Generalised Lévy Processes and their Applications in Insurance and Finance

Ji-Wook Jang

Actuarial Studies, University of New South Wales, Sydney, NSW 2052, Australia, e-mail: J.Jang@unsw.edu.au

Abstract For insurance risks, jump processes such as homogeneous/non-homogeneous Poisson process and Cox process have been used . In financial modelling, it has been observed that diffusion models are not robust enough to capture the appearance of jumps in underlying asset prices and interest rates. As a result, generalised Lévy processes, which are simply speaking, the combinations of Poisson process and Brownian motion have gained their popularity for modelling in insurance and finance. In this paper, considering a generalised Lévy process, we obtain the explicit expression of the Laplace transform of the distribution of a generalised Lévy process assuming that jump size follows the mixture of two exponential distribution, which is a special case of phase-type distributions. We derive the mean and variance of the aggregate accumulated claim amounts of insurance risk providing their numerical examples. European call option pricing and non-defaultable zero-coupon bond pricing are examined for its application in financial risks. The calculation of call option prices is illustrated using Transform Analysis technique from the financial option pricing literature.

Keywords: Generalised Lévy Processes; Piecewise deterministic Markov process; Aggregate accumulated claims; European call option; Zero-coupon bond; Transform Analysis.

1. Introduction

Let us consider a generalised Lévy process X_t (Bertoin, 1998; Sato, 1999; Barndorff-Nielsen et al., 2001 and Cont and Tankov, 2004), i.e.

$$dX_t = c(b + aX_t)dt + \sigma\sqrt{X_t}dB_t + dC_t \tag{1.1}$$

where $c \geq 0$, $b \geq 0$, $a \in \mathbb{R}$, $\sigma \geq 0$, B_t is a standard Brownian motion and

$$C_t = \sum_{i=1}^{N_t} Y_i \tag{1.2}$$

is a pure-jump process with N_t being the number of jumps up to time t and Y_i, $i = 1, 2, \cdots$, are their sizes.

Set $c = 0$ and $\sigma = 0$ in (1.1) and let Y_i, $i = 1, 2, \cdots$, be the claim amounts, which are assumed to be independent and identically distributed with distribution function $G(y)$ $(y > 0)$. Then (1.1) is equivalent to the loss process that can be easily found in classical actuarial risk theory, where we assume (often implicitly) that interest rates equal zero. If we let δ to be a constant risk-free force of interest rate, the aggregate accumulated claim amounts up to time t, M_t is given by

$$M_t = \sum_{i=1}^{N_t} Y_i e^{\delta(t-s_i)} \tag{1.3}$$

with s_i being the time of claim i. If we set $c = 1$, $b = 0$, $a = \delta$ and $\sigma = 0$ in (1.1), (1.3) can be expressed as

$$dM_t = \delta M_t dt + dC_t. \tag{1.4}$$

If we substitute δ with $-\delta$ in (1.4), it becomes the shot noise process, denoted by λ_t,

$$d\lambda_t = -\delta\lambda_t dt + dC_t \tag{1.5}$$

which has been used for an actuarial application as discounted aggregate claim amounts process (Jang 2004 and Jang and Krvavych 2004).

Taylor (1979) and Delbaen and Haezendonck (1987) extended the classical risk theory to consider the effect of the introduction of constant interest rate factors, leading to an explosion of literature in this subject in recent years (Willmot, 1989; Paulsen, 1998 for a survey). Most of these papers deal with the effect of constant interest rates on the probability of ruin. More papers dealt with the effect of constant interest rates in terms of premium setting can be found in Léveillé and Garrido (2001), Jang (2004) and Jang and Krvavych (2004).

In contrast, we will consider a stochastic interest rate to the aggregate claim amounts, denoted by L_t, as it is not deterministic in practice. So if we set $c = 1$, $b = 0$, $a = \mu$ and $\sigma > 0$ in (1.1), we extend our model from (1.4) to

$$dL_t = \mu L_t dt + \sigma\sqrt{L_t} dB_t + dC_t \tag{1.6}$$

We assume that the claim arrival process N_t follows a Poisson process with claim frequency rate ρ. It is also assumed that is independent of Y_i, $i = 1, 2, \cdots$. In order to obtain the explicit expressions of the moments of L_t, we employ mixture of exponential distribution for claim sizes.

If we set $\rho = 0$, $c = 1$ and $a < 0$, (1.1) becomes the celebrated Cox-Ingersoll-Ross (1985) model for interest rate, denoted by r_t, i.e.

$$dr_t = (b - ar_t)\, dt + \sigma\sqrt{r_t} dB_t \tag{1.7}$$

If set $\rho = 0$, $c = 1$ and $b = 0$, (1.1) becomes a square root process, denoted by S_t, i.e.

$$dS_t = aS_t dt + \sigma\sqrt{S_t} dB_t, \tag{1.8}$$

that can be used to model asset prices when the asset price volatility does not increase too much when S_t increases. If we set $\rho = 0$, $c = 1$ and $a = 0$, (1.1) becomes a squared Bessel process, denoted by Z_t, i.e.

$$dZ_t = bdt + \sigma\sqrt{Z_t} dB_t. \tag{1.9}$$

This paper is structured as follows. In Section 2, we derive the joint Laplace transform of the distribution of X_t and $\Psi_t = \int_0^t X_s ds$, applying the piecewise deterministic Markov processes theory. Section 3 contains the explicit expressions for the moments of X_t and its application in terms of insurance risk, i.e. the moments of the aggregate accumulated claim process, $\mathbb{E}(L_t)$, $\mathbb{E}(M_t)$, $\mathrm{Var}\,(L_t)$ and $\mathrm{Var}\,(M_t)$. The comparisons among the moments of the aggregate accumulated claim process with/without diffusion coefficient of σ, i.e. $\mathbb{E}(L_t)$ vs. $\mathbb{E}(M_t)$ and $\mathrm{Var}\,(L_t)$ vs. $\mathrm{Var}\,(M_t)$ are also made. As expected, if $\mu = \delta$, we witness that $\mathbb{E}(L_t) = \mathbb{E}(M_t)$ as $\mathbb{E}\left[\int_0^t \sigma\sqrt{X_s} dB_s\right] = 0$, but $\mathrm{Var}\,(L_t) \neq \mathrm{Var}\,(M_t)$. Numerical examples are provided to measure the differences between $\mathrm{Var}\,(L_t)$ and $\mathrm{Var}\,(M_t)$ by changing the value of σ. In section 4, we apply the results in section 2 to financial risks modelling, which are European call option pricing and zero-coupon government bond pricing. Transform Analysis technique from the financial option pricing literature is used for the illustration of the calculation of call option prices. Section 5 concludes.

2. The Laplace transform of the distribution of a generalised Lévy process

The piecewise deterministic Markov processes (PDMP) theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. Using this theory, we derive the Laplace transform of the distribution of a generalised Lévy process, X_t. The generator of the process (Ψ_t, X_t, t) acting on a function $f(\psi, x, t)$ belonging to its domain is given by

$$A\ f(\psi, x, t) = \frac{\partial f}{\partial t} + x\frac{\partial f}{\partial \psi} + c\,(b + ax)\,\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x \frac{\partial^2 f}{\partial x^2} + \rho\left[\int_0^\infty f(\psi, x + y, t)\, dG(y) - f(\psi, x, t)\right], \tag{2.1}$$

where $f : (0, \infty) \times (0, \infty) \times \mathbb{R}^+ \to (0, \infty)$. It is sufficient that $f(\psi, x, t)$ is differentiable w.r.t. ψ, x, t for all ψ, x, t and that for $\left|\int_0^\infty f(\psi, x + y, t)\, dG(y) - f(\psi, x, t)\right| < \infty$ for $f(\psi, x, t)$ to belong to the domain of the generator A.

Now let us obtain a suitable martingale to derive the Laplace transforms of the distribution of X_t and Ψ_t.

Lemma 2.1 *Considering a constant* k,

$$\exp\left[-\left\{\frac{\left(ca+\sqrt{c^2a^2+2\sigma^2\xi}\right)-\left(ca-\sqrt{c^2a^2+2\sigma^2\xi}\right)\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}t-k\right)}{\sigma^2\left\{1-\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}t-k\right)\right\}}\right\}X_t\right]\exp(-\xi\Psi_t)$$
$$\times\exp\left[\rho\int_0^t\left[1-\hat{g}\left\{\frac{\left(ca+\sqrt{c^2a^2+2\sigma^2\xi}\right)-\left(ca-\sqrt{c^2a^2+2\sigma^2\xi}\right)\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)}{\sigma^2\left\{1-\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)\right\}}\right\}\right]ds\right]$$
$$\times\exp\left[cb\int_0^t\left\{\frac{\left(ca+\sqrt{c^2a^2+2\sigma^2\xi}\right)-\left(ca-\sqrt{c^2a^2+2\sigma^2\xi}\right)\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)}{\sigma^2\left\{1-\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)\right\}}\right\}ds\right] \quad (2.2)$$

is a martingale where $\hat{g}(u)=\int\limits_0^\infty e^{-uy}dG(y)$.

Proof. From (2.1), $f(\psi,x,t)$ has to satisfy $Af=0$ for it to be a martingale. Setting $f(\psi,x,t)=e^{-A(t)x}e^{-\xi\psi}e^{B(t)}$ we get the equation

$$-xA'(t)+B'(t)-x\xi-cbA(t)-caxA(t)+\frac{1}{2}\sigma^2x\{A(t)\}^2+\rho\{\hat{g}\{A(t)\}-1\}=0 \quad (2.3)$$

and solving (2.3), we get

$$A(t)=\frac{\left(ca+\sqrt{c^2a^2+2\sigma^2\xi}\right)-\left(ca-\sqrt{c^2a^2+2\sigma^2\xi}\right)\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}t-k\right)}{\sigma^2\left\{1-\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}t-k\right)\right\}} \quad (2.4)$$

and

$$\begin{aligned}B(t) &= \rho\int_0^t\left[1-\hat{g}\left\{\frac{\left(ca+\sqrt{c^2a^2+2\sigma^2\xi}\right)-\left(ca-\sqrt{c^2a^2+2\sigma^2\xi}\right)\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)}{\sigma^2\left\{1-\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)\right\}}\right\}\right]ds\\ &\quad +cb\int_0^t\left\{\frac{\left(ca+\sqrt{c^2a^2+2\sigma^2\xi}\right)-\left(ca-\sqrt{c^2a^2+2\sigma^2\xi}\right)\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)}{\sigma^2\left\{1-\exp\left(\sqrt{c^2a^2+2\sigma^2\xi}s-k\right)\right\}}\right\}ds \quad (2.5)\end{aligned}$$

where k is a constant. Hence the result follows. ■

Corollary 2.2 *Considering constants* ν *and* ξ *such that* $\nu\geq0$ *and* $\xi\geq0$, *Then the joint Laplace transform of the distribution of* (X_t,Ψ_t) *is given by*

$$\begin{aligned}&\mathbb{E}\left(e^{-\nu X_t}e^{-\xi\Psi_t}|X_0\right)\\ &= \exp\left[-\{B_1(t)\}X_0\right]\exp\left[-\rho\int_0^t\left[1-\hat{g}\{B_1(s)\}\right]ds\right][C_1(t)]^{\frac{2cb}{\sigma^2}}. \quad (2.6)\end{aligned}$$

where

$$B_1(t)=\frac{\nu\left\{\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}\right\}+2\xi\left(1-e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}\right)}{\sigma^2\nu\left(1-e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}},$$

$$C_1(t)=\frac{2\sqrt{c^2a^2+2\sigma^2\xi}\exp\left(-\frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)}{2}t\right)}{\sigma^2\nu\left(1-e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}}$$

Proof. Set $A(T) = \nu \geq 0$ using (2.4), where $t < T$, then we have

$$k = \sqrt{c^2a^2 + 2\sigma^2\xi}T - \ln\left(\frac{ca + \sqrt{c^2a^2 + 2\sigma^2\xi} - \nu\sigma^2}{ca - \sqrt{c^2a^2 + 2\sigma^2\xi} - \nu\sigma^2}\right). \tag{2.7}$$

Substitute (2.7) into (2.2) and the result follows immediately. ■

Now we can easily derive the Laplace transforms of the distribution of X_t and Ψ_t.

Corollary 2.3 *The Laplace transform of the distribution of* X_t *is given by*

$$\begin{aligned}
& \mathbb{E}\left(e^{-\nu X_t}|X_0\right) \\
= \; & \exp\left[-\left\{\frac{2ca\nu\exp(cat)}{\sigma^2\nu\{\exp(cat)-1\}+2ca}\right\}X_0\right] \\
& \times\exp\left[-\rho\int_0^t\left[1-\hat{g}\left\{\frac{2ca\nu\exp(cas)}{\sigma^2\nu\{\exp(cas)-1\}+2ca}\right\}\right]ds\right] \\
& \times\left[\frac{2ca}{\sigma^2\nu\{\exp(cat)-1\}+2ca}\right]^{\frac{2cb}{\sigma^2}}.
\end{aligned} \tag{2.8}$$

and the Laplace transform of the distribution of Ψ_t *is given by*

$$\begin{aligned}
& \mathbb{E}\left(e^{-\xi\Psi_t}|X_0\right) \\
= \; & \exp\left[-\{B_2(t)\}X_0\right]\exp\left[-\rho\int_0^t\left[1-\hat{g}\{B_2(s)\}\right]ds\right][C_2(t)]^{\frac{2cb}{\sigma^2}},
\end{aligned} \tag{2.9}$$

where

$$B_2(t) = \frac{2\xi\left\{1-\exp\left(-\sqrt{c^2a^2+2\sigma^2\xi}s\right)\right\}}{\left\{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\exp\left(-\sqrt{c^2a^2+2\sigma^2\xi}s\right)\right\}},$$

$$C_2(t) = \frac{2\sqrt{c^2a^2+2\sigma^2\xi}\exp\left(-\frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)}{2}t\right)}{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\exp\left(-\sqrt{c^2a^2+2\sigma^2\xi}t\right)}.$$

Proof. Set $\xi = 0$ in (2.6), (2.8) follows immediately. Set $\nu = 0$ in (2.6), (2.9) follows immediately. ■

It will be interesting to find the Laplace transforms of the distribution of X_t and Ψ_t at time t, using a specific claim size distribution of $G(y)$ $(y > 0)$. We use the mixture of two exponential jump size distribution, which is a special case of phase-type distributions (Asmussen, 2000), i.e. $g(y) = \beta_1\alpha_1e^{-\alpha_1y} + \beta_2\alpha_2e^{-\alpha_2y}$, $y > 0$, $\alpha_1 > \alpha_2 > 0$ and $\beta_1 + \beta_2 = 1$.

Corollary 2.4 *Let the claim size distribution be the mixture of two exponential, i.e.* $g(y) = \beta_1\alpha_1e^{-\alpha_1y} + \beta_2\alpha_2e^{-\alpha_2y}$, $y > 0$, $\alpha_1 > \alpha_2 > 0$ and $\beta_1 + \beta_2 = 1$. *Then the joint Laplace transform of the distribution of* (X_t, Ψ_t) *is given by*

$$\begin{aligned}
& \mathbb{E}\left(e^{-\nu X_t}e^{-\xi\Psi_t}|X_0\right) \\
= \; & \exp\left[-\{B_1(t)\}X_0\right]\times e^{-\rho t}\times\{C_1(t)\}^{\frac{2cb}{\sigma^2}} \\
& \times\{J_1(t)\}^{\frac{\left\{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\sigma^2\nu\right\}\alpha_1\beta_1}{\alpha_1\left\{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\sigma^2\nu\right\}+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\nu+2\xi}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}} \\
& \times\{K_1(t)\}^{\frac{\left\{\sigma^2\nu-\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\right\}\alpha_1\beta_1}{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\nu-2\xi-\alpha_1\left\{\sigma^2\nu-\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\right\}}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}} \\
& \times\{\Gamma_1(t)\}^{\frac{\left\{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\sigma^2\nu\right\}\alpha_2\beta_2}{\alpha_2\left\{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+\sigma^2\nu\right\}+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\nu+2\xi}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}} \\
& \times\{\Pi_1(t)\}^{\frac{\left\{\sigma^2\nu-\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\right\}\alpha_2\beta_2}{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\nu-2\xi-\alpha_2\left\{\sigma^2\nu-\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\right\}}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}},
\end{aligned} \tag{2.10}$$

where

$$J_1(t) = \frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\left(\alpha_1 e^{\sqrt{c^2a^2+2\sigma^2\xi}t}+\nu\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\left(\nu e^{\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_1\right) + \left(2\xi+\alpha_1\sigma^2\nu\right)\left(e^{\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\sqrt{c^2a^2+2\sigma^2\xi}\left(\alpha_1+\nu\right)},$$

$$K_1(t) = \frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\left(\alpha_1 e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}+\nu\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\left(\nu e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_1\right) - \left(2\xi+\alpha_1\sigma^2\nu\right)\left(e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\sqrt{c^2a^2+2\sigma^2\xi}\left(\alpha_1+\nu\right)},$$

$$\Gamma_1(t) = \frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\left(\alpha_2 e^{\sqrt{c^2a^2+2\sigma^2\xi}t}+\nu\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\left(\nu e^{\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_2\right) + \left(2\xi+\alpha_2\sigma^2\nu\right)\left(e^{\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\sqrt{c^2a^2+2\sigma^2\xi}\left(\alpha_2+\nu\right)},$$

$$\Pi_1(t) = \frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\left(\alpha_2 e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}+\nu\right)+\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\left(\nu e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_2\right) - \left(2\xi+\alpha_2\sigma^2\nu\right)\left(e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\sqrt{c^2a^2+2\sigma^2\xi}\left(\alpha_2+\nu\right)}$$

and the Laplace transform of the distribution of X_t *is given by*

$$\begin{aligned}
&\mathbb{E}\left(e^{-\nu X_t}|X_0\right) \\
=\ & \exp\left[-\left\{\frac{2ca\nu e^{cat}}{\sigma^2\nu\left(e^{cat}-1\right)+2ca}\right\}X_0\right]\times\left[\frac{2ca}{\sigma^2\nu\left(e^{cat}-1\right)+2ca}\right]^{\frac{2cb}{\sigma^2}} \\
&\times\left\{\frac{2ca\left(\alpha_1+\nu e^{cat}\right)-\alpha_1\sigma^2\nu\left(1-e^{cat}\right)}{2ca\left(\alpha_1+\nu\right)}\right\}^{-\beta_1\frac{\rho}{ca}}\times\left\{\frac{2ca\left(\nu e^{cat}+\alpha_1\right)+\alpha_1\sigma^2\nu\left(e^{cat}-1\right)}{2ca\left(\alpha_1+\nu\right)}\right\}^{\sigma^2\left(\frac{\alpha_1\beta_1}{\alpha_1\sigma^2+2ca}\right)\frac{\rho}{ca}} \\
&\times\left\{\frac{2ca\left(\alpha_2+\nu e^{cat}\right)-\alpha_2\sigma^2\nu\left(1-e^{cat}\right)}{2ca\left(\alpha_2+\nu\right)}\right\}^{-\beta_2\frac{\rho}{ca}}\times\left\{\frac{2ca\left(\nu e^{cat}+\alpha_2\right)+\alpha_2\sigma^2\nu\left(e^{cat}-1\right)}{2ca\left(\alpha_2+\nu\right)}\right\}^{\sigma^2\left(\frac{\alpha_2\beta_2}{\alpha_2\sigma^2+2ca}\right)\frac{\rho}{ca}} \\
&,
\end{aligned} \tag{2.11}$$

and the Laplace transform of the distribution of Ψ_t *is given by*

$$\begin{aligned}
&\mathbb{E}\left(e^{-\xi\Psi_t}|X_0\right) \\
=\ & \exp\left[-\left\{B_2(t)\right\}X_0\right]\times e^{-\rho t}\times\left\{C_2(t)\right\}^{\frac{2cb}{\sigma^2}} \\
&\times\left\{J_2(t)\right\}^{\frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\alpha_1\beta_1}{\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+2\xi}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}}\times\left\{K_2(t)\right\}^{\frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\alpha_1\beta_1}{2\xi-\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}} \\
&\times\left\{\Gamma_2(t)\right\}^{\frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)\alpha_2\beta_2}{\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)+2\xi}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}}\times\left\{\Pi_2(t)\right\}^{\frac{\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)\alpha_2\beta_2}{2\xi-\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)}\frac{\rho}{\sqrt{c^2a^2+2\sigma^2\xi}}}
\end{aligned} \tag{2.12}$$

where

$$J_2(t) = \frac{\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)e^{\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)+2\xi\left(e^{\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\alpha_1\sqrt{c^2a^2+2\sigma^2\xi}},$$

$$K_2(t) = \frac{\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)-2\xi\left(e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\alpha_1\sqrt{c^2a^2+2\sigma^2\xi}},$$

$$\Gamma_2(t) = \frac{\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)e^{\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)+2\xi\left(e^{\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\alpha_2\sqrt{c^2a^2+2\sigma^2\xi}},$$

$$\Pi_2(t) = \frac{\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right)e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}+\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right)-2\xi\left(e^{-\sqrt{c^2a^2+2\sigma^2\xi}t}-1\right)}{2\alpha_2\sqrt{c^2a^2+2\sigma^2\xi}}.$$

Proof. If we use $g(y) = \beta_1\alpha_1 e^{-\alpha_1 y} + \beta_2\alpha_2 e^{-\alpha_2 y}$, $y > 0$, $\alpha_1 > \alpha_2 > 0$ and $\beta_1 + \beta_2 = 1$ in (2.6), (2.8) and (2.9), the results follow immediately. ■

In practice, we might need to employ one of the heavy-tailed distributions for jump sizes such as Pareto, Gumbel and Fréchet, etc. rather than exponential. However if it is the case, we will not be able to obtain the explicit expression of the Laplace transform of the distribution of a generalised Lévy process, X_t.

3. Generalised Lévy processes in insurance

Now let us derive the moments of a Lévy process X_t using the explicit expression derived in section 2 to model insurance risk. If we differentiate (2.11) with respect to ν and put $\nu = 0$, we can obtain the mean of X_t, i.e.

$$\mathbb{E}(X_t \mid X_0) = e^{cat}X_0 + \left(\frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2} + cb\right)\left(\frac{e^{cat}-1}{ca}\right) \tag{3.1}$$

and higher moments can be obtained by differentiating it further, i.e.

$$\operatorname{Var}(X_t \mid X_0) = \left[\begin{array}{c} \sigma^2\left\{e^{cat}X_0 + \frac{cb}{2}\left(\frac{e^{cat}-1}{ca}\right)\right\} \\ +\left(\frac{2ca+\alpha_1\sigma^2}{2}\right)\left(\frac{2cae^{cat}+\alpha_1\sigma^2\left(e^{cat}-1\right)}{2ca}+1\right)\left(\frac{\beta_1\rho}{ca\alpha_1^2} - \frac{\sigma^2\beta_1\rho}{ca(\alpha_1\sigma^2+2ca)\alpha_1}\right) \\ +\left(\frac{2ca+\alpha_2\sigma^2}{2}\right)\left(\frac{2cae^{cat}+\alpha_2\sigma^2\left(e^{cat}-1\right)}{2ca}+1\right)\left(\frac{\beta_2\rho}{ca\alpha_2^2} - \frac{\sigma^2\beta_2\rho}{ca(\alpha_2\sigma^2+2ca)\alpha_2}\right) \end{array}\right]\left(\frac{e^{cat}-1}{ca}\right). \tag{3.2}$$

Assume that $X_0 = 0$ and $c = 1$, $b = 0$ and $a = \mu$ in (3.1) and (3.2), they become

$$\mathbb{E}(L_t) = \left(\frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2}\right)\left(\frac{e^{\mu t}-1}{\mu}\right) \tag{3.3}$$

and

$$\operatorname{Var}(L_t) = \left[\begin{array}{c} \left(\frac{2\mu+\alpha_1\sigma^2}{2}\right)\left(\frac{2\mu e^{\mu t}+\alpha_1\sigma^2\left(e^{\mu t}-1\right)}{2\mu}+1\right)\left(\frac{\beta_1\rho}{\mu\alpha_1^2} - \frac{\sigma^2\beta_1\rho}{\mu(\alpha_1\sigma^2+2\mu)\alpha_1}\right) \\ +\left(\frac{2\mu+\alpha_2\sigma^2}{2}\right)\left(\frac{2\mu e^{\mu t}+\alpha_2\sigma^2\left(e^{\mu t}-1\right)}{2\mu}+1\right)\left(\frac{\beta_2\rho}{\mu\alpha_2^2} - \frac{\sigma^2\beta_2\rho}{\mu(\alpha_2\sigma^2+2\mu)\alpha_2}\right) \end{array}\right]\left(\frac{e^{\mu t}-1}{\mu}\right), \tag{3.4}$$

which are the mean and variance of the aggregate claim amounts accumulated via a stochastic interest rate up to time t. If we set $\beta_2 = 0$ (i.e. $\beta_1 = 1$) in (3.3) and (3.4), we can easily obtain the mean and variance of the aggregate accumulated claim amounts when the claim size distribution is exponential, i.e. $g(y) = \alpha_1 e^{-\alpha_1 y}$, $y > 0$, $\alpha_1 > 0$,

$$\mathbb{E}(L_t) = \frac{\rho}{\alpha_1}\left(\frac{e^{\mu t}-1}{\mu}\right) \tag{3.5}$$

and

$$\mathrm{Var}(L_t) = \left[\left(\frac{2\mu + \alpha_1\sigma^2}{2}\right)\left(\frac{2\mu e^{\mu t} + \alpha_1\sigma^2\left(e^{\mu t} - 1\right)}{2\mu} + 1\right)\left(\frac{\rho}{\mu\alpha_1^2} - \frac{\sigma^2\rho}{\mu\left(\alpha_1\sigma^2 + 2\mu\right)\alpha_1}\right)\right]\left(\frac{e^{\mu t} - 1}{\mu}\right). \quad (3.6)$$

If we set $\sigma = 0$ and $\mu = \delta$ in (3.3), (3.4) (3.5) and (3.6), we can also obtain the means and variances of the aggregate claim amounts accumulated via a constant risk-free force of interest rate up to time t, i.e.

$$\mathbb{E}(M_t) = \left(\frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2}\right)\left(\frac{e^{\delta t} - 1}{\delta}\right), \quad (3.7)$$

$$\mathrm{Var}(M_t) = \left(\frac{2\beta_1\rho}{\alpha_1^2} + \frac{2\beta_2\rho}{\alpha_2^2}\right)\left(\frac{e^{2\delta t} - 1}{2\delta}\right), \quad (3.8)$$

$$\mathbb{E}(M_t) = \frac{\rho}{\alpha_1}\left(\frac{e^{\delta t} - 1}{\delta}\right) \quad (3.9)$$

and

$$\mathrm{Var}(M_t) = \frac{2\rho}{\alpha_1^2}\left(\frac{e^{2\delta t} - 1}{2\delta}\right), \quad (3.10)$$

that can also be found in Léveillé and Garrido (2001), Jang (2004) and Jang and Krvavych (2004). As expected, if $\mu = \delta$, $\mathbb{E}(L_t) = \mathbb{E}(M_t)$ as $\mathbb{E}\left[\int_0^t \sigma\sqrt{X_s}dB_s\right] = 0$, but $\mathrm{Var}\left(L_t\right) \neq \mathrm{Var}\left(M_t\right)$. If we set $\delta = 0$ in (3.9) and (3.10), then we can obtain

$$\mathbb{E}(C_t) = \frac{\rho}{\alpha_1}t \quad (3.11)$$

and

$$\mathrm{Var}\left(C_t\right) = \frac{2\rho}{\alpha_1^2}t, \quad (3.12)$$

which are the moments of the compound Poisson process with exponential claim sizes.

Now let us illustrate the calculation of the moments of the aggregate accumulated claim amounts up to time t using the expressions above.

Example 3.1

The parameter values used to calculate the moments of the aggregate accumulated claim amounts using (3.3), (3.4), (3.7) and (3.8) are

$$\alpha_1 = 0.01,\ \alpha_2 = 0.009,\ \beta_1 = 0.7,\ \beta_2 = 0.3,\ \mu = \delta = 0.05,\ \rho = 50 \text{ and } t = 1$$

and the mean of the the aggregate accumulated claim amounts is given by

$$\mathbb{E}(L_t) = \mathbb{E}(M_t) = 5,298.$$

The calculations of variances of the aggregate accumulated claim amounts and their differences are shown in Table 3.1 by changing the values of diffusion coefficient of σ.

Table 3.1.

σ	$\mathrm{Var}(L_t)$	$\mathrm{Var}(M_t)$	$\mathrm{Var}(L_t) - \mathrm{Var}(M_t)$
0.0	1,125,700	1,125,700	0
0.5	1,126,400	1,125,700	700
0.6	1,126,700	1,125,700	1,000
0.7	1,127,000	1,125,700	1,300
0.8	1,127,500	1,125,700	1,800
0.9	1,127,900	1,125,700	2,200
1.0	1,128,400	1,125,700	2,700

As long as $\mu = \delta$, the insurance companies have the same mean of the the aggregate accumulated claim amounts even if they consider a stochastic interest rate to the aggregate claim amounts, which is assumed to be (1.6), as $\mathbb{E}\left[\int_0^t \sigma\sqrt{X_s}dB_s\right] = 0$. However the higher the value of diffusion coefficient of σ is, the higher the variance of the aggregate accumulated claim amounts is. Therefore if insurance companies employ mean-variance principle (Bühlmann, 1970, Gerber, 1979 and Goovaert et al., 1984) for their premium calculations, they become higher than those calculated using a deterministic interest rate δ and it is necessary for insurance companies to charge higher premiums when the interest rate expected to be more volatile than as usual.

4. Generalised Lévy processes in finance

Assuming that the underlying asset price follows a Lévy process of X_t, let us consider pricing on the European call option contract. The call option premium at present time 0, is given by

$$\mathbb{E}\left\{(X_t - K)^+ \mid X_0\right\}$$

and it can be expressed as

$$\begin{aligned} \mathbb{E}\left\{(X_t - K)^+\right\} &= \int_K^\infty (x-K)\, dF_{X_t|X_0}(x) \\ &= \mathbb{E}\left\{X_t\, I\left(X_t > K \mid X_0\right)\right\} - K\mathbb{E}\left\{I\left(X_t > K \mid X_0\right)\right\} \end{aligned} \tag{4.1}$$

where K is an exercise price, $I(\cdot)$ is the indicator function and we assume interest rates to be constant.

As it is not possible for us to obtain the distribution of X_t explicitly we employ transform analysis techniques developed by Heston (1993) and Duffie et al. (2000) to calculate the call option premium numerically. We highlight their methodology as applied to our problem below.

We know from section 2 the Laplace transform $\phi(-\nu)$ of X_t given X_0

$$\phi(-\nu) = \mathbb{E}\left(e^{-\nu X_t}|X_0\right)$$

and can consider the function

$$\widehat{\Xi}(w) = \int_{-\infty}^\infty e^{iwx} d\left(\int_0^x dF_{X_t|X_0}(u)\right)$$

and hence

$$\begin{aligned} \widehat{\Xi}(w) &= \int_{-\infty}^\infty e^{iwx} dF_{X_t|X_0}(x) \\ &= \mathbb{E}\left(e^{iwX_t} \mid X_0\right) = \phi(iw) \end{aligned}$$

and the standard Lévy inversion formula gives

$$\begin{aligned} \mathbb{E}\left\{I\left(X_t < K \mid X_0\right)\right\} &= \Pr\left(X_t < K \mid X_0\right) = F(K \mid X_0) \\ &= \frac{\widehat{\Xi}(0)}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{w}\,\mathrm{Im}\left(e^{-iwl}\widehat{\Xi}(w)\right) dw \end{aligned} \tag{4.2}$$

from which we can easily obtain $\Pr\left(X_t \geq K \mid X_0\right)$.

Consider now another function

$$\widehat{\Theta}(w) = \int_{-\infty}^\infty e^{iwx} d\left(\int_0^x u dF_{X_t|X_0}(u)\right).$$

Assume $\int |\Theta(w)|\, dw < \infty$ and we find that

$$\begin{aligned} \widehat{\Theta}(w) &= \int_{-\infty}^\infty e^{iwx}\ x\ dF_{X_t|X_0}(x) \\ &= E\left(X_t\ e^{iwX_t} \mid X_0\right), \end{aligned}$$

which can be calculated as follows. Differentiating $\xi(-\nu)$ with respect to $-\nu$ gives

$$
\begin{aligned}
&-\frac{\partial}{\partial \nu}\xi(-\nu) = \mathbb{E}\left(X_t\ e^{-\nu X_t} \mid X_0\right) \\
= \ & \exp\left(-\frac{2ca\nu e^{cat}}{\sigma^2\nu\left(e^{cat}-1\right)+2ca}X_0\right)\left(\frac{2ca}{\sigma^2\nu\left(e^{cat}-1\right)+2ca}\right)^{2c\frac{b}{\sigma^2}} \\
& \left(\frac{2ca\left(\alpha_1+\nu e^{cat}\right)-\alpha_1\sigma^2\nu\left(1-e^{cat}\right)}{2ca\left(\alpha_1+\nu\right)}\right)^{-\beta_1\frac{\rho}{ca}}\left(\frac{2ca\left(\alpha_1+\nu e^{cat}\right)+\alpha_1\sigma^2\nu\left(e^{cat}-1\right)}{2ca\left(\alpha_1+\nu\right)}\right)^{\sigma^2\alpha_1\frac{\beta_1}{\alpha_1\sigma^2+2ca}\frac{\rho}{ca}} \\
& \left(\frac{2ca\left(\alpha_2+\nu e^{cat}\right)-\alpha_2\sigma^2\nu\left(1-e^{cat}\right)}{2ca\left(\alpha_2+\nu\right)}\right)^{-\beta_2\frac{\rho}{ca}}\left(\frac{2ca\left(\alpha_2+\nu e^{cat}\right)+\alpha_2\sigma^2\nu\left(e^{cat}-1\right)}{2ca\left(\alpha_2+\nu\right)}\right)^{\sigma^2\alpha_2\frac{\beta_2}{\alpha_2\sigma^2+2ca}\frac{\rho}{ca}} \\
& \left[\begin{array}{c}
\frac{2cae^{cat}}{\sigma^2\nu(e^{cat}-1)+2ca}\left(1-\frac{\nu\left(e^{cat}-1\right)}{\sigma^2\nu(e^{cat}-1)+2ca}\sigma^2\right)X_0+\frac{2bc}{\sigma^2\nu(e^{cat}-1)+2ca}\left(e^{cat}-1\right) \\
+\frac{\beta_1\rho}{ca}\left(\frac{2cae^{cat}-\alpha_1\sigma^2\left(1-e^{cat}\right)}{2ca(\alpha_1+\nu e^{cat})-\alpha_1\sigma^2\nu(1-e^{cat})}-\frac{1}{(\alpha_1+\nu)}\right)-\frac{\sigma^2\alpha_1\beta_1\rho}{ca(\alpha_1\sigma^2+2ca)}\left(\frac{2cae^{cat}+\alpha_1\sigma^2\left(e^{cat}-1\right)}{2ca(\alpha_1+\nu e^{cat})+\alpha_1\sigma^2\nu(e^{cat}-1)}-\frac{1}{(\alpha_1+\nu)}\right) \\
+\frac{\beta_2\rho}{ca}\left(\frac{2cae^{cat}-\alpha_2\sigma^2\left(1-e^{cat}\right)}{2ca(\alpha_2+\nu e^{cat})-\alpha_2\sigma^2\nu(1-e^{cat})}-\frac{1}{(\alpha_2+\nu)}\right)-\frac{\sigma^2\alpha_2\beta_2\rho}{ca(\alpha_2\sigma^2+2ca)}\left(\frac{2cae^{cat}+\alpha_2\sigma^2\left(e^{cat}-1\right)}{2ca(\alpha_2+\nu e^{cat})+\alpha_2\sigma^2\nu(e^{cat}-1)}-\frac{1}{(\alpha_2+\nu)}\right)
\end{array}\right] \\
= \ & \eta(-\nu) \qquad (4.3)
\end{aligned}
$$

and hence

$$\widehat{\Theta}(w) = \eta(iw).$$

Since we now have a closed form formula for $\widehat{\Theta}(w)$, the inversion lemma gives

$$
\begin{aligned}
\mathbb{E}\left[X_t\ I\{X_t < K\} \mid X_0\right] &= \Theta(w) \\
&= \frac{\widehat{\Theta}(0)}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{w}\,\mathrm{Im}\left(e^{-iwx}\widehat{\Theta}(w)\right)dw,
\end{aligned}
$$

with

$$\widehat{\Theta}(0) = \eta(0) = \mathbb{E}(X_t \mid X_0),$$

which allows us to calculate the first term of (4.1) as

$$\mathbb{E}\{X_t\ I(X_t\ >\ K) \mid X_0\} = \mathbb{E}(X_t \mid X_0) - \mathbb{E}\{X_t\ I(X_t\ \le\ K) \mid X_0\}. \qquad (4.4)$$

If set $\rho = 0$, $c = 1$ and $b = 0$ in (3.1) and (3.2), it becomes the mean and variance of a square root process, i.e.

$$\mathbb{E}(S_t \mid S_0) = e^{at}S_0, \qquad (4.5)$$

$$\mathrm{Var}(S_t \mid S_0) = \sigma^2 e^{at} S_0\left(\frac{e^{at}-1}{a}\right). \qquad (4.6)$$

Now let us now illustrate the calculations of call option premiums using the methods derived above.

Example 4.1

The parameter values used to calculate the European call option premiums are

$$X_0 = 100,\ \alpha_1 = 0.2,\ \alpha_2 = 0.25,\ \beta_1 = 0.7,\ \beta_2 = 0.3,\ c = 1,\ a = 0.05,\ \rho = 4,\ \sigma = 0.8,\ b = 0.025 \text{ and } t = 1$$

and

$$\mathbb{E}(X_t) = e^{cat}X_0 + \left(\frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2} + cb\right)\left(\frac{e^{cat}-1}{ca}\right) = 123.43,$$

where its counterpart, assuming there are no jumps, is given by

$$e^{cat}X_0 + cb\left(\frac{e^{cat}-1}{ca}\right) = 105.15$$

and if $S_0 = X_0$,

$$\mathbb{E}(S_t \mid S_0) = e^{at} S_0 = 105.13.$$

Using *Matlab*, the call option premiums at each exercise price K with/without jumps are shown in Table 4.1.

Table 4.1.

Exercise price K	Call option premium with jumps	Call option premium without jumps
0	124.4302	105.1524
50	74.4305	55.1526
100	24.5848	6.4738
105.1524	19.7625	3.3134
124.4302	6.3663	0.0396
130	4.2113	0.0060

Having considered upward jumps only in X_t, we have higher call option premiums than those calculated assuming that there are no jumps in the underlying asset price.

For another application to finance, let us consider a zero-coupon bond paying \$100 at time t. Let $c = 1$ and $a < 0$ in (1.1) then we have another process, denoted by V_t,

$$dV_t = (b - aV_t)dt + \sigma\sqrt{V_t}dB_t + dC_t. \tag{4.7}$$

So from (2.12) the Laplace transform of the distribution of V_t is given by

$$\begin{aligned}
& \mathbb{E}\left(e^{-\nu V_t}|V_0\right) \\
= \; & \exp\left[-\left\{\frac{2a\nu e^{-at}}{\sigma^2\nu\left(1-e^{-at}\right)+2a}\right\}V_0\right] \\
& \times\left\{\frac{2a\left(\alpha_1+\nu e^{-at}\right)+\alpha_1\sigma^2\nu\left(1-e^{-at}\right)}{2a\left(\alpha_1+\nu\right)}\right\}^{\beta_1\frac{\rho}{a}} \\
& \times\left\{\frac{2a\left(\nu e^{-at}+\alpha_1\right)+\alpha_1\sigma^2\nu\left(1-e^{-at}\right)}{2a\left(\alpha_1+\nu\right)}\right\}^{-\sigma^2\left(\frac{\alpha_1\beta_1}{\alpha_1\sigma^2-2a}\right)\frac{\rho}{a}} \\
& \times\left\{\frac{2a\left(\alpha_2+\nu e^{-at}\right)+\alpha_2\sigma^2\nu\left(1-e^{-at}\right)}{2a\left(\alpha_2+\nu\right)}\right\}^{\beta_2\frac{\rho}{a}} \\
& \times\left\{\frac{2a\left(\nu e^{-at}+\alpha_2\right)+\alpha_2\sigma^2\nu\left(1-e^{-at}\right)}{2a\left(\alpha_2+\nu\right)}\right\}^{-\sigma^2\left(\frac{\alpha_2\beta_2}{\alpha_2\sigma^2-2a}\right)\frac{\rho}{a}} \\
& \times\left[\frac{2a}{\sigma^2\nu\left(1-e^{-at}\right)+2a}\right]^{\frac{2b}{\sigma^2}},
\end{aligned} \tag{4.8}$$

and from (3.1) and (3.2), we have its moments, i.e.

$$\mathbb{E}(V_t \mid V_0) = e^{-at}V_0 + \left(\frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2} + b\right)\left(\frac{1-e^{-at}}{a}\right) \tag{4.9}$$

and

$$\mathrm{Var}(V_t \mid V_0) = \left[\begin{array}{c} \sigma^2 e^{-at}V_0 + \frac{b}{2}\left(\frac{1-e^{-at}}{a}\right) \\ +\left(\frac{2a-\alpha_1\sigma^2}{2}\right)\left(1+\frac{\alpha_1\sigma^2\left(1-e^{-at}\right)+2ae^{-at}}{2a}\right)\left(\frac{\beta_1\rho}{a\alpha_1^2}-\frac{\sigma^2\beta_1\rho}{a(\alpha_1\sigma^2-2a)\alpha_1}\right) \\ +\left(\frac{2a-\alpha_2\sigma^2}{2}\right)\left(1+\frac{\alpha_2\sigma^2\left(1-e^{-at}\right)+2ae^{-at}}{2a}\right)\left(\frac{\beta_2\rho}{a\alpha_2^2}-\frac{\sigma^2\beta_2\rho}{a(\alpha_2\sigma^2-2a)\alpha_2}\right) \end{array}\right]\left(\frac{1-e^{-at}}{a}\right). \tag{4.10}$$

If we set $\sigma = 0$, $b = 0$ and $a = \delta$ in (4.8), (4.9) and (4.10), we can easily obtain the Laplace transform of the distribution of λ_t and its moments, i.e.

$$\mathbb{E}\left(e^{-\nu\lambda_t}|\lambda_0\right) = \exp\left(-\nu e^{-\delta t}\lambda_0\right)\left(\frac{\alpha_1 + \nu e^{-\delta t}}{\alpha_1 + \nu}\right)^{\frac{\beta_1\rho}{\delta}}\left(\frac{\alpha_2 + \nu e^{-\delta t}}{\alpha_2 + \nu}\right)^{\frac{\beta_2\rho}{\delta}}, \tag{4.11}$$

$$\mathbb{E}(\lambda_t \mid \lambda_0) = e^{-\delta t}\lambda_0 + \left(\frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2}\right)\left(\frac{1 - e^{-\delta t}}{\delta}\right) \tag{4.12}$$

and

$$\mathrm{Var}(\lambda_t \mid \lambda_0) = \left(\frac{2\beta_1\rho}{\alpha_1^2} + \frac{2\beta_2\rho}{\alpha_2^2}\right)\left(\frac{1 - e^{-2\delta t}}{2\delta}\right). \tag{4.13}$$

that can also be found in Jang (2004) and Jang and Krvavych (2004). If set $\rho = 0$ in (4.9) and (4.10), they become the mean and variance of the celebrated Cox-Ingersoll-Ross (1985) model for interest rate, i.e.

$$\mathbb{E}(r_t \mid r_0) = \frac{b}{a} + \left(r_0 - \frac{b}{a}\right)e^{-at} \tag{4.14}$$

and

$$\mathrm{Var}(r_t \mid r_0) = \sigma^2\left\{e^{-at}r_0 + \frac{b}{2}\left(\frac{1 - e^{-at}}{a}\right)\right\}\left(\frac{1 - e^{-at}}{a}\right). \tag{4.15}$$

If we define $W_t = \int\limits_0^t V_s ds$, then from (2.12) the Laplace transform of the distribution of W_t is given by

$$\begin{aligned} &\mathbb{E}\left(e^{-\xi W_t}|V_0\right) \\ = \;& \exp\left[-\left\{B_3(t)\right\}V_0\right] \times e^{-\rho t} \times \left\{C_3(t)\right\}^{\frac{2b}{\sigma^2}} \\ & \times \left\{J_3(t)\right\}^{\frac{\left(\sqrt{a^2+2\sigma^2\xi}+a\right)\alpha_1\beta_1}{\alpha_1\left(\sqrt{a^2+2\sigma^2\xi}+a\right)+2\xi}\frac{\rho}{\sqrt{a^2+2\sigma^2\xi}}} \times \left\{K_3(t)\right\}^{\frac{\left(\sqrt{a^2+2\sigma^2\xi}-a\right)\alpha_1\beta_1}{2\xi-\alpha_1\left(\sqrt{a^2+2\sigma^2\xi}-a\right)}\frac{\rho}{\sqrt{a^2+2\sigma^2\xi}}} \\ & \times \left\{\Gamma_3(t)\right\}^{\frac{\left(\sqrt{a^2+2\sigma^2\xi}+a\right)\alpha_2\beta_2}{\alpha_2\left(\sqrt{a^2+2\sigma^2\xi}+a\right)+2\xi}\frac{\rho}{\sqrt{a^2+2\sigma^2\xi}}} \times \left\{\Pi_3(t)\right\}^{\frac{\left(\sqrt{a^2+2\sigma^2\xi}-a\right)\alpha_2\beta_2}{2\xi-\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}-a\right)}\frac{\rho}{\sqrt{a^2+2\sigma^2\xi}}}, \end{aligned} \tag{4.16}$$

where

$$B_3(t) = \frac{2\xi\left\{1 - \exp\left(-\sqrt{a^2+2\sigma^2\xi}t\right)\right\}}{\left\{\left(\sqrt{a^2+2\sigma^2\xi}+a\right) + \left(\sqrt{a^2+2\sigma^2\xi}-a\right)\exp\left(-\sqrt{a^2+2\sigma^2\xi}t\right)\right\}},$$

$$J_3(t) = \frac{\alpha_1\left(\sqrt{a^2+2\sigma^2\xi}+a\right)e^{\sqrt{a^2+2\sigma^2\xi}t} + \alpha_1\left(\sqrt{a^2+2\sigma^2\xi}-a\right) + 2\xi\left(e^{\sqrt{a^2+2\sigma^2\xi}t} - 1\right)}{2\alpha_1\sqrt{a^2+2\sigma^2\xi}},$$

$$K_3(t) = \frac{\alpha_1\left(\sqrt{a^2+2\sigma^2\xi}-a\right)e^{-\sqrt{a^2+2\sigma^2\xi}t} + \alpha_1\left(\sqrt{a^2+2\sigma^2\xi}+a\right) - 2\xi\left(e^{-\sqrt{a^2+2\sigma^2\xi}t} - 1\right)}{2\alpha_1\sqrt{a^2+2\sigma^2\xi}},$$

$$\Gamma_3(t) = \frac{\alpha_2\left(\sqrt{a^2+2\sigma^2\xi}+a\right)e^{\sqrt{a^2+2\sigma^2\xi}t} + \alpha_2\left(\sqrt{a^2+2\sigma^2\xi}-a\right) + 2\xi\left(e^{\sqrt{a^2+2\sigma^2\xi}t} - 1\right)}{2\alpha_2\sqrt{a^2+2\sigma^2\xi}},$$

$$\Pi_3(t) = \frac{\alpha_2\left(\sqrt{a^2+2\sigma^2\xi}-a\right)e^{-\sqrt{a^2+2\sigma^2\xi}t} + \alpha_2\left(\sqrt{a^2+2\sigma^2\xi}+a\right) - 2\xi\left(e^{-\sqrt{a^2+2\sigma^2\xi}t} - 1\right)}{2\alpha_2\sqrt{a^2+2\sigma^2\xi}},$$

$$C_3(t) = \frac{2\sqrt{a^2+2\sigma^2\xi}\exp\left(-\frac{\left(\sqrt{a^2+2\sigma^2\xi}-a\right)}{2}t\right)}{\left(\sqrt{a^2+2\sigma^2\xi}+a\right)+\left(\sqrt{a^2+2\sigma^2\xi}-a\right)\exp\left(-\sqrt{a^2+2\sigma^2\xi}t\right)}.$$

Setting $\xi = 1$ in (4.16), we can easily calculate default-free zero-coupon bond price paying 100 at time t, i.e.

$$B(t,0) = \mathbb{E}[\$100 \times e^{-\int_0^t V_s ds} | V_0] \tag{4.17}$$

where $B(t,0)$ denotes the present value of 100 at time 0.

Now now illustrate the calculations of default-free zero-coupon bond prices using the methods derived above.

Example 4.2

The parameter values used to calculate the price of default-free zero-coupon bond using (4.17) are

$$V_0 = 0.05,\ \alpha_1 = 200,\ \alpha_2 = 250,\ \beta_1 = 0.7,\ \beta_2 = 0.3,\ a = \delta = 0.05,\ \rho = 3,\ \sigma = 0.8,\ b = 0.025 \text{ and } t = 1$$

and its price is given by

$$B(1,0) = \mathbb{E}[100 \times e^{-\int_0^1 V_s ds} | V_0] = 93.937,$$

where its counterpart using a deterministic interest rate δ is given by

$$100 \times e^{-0.05} = 95.123.$$

The calculations of prices of default-free zero-coupon bond by changing the values of jump frequency rate ρ are shown in Table 4.2.

Table 4.2.

ρ	$B(1,0)$
0	94.557 (CIR case)
1	94.350
2	94.143
3	93.937
4	93.731
5	93.526
10	92.506

The calculations of prices of default-free zero-coupon bond by changing the values of the magnitude of jump sizes, i.e. α_1 and α_2 are shown in Table 4.3.

Table 4.3.

α_1	α_2	$B(1,0)$
2	2.5	56.486
20	25	88.682
200	250	93.937
2000	2500	94.495

The calculations of prices of default-free zero-coupon bond by changing the values of diffusion coefficient of σ with are shown in Table 4.4.

Table 4.4.

σ	$B(1,0)$
0.01	93.428
0.1	93.437
0.5	93.640
0.8	93.937
10	98.806
∞	100

Having considered upward jumps only in V_t, we are expecting higher interest rate as time goes by. Therefore it is more attractive for investors to leave an amount of money e.g. in a savings account rather than purchasing a bond that pays guaranteed 100, regardless of the rate of interest at time t. So the higher ρ is, the lower the default-free zero-coupon bond price is (see Table 4.2). Also the bigger the magnitude of positive jump is, less attractive purchasing a bond that pays guaranteed 100 is. So the smaller α_1 and α_2 are, the lower the default-free zero-coupon bond price is (see Table 4.3). Same result as CIR case, the more volatile the interest rate, that means more uncertainty for future consumption is, more attractive purchasing a bond that pays guaranteed 100 is. So the higher σ is, the higher the default-free zero-coupon bond price is (see Table 4.4).

5. Conclusion

We derived the Laplace transform of the distribution of a generalised Lévy process applying the piecewise deterministic Markov processes theory and the martingale. In order to obtain its explicit expression, we assumed that the jump arrival process follows the Poisson process and jump sizes are assumed to be mixture of exponential. As insurance applications of a generalised Lévy process, we derived the explicit formulae of the means and variances of aggregate accumulated claims of insurance risk. As financial applications, we presented how it can be used to price default-free zero coupon bond and European call option with Transform Analysis technique.

It is of interest to obtain the Laplace transform of the distribution of a generalised Lévy process that has upward and downward jumps as we did not consider downward jumps in this paper. We can also easily replace the Poisson process with the Cox process as long as being independent holds between state variables and the arrival of jumps. We did not employ risk-neutral approach to price default-free zero coupon bond and European call option in this paper so for that purpose the Esscher transform (Gerber and Shiu, 1996) or Girsanov theorem (Karatzas and Shreve, 1991) can be considered.

References

Asmussen, S. (2000) : Ruin Probabilities, World Scientific, Singapore.

Barndorff-Nielsen, O.E., Mikosch, T. and Resnick, S.I. (2001) : Lévy Processes: Theory and Applications Birkhäuser, Boston, MA.

Bertoin, J. (1998) : Lévy processes, Cambridge University Press, UK.

Bühlmann, H. (1970) : Mathematical Methods in Risk Theory, Springer-Verlag, Berlin-Heidelberg.

Cont, R. and Tankov, P. (2004) : Financial Modelling With Jump Processes, Champan & Hall, Boca Raton, FL.

Cox, J. Ingersoll, J. and Ross, S. (1985) : A theory of the term structure of interest rates, Econometrica, 53/2, 385-407.

Davis, M. H. A. (1984) : Piecewise deterministic Markov processes: A general class of non diffusion stochastic models, J. R. Stat. Soc. B, 46, 353-388.

Delbaen, F., and Haezendonck, J. (1987) : Classical Risk Theory in an Economic Environment, Insurance: Mathematics and Economics, 6: 85-116.

Duffie, D., J. Pan, and K.Singleton. (2000) : Transform analysis and asset pricing for affine jump-diffusions, Econometrica, 68, 1343-1376.

Gerber, H. U. (1979) : An Introduction to Mathematical Risk Theory, S. S. Huebner Foundation for Insurance Education, Philadelphia.

Gerber, H. U. and Shiu, E. S. W. (1996) : Actuarial bridges to dynamic hedging and option pricing, Insurance: Mathematics and Economics, 18, 183-218.

Goovaerts, M. J., De Vylder F. and Haezendonck, J. (1984) : Insurance Premiums, North-Holland, Amsterdam.

Heston, S. (1993): A closed form solution for options with stochastic volatility with applications to bond and currency options, Review of Financial Studies, 6, 327-344.

Jang, J. (2004) : Martingale approach for moments of discounted aggregate claims, Journal of Risk and Insurance, 71/2, 201-211.

Jang, J. and Krvavych, Y. (2004); Arbitrage-free premium calculation for extreme losses using the shot noise process and the Esscher transform, Insurance: Mathematics & Economics, 35/1, 97-111.

Karatzas, I. and Shreve, S. (1991) : Brownian Motion and Stochastic Calculus, Springer, New York.

Léveillé, G. and Garrido, J. (2001) : Moments of Compound Renewal Sums with Discounted Claims, Insurance: Mathematics and Economics, 28: 217-231.

Paulsen, J. (1998) : Ruin Theory with Compounding Assets: A Survey. Insurance: Mathematics and Economics, 22: 3-16.

Sato, K. (1999) : Lévy processes and Infinitely Divisible Distributions, Cambridge University Press, UK.

Taylor, G. (1979) : Probability of ruin under inflationary conditions or under experience rating, ASTIN Bulletin, 10, 149-162.

Willmot, G. E. (1989) : The total claims distribution under inflationary conditions, Scandinavian Actuarial Journal, 1-12.