sigma_p^2} + \frac{\mu^2}{\sigma^2}$. Since we want the dividends to be less than the current surplus of the company we have,

$$l^*(t, x) = \begin{cases} 
\frac{x}{C}, & C > 1 \\
x, & 0 < C \leq 1 \\
0, & C < 0
\end{cases}$$
ANALYSIS OF THE RESULTS: The Dividend Payout

\[ l^*(t, x) = \begin{cases} \frac{x}{C}, & C > 1 \\ x, & 0 < C \leq 1 \\ 0, & C < 0 \end{cases} \]

- For a fixed \( t \), the company disburses a proportion of its surplus to its shareholders.

- The farther we are from the terminal date \( T \) this proportion gets smaller and vice-versa.

- There is a surplus barrier (which depends on \( t \)) beyond which it is optimal to distribute the current surplus as dividends and thus leads to company ruin. This is surprising but not completely unexpected.
ANALYSIS OF THE RESULTS: The Dividend Payout

\[ l^*(t, x) = \begin{cases} 
\frac{x}{C}, & C > 1 \\
x, & 0 < C \leq 1 \\
0, & C < 0 
\end{cases} \]

\[ C = \left( \frac{1 - \eta}{\delta - \eta \left( r_F + \frac{1}{2(1-\eta)\phi} \right)} \right) \cdot \left( 1 - e^{-\left( \frac{\delta - \eta \left( r_F + \frac{1}{2(1-\eta)\phi} \right)}{1-\eta} \right) \cdot (T-t)} \right) . \]

Clearly, one has to set the parameter values in \( C \) such that the optimal strategy presented here may be applied directly. It is therefore interesting to find what the constant \( C \) which determines the surplus barrier means. That is, it will be interesting to find how the exogeneous parameters in the model must be chosen so that the optimal strategy obtained here is applicable.
Simulations of $l$: Suppose our specific values are as follows, $\mu = 20$, $\sigma = 20$, $r_p = 8\%$, $r_F = 5\%$, $\sigma_p^2 = .03$, $\delta = 0.10$, $\eta = 0.80$ and $t = 0$. In Figure 3 the dashed line correspond to the $45^o$ line, the dotted line correspond to $l^*(t, x)$ with $T = 0.8$ while the solid line correspond to $l^*(t, x)$ with $T = 2$. 

Figure 3: $l^*(t, x)$