Credit derivatives pricing using the Cox process with shot noise intensity

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Abstract

As the building blocks for credit derivatives pricing, we need the price of the default-free zero-coupon bond, the price of the defaultable zero-coupon bonds with zero recovery and the value of a deterministic (or non-deterministic) payoff at recovery, which is some time after the default. In order to price the non-defaultable zero-coupon bond, we use a generalised Cox-Ingersoll-Ross (CIR) model (1985). We employ the Cox process with shot noise intensity to model the default time and derive the survival probability to price the defaultable zero-coupon bonds with zero recovery and the value of a deterministic payoff at recovery. In order to obtain the explicit expressions of three building blocks for credit derivatives pricing, we assume that the instantaneous rate of interest and the default intensity rate are independent each other. We also assume that the jump size of shot noise intensity follows an exponential distribution and the asymptotic distribution of the shot noise intensity is used not to have its initial value. As examples of credit derivatives pricing using these building blocks, we calculate the price of defaultable fixed-coupon bond and the market credit default swaps (CDS) rate numerically. The Esscher transform is used to achieve an absence of arbitrage opportunities in the market.

Keywords: The Cox process with shot noise process; Default time and survival probability; A generalised CIR model; Piecewise deterministic Markov process; Defaultable fixed-coupon bond; Market credit default swaps (CDS) rate; the Esscher transform.

1. **Introduction**

Since Merton (1974), one of the credit risk modelling developed is based on the default intensity of a counting process. Jarrow and Turnbull (1995) proposed to use the Poisson process and extended it further employing a discrete state space Markov chain in credit rating with Lando (1997). Lando (1998) examined it deeper introducing the Cox process where its intensity has finite state space. Similar approach was used by Duffie and Singleton (1999), where they considered the fractional reduction in market value that occurs at default with respect to risk neutral probability measure. The comparison between the classical approach proposed in Black and Scholes (1973) and Merton (1974) and the intensity-based approach can be found in Cooper and Martin (1996). Other works relating to credit risk we refer you Artzner and Delbaen (1995), Madan and Unal (1998), Bielecki and Rutkowski (2000), Elliot et al. (2000) and Schönbucher (2003).

In this paper we also employ the Cox process (Cox 1955; Grandell, 1976 and Brémaud 1981) to model the default time. Under a doubly stochastic Poisson process, or the Cox process, the intensity function is assumed to be stochastic. The doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process). Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one used by Dassios and Jang (2003).

Definition 1.1 *Let (Ω, F, P) be a probability space with information structure given by $F = \{\Im_t,\ t \in [0,T]\}$. Let N_t be a point process adapted to F. Let λ_t be a non-negative process adapted to F such that*

$$\int_0^t \lambda_s ds < \infty \quad \textit{almost surely} \quad (\textit{no explosions}).$$

If for all $0 \le t_1 \le t_2$ and $u \in \Re$

$$\mathbb{E}\left\{e^{iu\left(N_{t_2}-N_{t_1}\right)}|\Im^{\lambda}_{t_2}\right\} = \exp\left\{\left(e^{iu}-1\right)\int_{t_1}^{t_2} \lambda_s ds\right\} \tag{1.1}$$

then N_t is call a $\Im_t$-doubly stochastic Poisson process with intensity λ_t where $\Im_t^\lambda = \sigma\{\lambda_s;\ s \le t\}$.

Equation (1.1) gives us

$$\Pr\{N_{t_2} - N_{t_1} = k|\lambda_s; t_1 \le s \le t_2\} = \frac{\exp\left(-\int_{t_1}^{t_2} \lambda_s ds\right)\left(\int_{t_1}^{t_2} \lambda_s ds\right)^k}{k!}. \tag{1.2}$$

and

$$\Pr\{\tau_2 > t|\lambda_s; t_1 \le s \le t_2\} = \Pr\{N_{t_2} - N_{t_1} = 0|\lambda_s; t_1 \le s \le t_2\} = \exp\left(-\int_{t_1}^{t_2} \lambda_s ds\right) \tag{1.3}$$

where τ_k denotes the length of the time interval between the $(k-1)^{\text{th}}$ and k^{th} point. Now consider the process $\Lambda_t = \int_0^t \lambda_s ds$ (*the aggregated process*), then from (1.2) we can easily find that

$$\mathbb{E}\left(\theta^{N_{t_2} - N_{t_1}}\right) = \mathbb{E}\left\{e^{-(1-\theta)\left(\Lambda_{t_2} - \Lambda_{t_1}\right)}\right\}. \tag{1.4}$$

Equation (1.4) suggests that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of Λ_t, the aggregated process. It means that we just have to find the p.g.f. (probability generating function) of N_t to retrieve the m.g.f. (moment generating function) of Λ_t and vice versa.

From the Cox process of N_t, the survival probability is given by

$$\Pr(\tau_1 > t \mid \lambda_0) = \mathbb{E}\left\{\exp\left(-\int_0^t \lambda_s ds\right) \mid \lambda_0\right\} = \mathbb{E}\left(e^{-\Lambda_t} \mid \lambda_0\right) \tag{1.5}$$

where $\tau_1 \equiv \inf\{t :\ N_t = 1 \mid N_0 = 0\}$ is the default arrival time that is equivalent to the first jump arrival time of the Cox process N_t and the expectation is calculated under an appropriate probability measure.

If we assume that the interest rate process for a zero-coupon default-free bond, r_t follows a generalised Cox-Ingersoll-Ross (CIR) model (1985), i.e.

$$dr_t = c(b - ar_t)dt + \sigma\sqrt{r_t}dB_t, \tag{1.6}$$

where $a > 0$, $b > 0$ and $c > 0$, its price at time 0, paying 1 at time t is given by

$$B(0,t) = \mathbb{E}\left\{\exp\left(-\int_0^t r_s ds\right) \mid r_0\right\} = \mathbb{E}\left(e^{-R_t} \mid r_0\right) \tag{1.7}$$

where $B(0,t)$ denotes the price of a default -free zero-coupon bond, $R_t = \int_0^t r_s ds$, $\Im_t^r = \sigma\{r_s;\ s \le t\}$ and the expectation is calculated under an appropriate probability measure.

Assuming that r_t and λ_t are independent, the price of a zero-coupon defaultable bond paying $1_{(\tau_1 > t)}$ at time t is given by

$$\begin{aligned}
\overline{B}(0,t) &= \mathbb{E}\left\{\exp\left(-\int_0^t r_s ds\right) 1_{(\tau_1 > t)} \mid r_0,\ \lambda_0\right\} = \mathbb{E}\left[\exp\left\{-\int_0^t (r_s + \lambda_s)\, ds\right\} \mid r_0,\ \lambda_0\right] \\
&= \mathbb{E}\left(e^{-R_t}\ e^{-\Lambda_t} \mid r_0, \lambda_0\right) \\
&= \mathbb{E}\left(e^{-R_t} \mid r_0\right)\ \mathbb{E}\left(e^{-\Lambda_t} \mid \lambda_0\right)
\end{aligned} \tag{1.8}$$

where $\overline{B}(0,t)$ denotes the price of a defaultable zero-coupon bond and the expectation is calculated under an appropriate probability measure.

In reality, the lenders (i.e. the buyers of defaultable bonds) can receive the part (or whole) of coupon payments and principle after the liquidation of borrowers' assets. So we simply consider the recovery of par model introduced by Duffie (1998). For fractional recovery, we refer you Duffie and Singleton (2003).

The value of a deterministic payoff, 1 that is paid at t_{k+1} if and only if a default happens in $]t_k, t_{k+1}]$, denoted by $e(0, t_k, t_{k+1})$, is given by

$$\begin{aligned} e(0,t_k,t_{k+1}) &= \mathbb{E}\left[\exp\left(-\int_0^{t_{k+1}} r_s ds\right)\left(1_{\{N_{t_k}=0\}} - 1_{\{N_{t_{k+1}}=0\}}\right) \mid r_0,\ \lambda_0\right] \\ &= \mathbb{E}\left[\exp\left(-\int_0^{t_{k+1}} r_s ds\right)\left\{\exp\left(-\int_0^{t_k} \lambda_s ds\right) - \exp\left(-\int_0^{t_{k+1}} \lambda_s ds\right)\right\} \mid r_0, \lambda_0\right] \\ &= \mathbb{E}\left(e^{-R_{t_{k+1}}} \mid r_0\right)\left\{\mathbb{E}\left(e^{-\Lambda_{t_k}} \mid \lambda_0\right) - \mathbb{E}\left(e^{-\Lambda_{t_{k+1}}} \mid \lambda_0\right)\right\} \end{aligned} \tag{1.9}$$

where $0 = t_0 < t_k < t_{k+1}$ and the expectation is calculated under an appropriate probability measure.

Now using (1.7), (1.8) and (1.9), that are building blocks for credit derivatives pricing, we can price a defaultable fixed-coupon bond at time 0, denoted by $\overline{C}(0)$, as

$$\overline{C}(0) = \sum_{k=1}^{N} \overline{c}_n\ \overline{B}(0, t_{k_n}) + \overline{B}(0, t_{k_N}) + \pi \sum_{k=1}^{k_N} e(0, t_{k-1}, t_k) \tag{1.10}$$

where $\overline{c}_n = \overline{c} \times (t_{k_n} - t_{k_{n-1}})$ are coupon payments at t_{k_n} $(n = 1, 2, \cdots, N)$, $t_{k_1} < t_{k_2} < \cdots < t_{k_N}$ and π is a deterministic recovery rate. We can also calculate the market credit default swaps (CDS) rate, denoted by $\overline{s}$,as

$$\overline{s} = (1 - \pi) \frac{\sum\limits_{k=1}^{k_n} e(0, t_{k-1}, t_k)}{\sum\limits_{k=1}^{N} (t_{k_{n+1}} - t_{k_n}) \overline{B}(0, t_{k_n})}. \tag{1.11}$$

The paper is structured as follows. In section 2, we introduce the shot noise process as an default intensity of the Cox process. We derive the Laplace transform of shot noise process, λ_t and aggregated process, Λ_t by piecewise deterministic Markov processes (PDMP) theory. All proofs are referred to Dassios and Jang (2003) where they used the Cox process with shot noise intensity for the pricing of reinsurance contract. Section 3 deals with risk neutrality. We examine how the dynamics of process λ_t and Λ_t change after changing probability measure obtained via the Esscher transform. For simplicity, we use zero-coupon default-free bond price with respect to the original measure. In section 4, we illustrate the calculations of defaultable fixed-coupon bond prices and the market credit default swaps (CDS) rates using the asymptotic distribution of shot noise process and exponential jump size distribution. Section 5 concludes.

2. Shot noise process and aggregated process

In practice, there are primary events such as the governments' fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the romours of mergers and acquisitions among firms and September 11 WTC catastrophe etc. that affect the value of the firm's risky debt and may lead to the default as the worst case. So we may assume that the number of defaults out of primary events follows the Poisson process. However it is inadequate as it has deterministic intensity, where the survival probability follows an exponential distribution. Therefore an alternative default intensity process needs to be used to generate the default arrival process.

One of the processes that can be used to measure the impact of primary events is the shot noise process (Cox & Isham 1980, 1986; Klüppelberg and Mikosch 1995 and Dassios and Jang 2003). The shot noise process is particularly useful in the default arrival process as it measures the frequency, magnitude and time period needed to go back to the previous level of intensity immediately after primary events occur. As time passes, the shot noise process decreases as all firms in the market do their best to avoid being in bankruptcy after the arrival of one of the primary events. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of the doubly stochastic Poisson process to measure the time to default due to primary events, i.e. we will use it as an intensity function to generate the Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

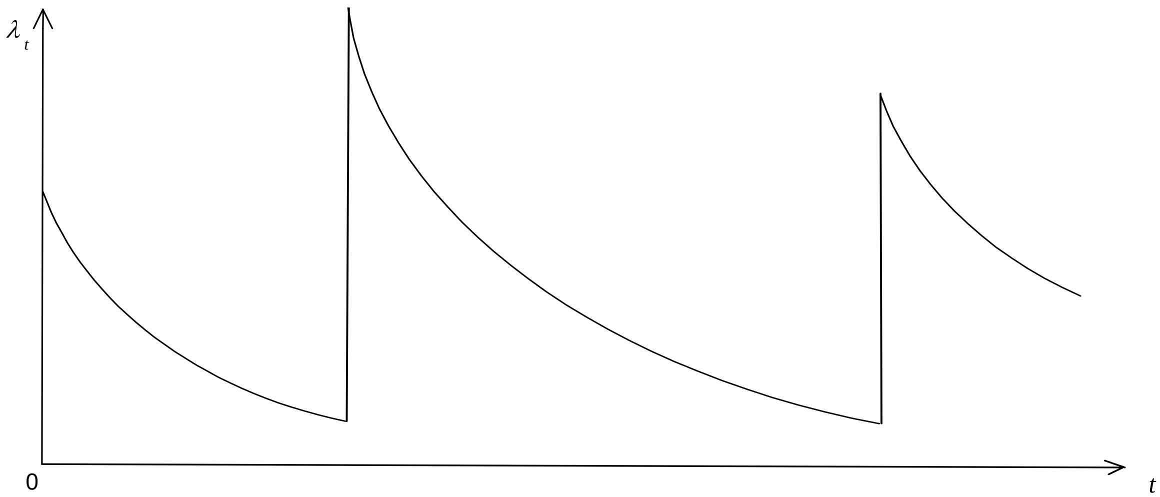

Figure 1. Graph illustrating a shot noise process

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i=1}^{M_t} Y_i e^{-\delta(t-S_i)}$$

where:

· λ_0 is the initial value of λ_t.

· $\{Y_i\}_{i=1,2,\cdots}$ is a sequence of independent and identically distributed random variables with distribution function $G(y)$ $(y > 0)$ and $E(Y) = \mu_1$.

· $\{S_i\}_{i=1,2,\cdots}$ is the sequence representing the event times of a Poisson process with constant intensity ρ.

· δ is the rate of exponential decay.

We also make the additional assumption that the Poisson process M_t and the sequences $\{Y_i\}_{i=1,2,\cdots}$ are independent of each other. Some works of insurance application using shot noise process can be found in Klüppelberg and Mikosch (1995), Dassios and Jang (2003), Jang (2004) and Jang and Krvavych (2004).

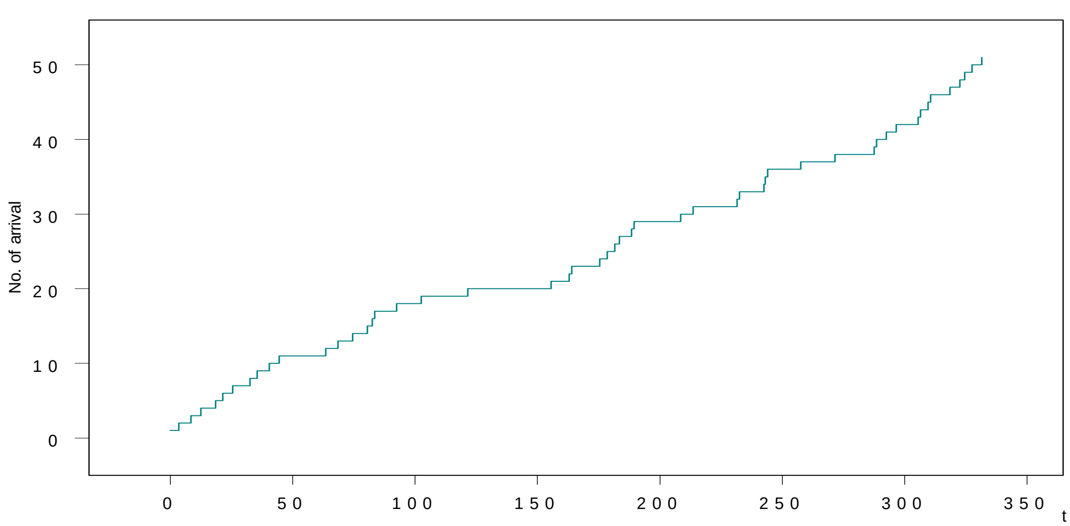

Figure 2. Graph illustrating the Cox process with shot noise intensity

The piecewise deterministic Markov processes (PDMP) theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. From now on, we present definitions and important properties of shot noise and aggregated processes with the aid of piecewise deterministic processes theory (Dassios 1987; Dassios and Embrechts 1989; Dassios and Jang 2003 and Rolski et al. 1998). This theory is used to derive the Laplace transform of shot noise process, λ_t and aggregated process , Λ_t.

The three parameters of the shot noise process described are homogeneous in time. We are now going to generalise the shot noise process by allowing the parameters to depend on time. The rate of jump arrivals, $\rho(t)$, is bounded on all intervals $[0,t)$ (no explosions). $\delta(t)$ is the rate of decay and the distribution function of jump sizes at any time t is $G(y;t)$ $(y>0)$ with $E(Y;t)=\mu_1(t)=\int_0^\infty y dG(y;t)$. We assume that $\delta(t)$, $\rho(t)$ and $G(y;t)$ are all Riemann integrable functions of t and are all positive. The generator of the process $(\Lambda_t, N_t, \lambda_t, t)$ acting on a function $f(\Lambda,n,\lambda,t)$ belonging to its domain is given by

$$
\begin{aligned}
A\ f(\Lambda,n,\lambda,t) &= \frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial \Lambda}+\lambda\left[f(\Lambda,n+1,\lambda,t)-f(\Lambda,n,\lambda,t)\right]-\delta(t)\lambda\frac{\partial f}{\partial \lambda} \\
&\quad+\rho(t)\left[\int_0^\infty f(\Lambda,n,\lambda+y,t)\,dG(y;t)-f(\Lambda,n,\lambda,t)\right]. \qquad (2.4)
\end{aligned}
$$

For $f(\Lambda,n,\lambda,t)$ to belong to the domain of the generator A, it is sufficient that $f(\Lambda,n,\lambda,t)$ is differentiable w.r.t. Λ, λ, t for all Λ, n, λ, t and that

$$
\left|\int_0^\infty f(\cdot,\cdot,\lambda+y,\cdot)\,dG(y;t)-f(\cdot,\cdot,\lambda,\cdot)\right|<\infty.
$$

Let us find a suitable martingale in order to derive the Laplace transforms of the distribution of Λ_t and λ_t.

Lemma 2.2 *Considering constants k and v such that $k\geq 0$ and $v\geq 0$,*

$$
\begin{aligned}
&\exp(-v\Lambda_t)\cdot\exp\left[-\left\{ke^{\delta t}-ve^{\delta t}\int_0^t e^{-\delta r}dr\right\}\lambda_t\right] \\
&\times\exp\left[\rho\int_0^t\left[1-\hat{g}\left\{ke^{\delta s}-ve^{\delta t}\int_0^s e^{-\delta r}dr;s\right\}\right]ds\right] \qquad (2.5)
\end{aligned}
$$

is a martingale where $\hat{g}(u;s)=\int_0^\infty e^{-u\ y}dG(y;s)$ and $\triangle(t)=\int_0^t\delta(s)\,ds$.

Proof. See Dassios and Jang (2003). ■

Let us assume that $\delta(t)=\delta$ throughout the rest of this paper.

Corollary 2.3 *Let $v_1\geq 0$ and $v_2\geq 0$. Then*

$$
\begin{aligned}
&\mathbb{E}\left\{e^{-v_1(\Lambda_{t_2}-\Lambda_{t_1})}e^{-v_2\lambda_{t_2}}|\Lambda_{t_1},\lambda_{t_1}\right\} \\
=\ &\exp\left[-\left\{\frac{v_1}{\delta}+\left(v_2-\frac{v_1}{\delta}\right)e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right] \\
&\times\exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{\frac{v_1}{\delta}+\left(v_2-\frac{v_1}{\delta}\right)e^{-\delta(t_2-s)};s\right\}\right]ds\right]. \qquad (2.6)
\end{aligned}
$$

Proof. See Dassios and Jang (2003). ■

Now we can easily obtain the Laplace transforms of the distribution of λ_t, Λ_t.

Corollary 2.4 *The Laplace transforms of the distribution of λ_t and Λ_t are given by*

$$\begin{aligned}\mathbb{E}\left\{e^{-v\lambda_{t_2}}|\lambda_{t_1}\right\} &= \exp\left[-ve^{-\delta(t_2-t_1)}\lambda_{t_1}\right] \\ &\quad\times\exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{ve^{-\delta(t_2-s)};s\right\}\right]ds\right],\end{aligned} \tag{2.7}$$

$$\begin{aligned}&\mathbb{E}\left\{e^{-v\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}|\lambda_{t_1}\right\} \\ =\ &\exp\left[-\frac{v}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right] \\ &\times\exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{\frac{v}{\delta}\left(1-e^{-\delta(t_2-s)}\right);s\right\}\right]ds\right].\end{aligned} \tag{2.8}$$

Proof. See Dassios and Jang (2003). ∎

Let us obtain the asymptotic distributions of λ_t at time t from (2.8), provided that the process started sufficiently far in the past. In this context we interpret it as the limit when $t\to-\infty$. In other words, if we know λ at $'-\infty'$ and no information between $'-\infty'$ to present time t, $'-\infty'$ asymptotic distribution of λ_t can be used as the distribution of λ_t.

Lemma 2.5 *Assume that $\lim\limits_{t\to-\infty}\rho(t)=\rho$ and $\lim\limits_{t\to-\infty}\mu_1(t)=\mu_1$. Then the $'-\infty'$ asymptotic distribution of λ_t has Laplace transform*

$$\mathbb{E}\left(e^{-v\lambda_{t_1}}\right)=\exp\left[-\int_{-\infty}^{t_1}\rho(s)\left[1-\hat{g}\left\{ve^{-\delta(t_1-s)};s\right\}\right]ds\right]. \tag{2.9}$$

Proof. See Dassios and Jang (2003). ∎

It will be interesting to find the Laplace transforms of the distribution of λ_t and Λ_t at time t, using a specific jump size distribution of $G(y;t)$ $(y>0)$. We use an exponential jump size distribution, i.e. $g(y;t)=\left(\alpha+\gamma e^{\delta t}\right)e^{-\left(\alpha+\gamma e^{\delta t}\right)y}$, $y>0$, $-\alpha e^{-\delta t}<\gamma\le 0$. Let us assume that $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$. The reason for this particular assumption will become apparent later when we change the probability measure.

Theorem 2.6 *Let the jump size distribution be exponential, i.e. $g(y;t)=\left(\alpha+\gamma e^{\delta t}\right)\exp\left\{-\left(\alpha+\gamma e^{\delta t}\right)y\right\}$, $y>0$, $\alpha e^{-\delta t}<\gamma\le 0$, and assume that $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$. Then*

$$\begin{aligned}\mathbb{E}\left\{e^{-v\lambda_{t_1}}|\lambda_{t_0}\right\} &= \exp\left\{-v\lambda_{t_0}e^{-\delta(t_1-t_0)}\right\}\cdot\left(\frac{\gamma e^{\delta t_0}+\alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0}+\alpha}\right)^{\frac{\rho}{\delta}} \\ &\quad\times\left(\frac{\gamma e^{\delta t_0}+ve^{-\delta(t_1-t_0)}+\alpha}{\gamma e^{\delta t_0}+(v+\alpha)e^{-\delta(t_1-t_0)}}\right)^{\frac{\rho}{\delta}},\end{aligned} \tag{2.10}$$

$$\begin{aligned}\mathbb{E}\left\{e^{-v\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}|\lambda_{t_1}\right\} &= \exp\left[-\frac{v}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right] \\ &\quad\times\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}} \\ &\quad\times\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}}\end{aligned} \tag{2.11}$$

If λ_t is $'-\infty'$ asymptotic,

$$\mathbb{E}\left(e^{-v\lambda_{t_1}}\right)=\left(\frac{\gamma+\alpha e^{-\delta t_1}}{\gamma+(v+\alpha)e^{-\delta t_1}}\right)^{\frac{\rho}{\delta}}, \tag{2.12}$$

$$\mathbb{E}\left\{e^{-v\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}\right\} = \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}\right)^{\frac{\rho}{\delta}} \times\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}} \tag{2.13}$$

Proof. See Dassios and Jang (2003). ■

3. No-arbitrage, the Esscher transform and change of probability measure

In general, the Esscher transform is defined as a change of probability measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The parameters involved for the Esscher transform are determined so that the process is a martingale under the new probability measure. We will examine an equivalent martingale probability measure obtained via the Esscher transform (Gerber and Shiu 1996 and Dassios and Jang 2003).

If the market is complete, the fair price of a contingent claim is the expectation with respect to exactly one equivalent martingale probability measure, i.e. by assuming that there is an absence of arbitrage opportunities in the market. For example, when the underlying stochastic process follows geometric Brownian motion or homogeneous Poisson process, we can obtain the fair price with respect to a unique equivalent martingale probability measure. However, as the underlying stochastic process for default arrival process is the Cox process with shot noise intensity, we will have infinitely many equivalent martingale probability measures. In other words, we will have several choices of equivalent martingale probability measures to decide credit derivatives prices as the market is incomplete. It is not the purpose of this paper to decide which is the appropriate one to use. The attractive thing about the Esscher transform is that it provides us with at least one equivalent martingale probability measure in incomplete market situations.

We here offer the definition of the Esscher transform that is adopted from Gerber & Shiu (1996).

Definition 3.1 *Let X_t be a stochastic process and h^* a real number. For a measurable function f, the expectation of the random variable $f(X_t)$ with respect to the equivalent martingale probability measure is*

$$\mathbb{E}^*\left[f\left(X_t\right)\right]=\mathbb{E}\left[f\left(X_t\right)\frac{e^{h^*X_t}}{\mathbb{E}\left(e^{h^*X_t}\right)}\right]=\frac{\mathbb{E}\left[f\left(X_t\right)e^{h^*X_t}\right]}{\mathbb{E}\left[e^{h^*X_t}\right]}, \tag{3.1}$$

where the process $e^{h^*X_t}$ is a martingale and $\mathbb{E}\left(e^{h^*X_t}\right)<\infty$.

From definition 3.1, we need to obtain a martingale that can be used to define a change of probability measure, i.e. it can be used to define the Radon-Nikodym derivative $\frac{dP^*}{dP}$ where P is the original probability measure and P^* is the equivalent martingale probability measure with parameters involved.

Assuming that default and primary events do not occur at the same time, the generator of the process $(\Lambda_t, N_t, \lambda_t, M_t, t)$ acting on a function $f(\Lambda, n, \lambda, m, t)$ belonging to its domain is given by

$$\begin{aligned} & A\ f(\Lambda,n,\lambda,m,t) \\ = & \frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial \Lambda}+\lambda\left[f\left(\Lambda,n+1,\lambda,m,t\right)-f\left(\Lambda,n,\lambda,m,t\right)\right] \\ & -\delta\lambda\frac{\partial f}{\partial\lambda}+\rho\left[\int_0^\infty f\left(\Lambda,n,\lambda+y,m+1,t\right)dG\left(y\right)-f\left(\Lambda,n,\lambda,m,t\right)\right]. \end{aligned} \tag{3.2}$$

Clearly, for $f\left(\Lambda,n,\lambda,m,t\right)$ to belong to the domain of the generator A, it is essential that $f\left(\Lambda,n,\lambda,m,t\right)$ is differentiable w.r.t. Λ, λ, t for all Λ, n, λ, m, t and that

$$\left|\int_0^\infty f\left(.,\lambda+y,.\right)dG\left(y\right)-f\left(.,\lambda,.\right)\right|<\infty.$$

Lemma 3.2 *Considering constants θ^*, ψ^* and γ^* such that $\theta^*\geq 1$, $\psi^*\geq 1$ and $\gamma^*\leq 0$,*

$$\theta^{*N_t}\exp\left\{-\left(\theta^*-1\right)\Lambda_t\right\}\psi^{*M_t}\exp\left(-\gamma^*\lambda_t e^{\delta t}\right)\exp\left[\rho\int_0^t\left\{1-\psi^*\hat{g}\left(\gamma^*e^{\delta s}\right)\right\}ds\right] \tag{3.3}$$

is a martingale.

Proof. From (3.2), $f(\Lambda, n, \lambda, m, t)$ has to satisfy $Af = 0$ for $f(\Lambda_t, N_t, \lambda_t, M_t, t)$ to be a martingale. Trying $\theta^{*n} e^{\phi^* \Lambda} \psi^{*m} \exp\left(-\gamma^* \lambda e^{\delta t}\right) e^{A(t)}$ we get the equation

$$A'(t) + \lambda \phi^* + \lambda \{\theta^* - 1\} + \rho \left\{\psi^* \hat{g}\left(\gamma^* e^{\delta t}\right) - 1\right\} = 0 \tag{3.4}$$

and solving (3.4) we get

$$\phi^* = -\{\theta^* - 1\} \text{ and } A(t) = \rho \int_0^t \left\{1 - \psi^* \hat{g}\left(\gamma^* e^{\delta s}\right)\right\} ds$$

and the result follows. ■

Let us examine how the generator A^* of the process $(\Lambda_t, N_t, \lambda_t, M_t, t)$ acting on a function $f(\Lambda, n, \lambda, m, t)$ with respect to the equivalent martingale probability measure can be obtained.

Lemma 3.3 *Let v^* be a nonnegative constant. Assuming that $f(\Lambda, n, \lambda, m, t) = f(\lambda, t)$ for all Λ, n, m and that $e^{-v^* X_t}$ is a martingale with X_t an adapted process. The generator A^* of the process (λ_t, t) acting on a function $f(\lambda, t)$ with respect to the equivalent martingale measure is given by*

$$A^* \ f(\lambda, t) = \frac{A \ \left\{f(\lambda, t) \ e^{-v^* X_t}\right\}}{e^{-v^* X_0}}. \tag{3.5}$$

Proof. The generator of the process (λ_t, t) acting on a function $f(\lambda, t)$ with respect to the equivalent martingale probability measure is

$$A^* \ f(\lambda, t) = \lim_{dt \downarrow 0} \frac{\mathbb{E}^* \left[f(\lambda_{t+dt}, t + dt) \mid \lambda_t = \lambda\right] - f(\lambda, t)}{dt}. \tag{3.6}$$

We will use $\frac{e^{-v^* X_t}}{\mathbb{E}\left(e^{-v^* X_t}\right)}$ as the Radon-Nikodym derivative to define equivalent martingale probability measure. Hence, the expected value of $f(\lambda_{t+dt}, t + dt)$ given $\lambda_t = \lambda$ with respect to the equivalent martingale probability measure is

$$\mathbb{E}^* \left[f(\lambda_{t+dt}, t + dt) \mid \lambda_t = \lambda\right] = \frac{\mathbb{E}\left[f(\lambda_{t+dt}, t + dt) e^{-v^* X_{t+dt}} \mid \lambda_t = \lambda\right]}{\mathbb{E}\left(e^{-v^* X_{t+dt}} \mid \lambda_t = \lambda\right)}. \tag{3.7}$$

Since $e^{-v^* X_t}$ in (3.7) is a martingale, it becomes

$$\begin{aligned} & \mathbb{E}^* \left[f(\lambda_{t+dt}, t + dt) \mid \lambda_t = \lambda\right] \\ = & \frac{f(\lambda, t) e^{-v^* \lambda} + \int_t^{t+dt} \mathbb{E}\left[A \ f(\lambda_s, s) e^{-v^* X_s} \mid \lambda_t = \lambda\right] ds}{e^{-v^* X_0}}. \end{aligned} \tag{3.8}$$

Set (3.8) in (3.6) then

$$A^* \ f(\lambda, t) = \frac{1}{e^{-v^* X_0}} \lim_{dt \downarrow 0} \int_t^{t+dt} \frac{\mathbb{E}\left[A \ f(\lambda_s, s) e^{-v^* X_s} \mid \lambda_t = \lambda\right] ds}{dt}. \tag{3.9}$$

Therefore, from Dynkin's formula, (3.5) follows immediately. ■

Now let us look at how the dynamics of process λ_t and Λ_t change after changing probability measure by obtaining the generator $\mathbf{A}^*$ of the process $(\Lambda_t, N_t, \lambda_t, M_t, t)$ acting on a function $f(\Lambda, n, \lambda, m, t)$ with respect to the equivalent martingale probability measure. This is the key result that we require to establish an arbitrage-free price under our equivalent martingale measure. As Dassios and Jang (2003) have shown the change of dynamics of process λ_t and Λ_t with respect to the equivalent martingale probability measure, we offer the theorem adopted from their studies (see theorem 3.5 in section 3).

Theorem 3.4 *Consider constants θ^*, ψ^* and γ^* such that $\theta^* \geq 1$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Suppose that $\hat{g}\left(\gamma^* e^{\delta t}\right) < \infty$. Then*

$$\begin{aligned} & A^* \ f(\Lambda, n, \lambda, m, t) \\ = \ & \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} + \theta^* \lambda \left\{ f\left(\Lambda, n+1, \lambda, m, t\right) - f\left(\Lambda, n, \lambda, m, t\right) \right\} \\ & - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho^*\left(t\right) \left\{ \int_0^\infty f\left(\Lambda, n, \lambda + y, m+1, t\right) dG^*\left(y; t\right) - f\left(\Lambda, n, \lambda, m, t\right) \right\} \end{aligned} \tag{3.10}$$

where $\rho^*\left(t\right) = \rho \psi^* \hat{g}\left(\gamma^* e^{\delta t}\right)$ *and* $dG^*\left(y; t\right) = \frac{\exp\left(-\gamma^* e^{\delta t} y\right) dG(y)}{\hat{g}\left(\gamma^* e^{\delta t}\right)}$.

Proof. See Dassios and Jang (2003). ■

Theorem 3.4 yields the following:

(i) The intensity function λ_t has changed to $\theta^* \lambda_t$;

(ii) The rate of jump arrival ρ has changed to $\rho^*\left(t\right) = \rho \psi^* \hat{g}\left(\gamma^* e^{\delta t}\right)$ (it now depends on time);

(iii) The jump size measure $dG\left(y\right)$ has changed to $dG^*\left(y; t\right) = \frac{\exp\left(-\gamma^* e^{\delta t} y\right) dG(y)}{\hat{g}\left(\gamma^* e^{\delta t}\right)}$ (it now depends on time).

In other words, the risk-neutral Esscher measure is the measure with respect to which N_t becomes the Cox process with parameter $\theta^* \lambda_t$ where three parameters of the shot noise process λ_t are δ, $\rho^*\left(t\right) = \rho \psi^* \hat{g}\left(\gamma^* e^{\delta t}\right)$, $dG^*\left(y; t\right) = \frac{\exp\left(-\gamma^* e^{\delta t} y\right) dG(y)}{\hat{g}\left(\gamma^* e^{\delta t}\right)}$.

In practice, the banks and traders need to calculate credit derivatives prices using $\theta^* > 1$, $\psi^* > 1$ and $\gamma^* < 0$. This results in banks and traders assuming that there will be a higher value of intensity itself, more primary events occurring in a given period of time and a higher value of jump size of intensity. These assumptions are necessary, as the banks and traders have to consider the risks involved in incomplete market. If $\theta^* = 1$, $\psi^* = 1$, and $\gamma^* = 0$ then non-arbitrage free credit derivatives price is calculated without considering any risks involved in incomplete market. However, as expected, we have quite a flexible family of equivalent probability measures by the combination of θ^*, ψ^* and γ^*. It means that the banks and traders have various ways of obtaining an non-arbitrage credit derivatives price (i.e. by changing equivalent martingale probability measures using the combination of θ^*, ψ^* and γ^*).

Now let us derive the Laplace transform of the '$-\infty$' asymptotic distribution of Λ_t with respect to the equivalent martingale probability measure, i.e. $\mathbb{E}^*\left(e^{-\Lambda_t}\right)$. We will assume that the jump size distribution is exponential, i.e. $g\left(y\right) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$ and that λ_t is '$-\infty$' asymptotic. Therefore we can obtain that $g^*\left(y; t\right) = \left(\alpha + \gamma^* e^{\delta t}\right) \exp\left\{-\left(\alpha + \gamma^* e^{\delta t}\right) y\right\}$, $y > 0$, $-\alpha e^{-\alpha y} < \gamma^* \leq 0$ and $t < \frac{1}{\delta} \ln\left(-\frac{\alpha}{\gamma^*}\right)$ since $dG^*\left(y; t\right) = \frac{\exp\left(-\gamma^* e^{\delta t} y\right) dG(y)}{\hat{g}\left(\gamma^* e^{\delta t}\right)}$.

Corollary 3.5 *Let the jump size distribution be exponential. Consider constants ν, θ^*, ψ^* and γ^* such that $v \geq 0$, $\theta^* \geq 1$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Furthermore if λ_t is '$-\infty$' asymptotic, then*

$$\begin{aligned} \mathbb{E}^*\left\{e^{-v\left(\Lambda_{t_2} - \Lambda_{t_1}\right)}\right\} \ = \ & \left(\frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}}{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^* \nu}{\delta}\left(1 - e^{-\delta(t_2 - t_1)}\right)} \right)^{\frac{\psi^* \rho}{\delta}} \\ & \times \left(\frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^* \nu}{\delta}\left(1 - e^{-\delta(t_2 - t_1)}\right)}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}} \right)^{\frac{\alpha \psi^* \rho}{\delta \alpha + \theta^* \nu}} \end{aligned} \tag{3.11}$$

where $0 < t_1 < t_2 < t$.

Proof. From Theorem 3.4, (1.4) and (2.13), the result follows immediately. ■

In order to obtain an arbitrage-free zero-coupon default-free bond price, we can employ the Girsanov theorem (Karatzas and Shreve 1991; Revuz and Yor 1991 and Protter 1992) to change measure. However for simplicity, we will use a zero-coupon default-free bond price at time 0 with respect to the original measure. Hence from CIR (1985), assuming that $c = 1$, we can easily obtain a zero-coupon default-free bond price at time 0, paying 1 at time t,

$$
\begin{aligned}
B(0,t) &= \mathbb{E}\left\{ \exp\left(-\int_0^t r_s ds \right) \mid r_0 \right\} = \mathbb{E}\left(e^{-R_t} \mid r_0 \right) \\
&= \exp\left[-\left\{ \frac{2\left\{1 - \exp\left(-\sqrt{a^2+2\sigma^2}t\right)\right\}}{\left\{\left(\sqrt{a^2+2\sigma^2}+\xi^\star a\right) + \left(\sqrt{a^2+2\sigma^2}-a\right)\exp\left(-\sqrt{a^2+2\sigma^2}t\right)\right\}} \right\} r_0 \right] \\
&\quad \times \left\{ \frac{2\sqrt{a^2+2\sigma^2}\exp\left(-\frac{\left(\sqrt{a^2+2\sigma^2}-a\right)}{2}t\right)}{\left(\sqrt{a^2+2\sigma^2}+a\right) + \left(\sqrt{a^2+2\sigma^2}-a\right)\exp\left(-\sqrt{a^2+2\sigma^2}t\right)} \right\}^{\frac{2b}{\sigma^2}}.
\end{aligned} \tag{3.12}
$$

4. **Pricing of credit derivatives using three building blocks**

Now let us look at the credit derivatives prices at present time 0, i.e. the price of defaultable fixed-coupon bond and the market credit default swaps (CDS) rate, assuming that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale probability measure, $Q = P^* \times P$, within the expressions of three building blocks in Section 1. Therefore, from (1.8), (1.9), (1.10), (1.11) (3.11) and (3.12) the price of a zero-coupon corporate (defaultable) bond paying $1_{(\tau_1 > t)}$ at time t is given by

$$
\begin{aligned}
\overline{B}^Q(0,t) &= \mathbb{E}^Q\left\{ \exp\left(-\int_0^t r_s ds \right) 1_{(\tau_1 > t)} \mid r_0 \right\} = \mathbb{E}^Q\left[\exp\left\{ -\int_0^t (r_s + \lambda_s)\, ds \right\} \mid r_0 \right] \\
&= \mathbb{E}^Q\left(e^{-R_t}\, e^{-\Lambda_t} \mid r_0 \right) \\
&= \mathbb{E}\left(e^{-R_t} \mid r_0 \right)\ \mathbb{E}^*\left(e^{-\Lambda_t} \right)
\end{aligned} \tag{4.2}
$$

and the value of a deterministic payoff 1 that is paid at t_{k+1} if and only if a default happens in $]t_k, t_{k+1}]$ is given by

$$
\begin{aligned}
e^Q(0,t_k,t_{k+1}) &= \mathbb{E}^Q\left[\exp\left(-\int_0^{t_{k+1}} r_s ds \right) \left(1_{\{N_{t_k}=0\}} - 1_{\{N_{t_{k+1}}=0\}} \right) \mid r_0 \right] \\
&= \mathbb{E}^Q\left[\exp\left(-\int_0^{t_{k+1}} r_s ds \right) \left\{ \exp\left(-\int_0^{t_k} \lambda_s ds \right) - \exp\left(-\int_0^{t_{k+1}} \lambda_s ds \right) \right\} \mid r_0 \right] \\
&= \mathbb{E}\left(e^{-R_{t_{k+1}}} \mid r_0 \right) \left\{ \mathbb{E}^*\left(e^{-\Lambda_{t_k}} \right) - \mathbb{E}^*\left(e^{-\Lambda_{t_{k+1}}} \right) \right\}
\end{aligned} \tag{4.3}
$$

and the price a defaultable fixed-coupon bond is given by

$$
\overline{C}^Q(0) = \sum_{k=1}^{N} \overline{c}_n\, \overline{B}^Q(0,t_{k_n}) + \overline{B}^Q(0,t_{k_N}) + \pi \sum_{k=1}^{k_N} e^Q(0,t_{k-1},t_k) \tag{4.4}
$$

and the market credit default swaps (CDS) rate is given by

$$
\overline{s}^Q = (1-\pi)\, \frac{\sum_{k=1}^{k_n} e^Q(0,t_{k-1},t_k)}{\sum_{k=1}^{N} (t_{k_{n+1}} - t_{k_n}) \overline{B}^Q(0,t_{k_n})}. \tag{4.5}
$$

It has been assumed implicitly that the frequency and magnitude of primary events and time period needed to go back to the previous level of intensity immediately after primary events occur are the same among all firms. However some of these primary events might not affect at all to a specific firm e.g. 'AAA' rating firm. Also even if primary events affect firms' default intensities, their magnitude should be different to each firms. Time period needed to go back to the previous level of intensity also need to be discriminated among firms. Therefore we denote the survival probability of the firm i by

$$\mathbb{E}^* \left(e^{-\Lambda_t^i} \right).$$

Now now illustrate the calculations of defaultable fixed-coupon bond prices and the market credit default swaps (CDS) rates using the expressions derived above.

Example 4.1

The parameter values used to calculate (4.4) and (4.5) are

$$r_0 = 0.05,\ a = 0.05,\ b = 0.025 \text{ and } \sigma = 0.8 \text{ for } r_t$$

and

$$\psi^* = 1.1,\ \gamma^* = -0.1,\ \theta^* = 1.1,\ \alpha^i = 10,\ \delta^i = 0.5 \text{ and } \rho^i = 4 \text{ for } \lambda_t$$

and

$$\overline{c} = 5\% \quad \text{and} \quad \pi = 50\%$$

and

$$N = 2,\ t_{k_0} = 0,\ t_{k_1} = 0.5,\ t_{k_2} = 1.$$

Then an arbitrage-free defaultable fixed-coupon bond price is given by

$$\overline{C_i}^Q(0) = 0.024357 + 0.37052 + 0.28753 = 0.68241$$

and an arbitrage-free credit default swaps (CDS) rate is given by

$$\overline{s}_i^Q = (1 - 0.5) \left(\frac{0.57506}{0.48715} \right) = 0.59023 = 5,902.3\text{bp}.$$

Example 4.2

We will now examine the effect on arbitrage-free defaultable fixed-coupon bond price and credit default swaps (CDS) rate caused by changes in the value of α^i, δ^i and ρ^i. The calculation of arbitrage-free defaultable fixed-coupon bond prices and credit default swaps (CDS) rates are shown in Table 4.1 and Table 4.2 respectively, assuming other parameter values are the same as in Example 4.1.

Table 4.1.

	$\overline{C_i}^Q(0)$
$\alpha^i = 1$	0.47337
$\alpha^i = 20$	0.80033
$\delta^i = 0.1$	0.47981
$\delta^i = 4$	0.92659
$\rho^i = 0$	0.99354
$\rho^i = 8$	0.55836

Table 4.2.

	$\overline{s}_i^Q$
$\alpha^i = 1$	704,280bp
$\alpha^i = 20$	2,647.4bp
$\delta^i = 0.1$	94,499bp
$\delta^i = 4$	718.74bp
$\rho^i = 0$	0bp
$\rho^i = 8$	15,399bp

5. **Conclusion**

Using the intensity-based framework, i.e. employing the Cox process with shot noise intensity, we examined how it could be used to calculate the price of defaultable fixed-coupon bond and the market credit default swaps (CDS) rate. In order to obtain an arbitrage-free prices, we changed the probability measure via the Esscher transform, where we showed how the dynamics of the shot noise process λ_t and the aggregated process Λ_t changed with respect to an equivalent martingale probability measure. We witnessed that there are various ways to quantify the risk involved when the market is incomplete. By discriminating of three parameters of the shot noise intensity, i.e. the frequency and magnitude of primary events and time period needed to go back to the previous level of intensity immediately after primary events occur for each firm, we illustrated the calculations of arbitrage-free defaultable fixed-coupon bond prices and the market credit default swaps (CDS) rates.

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