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# Martingale Approach for Moments of Discounted Aggregate Claims

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## Abstract

We examine the Laplace transform of the distribution of the shot noise process using the martingale. Applying the piecewise deterministic Markov processes theory and using the relationship between the shot noise process and the accumulated/discounted aggregate claims process, the Laplace transform of the distribution of the accumulated aggregate claims is obtained. Assuming that the claim arrival process follows the Poisson process and claim sizes are assumed to be exponential and mixture of exponential, we derive the explicit expressions of the actuarial net premiums and variances of the discounted aggregate claims, which are the annuities paid continuously. Numerical examples are also provided based on them.


## Introduction

Let $X_i$, $i = 1, 2, \ldots$, be the claim amount, which are assumed to be independent and identically distributed with distribution function $H(x)$ $(x > 0)$. The accumulated value of aggregate claims up to time $t$, $L_t$ is given by

$$L_t = X_1 e^{\delta(t-s_1)} + X_2 e^{\delta(t-s_2)} + \cdots + X_{N_t-1} e^{\delta(t-s_{N_t-1})} + X_{N_t} e^{\delta(t-s_{N_t})}, \tag{1}$$

where $\delta$ is the instantaneous rate of net interest, $s_i$'s are time points at which claims occur $(s_i < t < \infty)$ and $N_t$ is the number of claims up to time $t$. If we multiply $e^{-\delta t}$ both sides in Equation (1), the discounted value of aggregate claims up to time $t$, $L_t^0$ is given by

$$L_t^0 = X_1 e^{-\delta s_1} + X_2 e^{-\delta s_2} + \cdots + X_{N_t-1} e^{-\delta s_{N_t-1}} + X_{N_t} e^{-\delta s_{N_t}}, \tag{2}$$

where $L_t^0 = e^{-\delta t} L_t$.

If we ignore the effect of the rate of net interest $\delta$, considering the claim inflation experienced cancels out interest earned, Equations (1) and (2) are equivalent to the

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classical risk model (Bühlmann, 1970; Gerber, 1979; Beard et al., 1984). However, in practice, the instantaneous rate of interest might be more variable than the claims themselves, and inflation does not merely cancel the interest earned. Hence, it is of interest to obtain the distribution of the accumulated aggregate claims up to time $t$, $L_t$.

Unfortunately, it is known that it is not possible for us to obtain the distribution of the accumulated/discounted aggregate claims explicitly. Dufresne (1990) and Milevsky (1997) proved that the continuous temporary annuity has an inverted Gamma distribution when the time horizon goes to infinity using a stochastic Weiner rate of interest. Goovaerts et al. (2000) also presented a computable approximation for the distribution function of the present value of a sequence of cash flows that are discounted using a stochastic return process. Therefore in this article, applying the piecewise deterministic Markov processes theory and using the relationship between the shot noise process and accumulated/discounted aggregate claims process, we find the Laplace transform of the distribution of the accumulated aggregate claims.

Assuming that the claim arrival process follows the Poisson process and claim sizes are assumed to be exponential and mixture of exponential, we also obtain the explicit expressions of the mean (i.e., the actuarial net premium) and variance of the discounted aggregate claims, i.e., $E(L_t^0)$ and $Var(L_t^0)$. Interestingly, we witness that they can be expressed in terms of an annuity paid continuously. Numerical examples are also provided based on the explicit expressions of the actuarial net premiums and variances of the discounted aggregate claims.

## SHOT NOISE PROCESS AND ITS GENERATOR

The shot noise process can be used in many diverse fields. In particular, it attracts us as it can be applied in financial and insurance field. The shot noise process is particularly useful as it measures the frequency, magnitude, and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases until another event occurs which will result in a positive jump in the shot noise process. We will adopt the shot noise process used by Cox and Isham (1980):

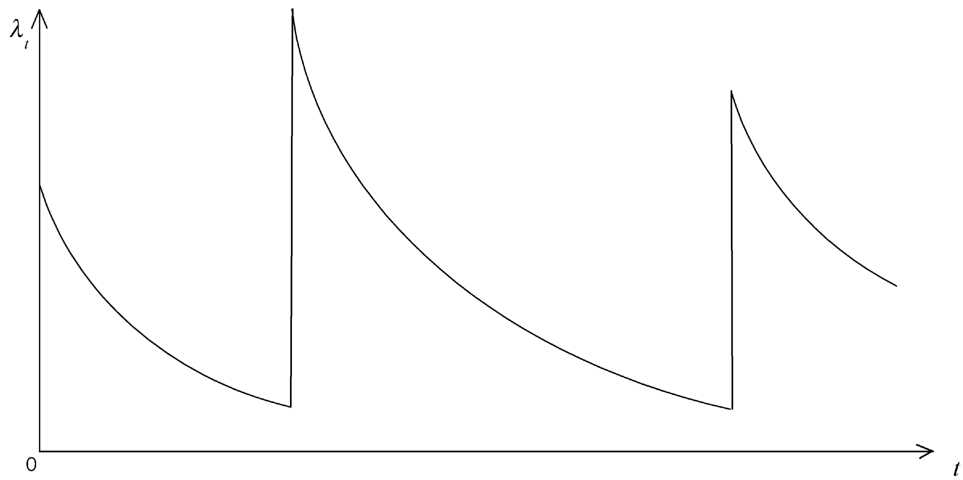
$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i < t}} y_i e^{-\delta(t-s_i)}, \tag{3}$$

where, $\lambda_0$ is the initial value of $\lambda$, $y_i$ is the jump size of primary event $i$, where $E(y_i) < \infty$, $s_i$ is the time at which primary event $i$ occurs, where $s_i < t < \infty$, $\delta$ is the exponential decay, and $\rho$ is the rate of primary event arrival. This is illustrated in Figure 1.

Some works of insurance application using shot noise process can be found in Dassios (1987), Klüppelberg and Mikosch (1995), Jang (1998), and Dassios and Jang (2003).

The piecewise deterministic Markov processes theory developed by Davis (1984) is a powerful mathematical tool for examining nondiffusion models. The shot noise process is an example of piecewise deterministic Markov process. Therefore we can present definitions and important properties of the shot noise process with the aid of this theory (Dassios, 1987; Dassios and Embrechts, 1989; Rolski et al.,

**FIGURE 1**
Graph Illustrating a Shot Noise Process




1998). It is used to derive the Laplace transform of the distribution of the shot noise process.

The three parameters of the shot noise process described are homogeneous in time. The decay is exponential $\delta$, which is a constant, can never reach 0 and the frequency of jump arrivals follows a Poisson distribution with $\rho$. We will have generally distributed jump sizes with density function $g(y)$ and distribution function $G(y)(y > 0)$. We can also generalize the shot noise process by allowing three parameters to depend on time (Dassios, 1987; Jang, 1998; Dassios and Jang, 2003). However, throughout the article, for simplicity, we assume the parameters to be homogeneous in time.

The generator of the process $(\lambda_t, t)$ acting on a function $f(\lambda, t)$ belonging to its domain is given by

$$Af(\lambda, t) = \frac{\partial f}{\partial t} - \delta\lambda \frac{\partial f}{\partial \lambda} + \rho \left\{ \int_0^\infty f(\lambda + y, t)\, dG(y) - f(\lambda, t) \right\}, \tag{4}$$

where $f : (0, \infty) \times \mathbb{R}^+ \to (0, \infty)$. It is sufficient that $f(\lambda, t)$ is differentiable w.r.t. $\lambda$, $t$ for all $\lambda$, $t$ and that $|\int_0^\infty f(\lambda + y, t)\, dG(y) - f(\lambda, t)| < \infty$ for $f(\lambda, t)$ to belong to the domain of the generator $A$.

Now let us find a suitable martingale in order to derive the Laplace transform of the distribution of $\lambda_t$ at time $t$, considering a constant such that $\nu \geq 0$. From Equation (4), $f(\lambda, t)$ has to satisfy $Af = 0$ for it to be a martingale. Setting $f(\lambda, t) = e^{-A(t)\lambda} e^{B(t)}$, we get the equation

$$-\lambda A'(t) + B'(t) + \delta\lambda A(t) + \rho[\hat{g}\{A(t)\} - 1] = 0 \tag{5}$$

and solving it we obtain

$$A(t) = \nu e^{\delta t} \quad \text{and} \quad B(t) = \left[\rho \int_0^t \{1 - \hat{g}(\nu e^{\delta s})\}\, ds\right].$$

Hence,

$$\exp\left(-\nu \lambda_t e^{\delta t}\right) \exp\left[\rho \int_0^t \{1 - \hat{g}(\nu e^{\delta s})\}\, ds\right] \tag{6}$$

is a martingale where $\hat{g}(\nu) = \int_0^\infty e^{-\nu y}\, dG(y)$.

## THE LAPLACE TRANSFORM OF THE DISTRIBUTION OF THE SHOT NOISE PROCESS

We can easily obtain the Laplace transform of the distribution of $\lambda_t$ at time $t$ using the martingale found above. From Equation (6), for a fixed time $t^*$ and a fixed constant $\nu^* \geq 0$, we have

$$E\left[\exp\left\{-\nu^* \lambda_{t^*} e^{\delta t^*}\right\} \mid \lambda_0\right] = \exp\left(-\nu^* \lambda_0\right) \exp\left[-\rho \int_0^{t^*} \{1 - \hat{g}(\nu^* e^{\delta s})\}\, ds\right]$$

and setting $\nu^* = \nu e^{-\delta t^*}$ we also have

$$E\left\{e^{-\nu \lambda_{t^*}} \mid \lambda_0\right\} = \exp\left\{-\nu \lambda_0 e^{-\delta t^*}\right\} \exp\left[-\rho \int_0^{t^*} \{1 - \hat{g}(\nu e^{-\delta t^*} e^{\delta s})\}\, ds\right]. \tag{7}$$

Since Equation (7) holds for an arbitrary fixed $t^*$, it holds for all $t \geq 0$ and the Laplace transform of the distribution of $\lambda_t$ at time $t$ is given by

$$E\left\{e^{-\nu \lambda_t} \mid \lambda_0\right\} = \exp\left(-\nu \lambda_0 e^{-\delta t}\right) \exp\left[-\rho \int_0^t \{1 - \hat{g}(\nu e^{-\delta s})\}\, ds\right], \tag{8}$$

where $\nu \geq 0$. If $t \to \infty$, the Laplace transform of the asymptotic (stationary) distribution of $\lambda_t$ is given by $\exp[-\rho \int_0^\infty \{1 - \hat{g}(\nu e^{-\delta s})\}\, ds]$.

Now let us examine the Laplace transforms of the distribution of shot noise process with specific jump size distributions of $G(y)(y > 0)$.

Firstly, let us obtain the Laplace transform of the distribution of $\lambda_t$ if the jump size distribution is the mixture of two exponential, i.e., $g(y) = \beta_1 \alpha_1 e^{-\alpha_1 y} + \beta_2 \alpha_2 e^{-\alpha_2 y}$, $y > 0$, $\alpha_1 > \alpha_2 > 0$, and $\beta_1 + \beta_2 = 1$. The Laplace transform of mixture of two exponential is given by

$$\beta_1 \left(\frac{\alpha_1}{\alpha_1 + \nu}\right) + \beta_2 \left(\frac{\alpha_2}{\alpha_2 + \nu}\right).$$

Hence, from Equation (8), the Laplace transform of mixture of two exponential with parameter $\nu e^{-\delta s}$, $\hat{g}(\nu e^{-\delta s})$ is given by

$$\beta_1\left(\frac{\alpha_1}{\alpha_1+\nu e^{-\delta s}}\right)+\beta_2\left(\frac{\alpha_2}{\alpha_2+\nu e^{-\delta s}}\right),$$

$$\begin{aligned}\int_0^t \{1-\hat{g}(\nu e^{-\delta s})\}\,ds &= \int_0^t\left\{1-\beta_1\left(\frac{\alpha_1}{\alpha_1+\nu e^{-\delta s}}\right)-\beta_2\left(\frac{\alpha_2}{\alpha_2+\nu e^{-\delta s}}\right)\right\}ds\\ &= t-\frac{\beta_1}{\delta}\ln\left[\frac{\alpha_1 e^{\delta t}+\nu}{\alpha_1+\nu}\right]-\frac{\beta_2}{\delta}\ln\left[\frac{\alpha_2 e^{\delta t}+\nu}{\alpha_2+\nu}\right]\end{aligned}$$

and

$$\exp\left[-\rho\int_0^t\{1-\hat{g}(\nu e^{-\delta s})\}\,ds\right]=e^{-\rho t}\left(\frac{\alpha_1 e^{\delta t}+\nu}{\alpha_1+\nu}\right)^{\frac{\beta_1\rho}{\delta}}\left(\frac{\alpha_2 e^{\delta t}+\nu}{\alpha_2+\nu}\right)^{\frac{\beta_2\rho}{\delta}}.$$

Therefore, the Laplace transform of the distribution of $\lambda_t$ with the mixture of two exponential is given by

$$E\left\{e^{-\nu\lambda_t}\,\middle|\,\lambda_0\right\}=e^{-\nu\lambda_0 e^{-\delta t}}\left\{\frac{\alpha_1 e^{-\delta\left(\frac{1}{\beta_1}-1\right)t}+\nu e^{-\frac{\delta}{\beta_1}t}}{\alpha_1+\nu}\right\}^{\frac{\beta_1\rho}{\delta}}\left(\frac{\alpha_2 e^{\delta t}+\nu}{\alpha_2+\nu}\right)^{\frac{\beta_2\rho}{\delta}} \tag{9}$$

and its asymptotic (stationary) distribution is given by

$$\left(\frac{\alpha_1}{\alpha_1+\nu}\right)^{\frac{\beta_1\rho}{\delta}}\left(\frac{\alpha_2}{\alpha_2+\nu}\right)^{\frac{\beta_2\rho}{\delta}}, \tag{10}$$

where $\nu \geq 0$.

Secondly, if we set $\beta_2=0$ (i.e., $\beta_1=1$) in Equation (9), we can easily obtain the Laplace transform of the distribution of $\lambda_t$ when the jump size distribution is exponential, i.e., $g(y)=\alpha_1 e^{-\alpha_1 y}$, $y>0$, $\alpha_1>0$,

$$E\left\{e^{-\nu\lambda_t}\,\middle|\,\lambda_0\right\}=e^{-\nu\lambda_0 e^{-\delta t}}\left(\frac{\alpha_1+\nu e^{-\delta t}}{\alpha_1+\nu}\right)^{\frac{\rho}{\delta}} \tag{11}$$

and its asymptotic (stationary) distribution is given by

$$\left(\frac{\alpha_1}{\alpha_1+\nu}\right)^{\frac{\rho}{\delta}}, \tag{12}$$

where $\nu \geq 0$. From Equation (12), we can easily find that shot noise process at a fixed time follows a gamma distribution, i.e.,

$$\frac{\alpha_1^{\frac{\rho}{\delta}} \lambda^{\frac{\rho}{\delta}-1} e^{-\alpha_1 \lambda}}{\Gamma\left(\frac{\rho}{\delta}\right)},$$

and its mean and variance are given by

$$\frac{\rho}{\alpha_1 \delta} \quad \text{and} \quad \frac{\rho}{\alpha_1^2 \delta}. \tag{13}$$

Finally, other distributions such as gamma, log-normal, and Pareto, etc., can also be applied for jump size distribution of $G(y)(y > 0)$ in Equation (8). As an example, if the jump size distribution is gamma, i.e., $g(y) = \frac{\alpha^\beta y^{\beta-1} e^{-\alpha y}}{\Gamma(\beta)}$, $y > 0, \alpha > 0, \beta > 0$, the Laplace transform of the distribution of $\lambda_t$ is given by

$$E\left\{e^{-\nu \lambda_t} \mid \lambda_0\right\} = e^{-\nu \lambda_0 e^{-\delta t}} \exp\left[-\rho \int_0^t \left\{1 - \left(\frac{\alpha}{\alpha + \nu e^{-\delta s}}\right)^\beta\right\} ds\right], \tag{14}$$

where $\nu \geq 0$.

## THE LAPLACE TRANSFORM OF THE DISTRIBUTION OF ACCUMULATED AGGREGATE CLAIMS AND ITS APPLICATION TO PREMIUM CALCULATION

Let us examine how the results we have obtained in the previous section can be applied in deriving the Laplace transform of distribution of accumulated aggregate claims covered by an insurance contract.

If we set $-\delta$ to $\delta$ in Equation (3), it becomes

$$\xi_t = \xi_0 e^{\delta t} + \sum_{\substack{\text{all } i \\ s_i < t}} y_i e^{\delta(t - s_i)}. \tag{15}$$

Interestingly, we can see that it is equivalent to Equation (1) if we substitute $\xi$ with $L$ and $y$ with $X$ in Equation (15), assuming that $\xi_0$, that can be considered as the total claims up to present time 0, is 0. Therefore the decay rate $\delta$ in Equation (3) is the rate of interest in Equation (15).

Similar to the previous section, based on

$$\mathrm{A} f(\xi, t) = \frac{\partial f}{\partial t} + \delta \xi \frac{\partial f}{\partial \xi} + \rho \left\{\int_0^\infty f(\xi + x, t)\, dH(x) - f(\xi, t)\right\},$$

we can easily derive the Laplace transform of the distribution of $\xi_t$ at time $t$, i.e.,

$$E\left\{e^{-\nu \xi_t} \mid \xi_0\right\} = \exp\left(-\nu \xi_0 e^{\delta t}\right) \exp\left[-\rho \int_0^t \{1 - \hat{h}(\nu e^{\delta s})\}\, ds\right]. \tag{16}$$

Assuming that $\xi_0 = 0$, the Laplace transform of the distribution of the accumulated aggregate claims is given by

$$\exp\left[-\rho \int_0^t \{1 - \hat{h}(\nu e^{\delta s})\}\, ds\right]. \tag{17}$$

If the claim size distribution is exponential, i.e., $h(x) = \alpha_1 e^{-\alpha_1 x}, x > 0, \alpha_1 > 0$, we have

$$\left(\frac{\alpha_1 + \nu e^{\delta t}}{\alpha_1 + \nu}\right)^{-\frac{\rho}{\delta}}. \tag{18}$$

If we differentiate Equation (18) with respect to $\nu$ and put $\nu = 0$, we can obtain the mean of the accumulated aggregate claims, i.e.,

$$E(L_t) = \frac{\rho}{\alpha_1 \delta}(e^{\delta t} - 1) = \frac{\rho}{\alpha_1}\left(\frac{e^{\delta t} - 1}{\delta}\right) = \frac{\rho}{\alpha_1}\bar{s}_{\overline{t|}} \quad (\text{at } \delta) \tag{19}$$

and the higher moments can be obtained by differentiating it further, i.e.,

$$\text{var}(L_t) = \frac{2\rho}{2\alpha_1^2 \delta}(e^{2\delta t} - 1) = \frac{2\rho}{\alpha_1^2}\left(\frac{e^{2\delta t} - 1}{2\delta}\right) = \frac{2\rho}{\alpha_1^2}\bar{s}_{\overline{t|}} \quad (\text{at } 2\delta). \tag{20}$$

If we multiply $e^{-\delta t}$ both sides in Equation (19) and $e^{-2\delta t}$ in Equation (20), we can easily obtain the mean of the discounted aggregate claims (i.e., the actuarial net premium),

$$E\left(L_t^0\right) = \frac{\rho}{\alpha_1}\bar{a}_{\overline{t|}} \quad (\text{at } \delta) \tag{21}$$

and the variance of the discounted aggregate claims, i.e.,

$$\text{var}\left(L_t^0\right) = \frac{2\rho}{\alpha_1^2}\bar{a}_{\overline{t|}} \quad (\text{at } 2\delta). \tag{22}$$

From Equation (21), we can find that the actuarial net premium at time 0, $E(L_t^0)$ is equivalent to an annuity paid continuously (Léveillé and Garrido, 2001). If $t \to \infty$, the mean and variance of the discounted aggregate claims are given by

$$\frac{\rho}{\alpha_1 \delta} \quad \text{and} \quad \frac{\rho}{\alpha_1^2 \delta}$$

which are equivalent to Equation (13).

If the claim size distribution is the mixture of two exponential, i.e., $h(x) = \beta_1 \alpha_1 e^{-\alpha_1 x} + \beta_2 \alpha_2 e^{-\alpha_2 x}, x > 0, \alpha_1 > \alpha_2 > 0$ and $\beta_1 + \beta_2 = 1$, the Laplace transform of the distribution of the accumulated aggregate claims is given by

$$\left\{ \frac{\alpha_1 e^{\delta\left(\frac{1}{\beta_1}-1\right)t} + \nu e^{\frac{\delta}{\beta_1}t}}{\alpha_1 + \nu} \right\}^{-\frac{\beta_1 \rho}{\delta}} \left( \frac{\alpha_2 e^{-\delta t} + \nu}{\alpha_2 + \nu} \right)^{-\frac{\beta_2 \rho}{\delta}}. \tag{23}$$

If we differentiate Equation (23) with respect to $\nu$ and put $\nu = 0$, we can obtain the mean of the accumulated aggregate claims, i.e.,

$$E(L_t) = \left( \beta_1 \frac{\rho}{\alpha_1} + \beta_2 \frac{\rho}{\alpha_2} \right) \bar{s}_{\overline{t|}} \quad (\text{at } \delta) \tag{24}$$

and the higher moments can be obtained by differentiating it further, i.e.,

$$\text{var}(L_t) = \left( \beta_1 \frac{2\rho}{\alpha_1^2} + \beta_2 \frac{2\rho}{\alpha_2^2} \right) \bar{s}_{\overline{t|}} \quad (\text{at } 2\delta). \tag{25}$$

If we multiply $e^{-\delta t}$ both sides in Equation (24) and $e^{-2\delta t}$ in Equation (25), we have

$$E\left(L_t^0\right) = \left( \beta_1 \frac{\rho}{\alpha_1} + \beta_2 \frac{\rho}{\alpha_2} \right) \bar{a}_{\overline{t|}} \quad (\text{at } \delta) \tag{26}$$

and

$$\text{var}\left(L_t^0\right) = \left( \beta_1 \frac{2\rho}{\alpha_1^2} + \beta_2 \frac{2\rho}{\alpha_2^2} \right) \bar{a}_{\overline{t|}} \quad (\text{at } 2\delta). \tag{27}$$

From Equation (26), we can find that the actuarial net premium at time 0, $E(L_t^0)$ is equivalent to the mixture of an annuity paid continuously with weights $\beta_1$ and $\beta_2$. If $t \to \infty$, the mean and variance of the discounted aggregate claims are given by

$$\left( \beta_1 \frac{\rho}{\alpha_1} + \beta_2 \frac{\rho}{\alpha_2} \right) \frac{1}{\delta} \quad \text{and} \quad \left( \beta_1 \frac{\rho}{\alpha_1^2} + \beta_2 \frac{\rho}{\alpha_2^2} \right) \frac{1}{\delta}.$$

Similar to Equations (26) and (27), we can easily extend to the mixture of three exponential claim size distribution, i.e., $h(x) = \beta_1 \alpha_1 e^{-\alpha_1 x} + \beta_2 \alpha_2 e^{-\alpha_2 x} + \beta_3 \alpha_3 e^{-\alpha_3 x}$, $x > 0$, $\alpha_1 > \alpha_2 > \alpha_3 > 0$ and $\beta_1 + \beta_2 + \beta_3 = 1$, then we have

$$E\left(L_t^0\right) = \left( \beta_1 \frac{\rho}{\alpha_1} + \beta_2 \frac{\rho}{\alpha_2} + \beta_3 \frac{\rho}{\alpha_3} \right) \bar{a}_{\overline{t|}} \quad (\text{at } \delta) \tag{28}$$

and

$$\text{var}\left(L_t^0\right) = \left( \beta_1 \frac{2\rho}{\alpha_1^2} + \beta_2 \frac{2\rho}{\alpha_2^2} + \beta_3 \frac{2\rho}{\alpha_3^2} \right) \bar{a}_{\overline{t|}} \quad (\text{at } 2\delta). \tag{29}$$

If $t \to \infty$, the mean and variance of the discounted aggregate claims are given by

$$\left(\beta_1 \frac{\rho}{\alpha_1} + \beta_2 \frac{\rho}{\alpha_2} + \beta_3 \frac{\rho}{\alpha_3}\right) \frac{1}{\delta} \quad \text{and} \quad \left(\beta_1 \frac{\rho}{\alpha_1^2} + \beta_2 \frac{\rho}{\alpha_2^2} + \beta_3 \frac{\rho}{\alpha_3^2}\right) \frac{1}{\delta}.$$

If the claim size distribution is gamma, i.e., $h(x) = \frac{\alpha^\beta x^{\beta-1} e^{-\alpha x}}{\Gamma(\beta)}$, $x > 0, \alpha > 0$, and $\beta > 0$, the Laplace transform of the distribution of the accumulated aggregate claims is given by

$$\exp\left[-\rho \int_0^t \left\{1 - \left(\frac{\alpha}{\alpha + \nu e^{\delta s}}\right)^\beta\right\} ds\right]. \tag{30}$$

However, it is not possible to obtain the explicit expression of the above Laplace transform to derive its mean and variance.

Now let us illustrate the calculation of premiums and variances using the formulae derived above, assuming that the claim size distributions are exponential, mixture of two exponential and mixture of three exponential.

**Example 1:** The parameter values used to calculate Equations (21), (22), (26)–(29) are $\alpha_1 = 0.01$, $\alpha_2 = 0.009$, $\alpha_3 = 0.008$, $\delta = 0.05$, $\rho = 50$, and $t = 1$. The calculations of the actuarial net premiums and variances of the discounted aggregate claims are shown in Table 1.

**TABLE 1**
The Actuarial Net Premiums and Variances of the Discounted Aggregate Claims

| | Mean | Standard Deviation | Variance |
|---|---|---|---|
| Exponential | 4,877.1 | 975.51 | 951,630 |
| Mixture of two exponential ($\beta_1 = 0.7$ and $\beta_2 = 0.3$) | 5,039.6 | 1,009.3 | 1,018,600 |
| Mixture of three exponential ($\beta_1 = 0.7$, $\beta_2 = 0.2$, and $\beta_3 = 0.1$) | 5,107.4 | 1,024.6 | 1,049,800 |

**TABLE 2**
The Actuarial Net Premiums at Each Value of the Instananeous Rate of Interest

| $\delta$ | Exponential | Mixture of Two Exponential ($\beta_1 = 0.7$ and $\beta_2 = 0.3$) | Mixture of Three Exponential ($\beta_1 = 0.7$, $\beta_2 = 0.2$, and $\beta_3 = 0.1$) |
|---|---|---|---|
| 0.05 | 4,877.1 | 5,039.6 | 5,107.4 |
| 0.06 | 4,853.0 | 5,014.7 | 5,082.1 |
| 0.07 | 4,829.0 | 4,990.0 | 5,057.0 |
| 0.08 | 4,805.2 | 4,965.4 | 5,032.1 |
| 0.09 | 4,781.6 | 4,941.0 | 5,007.4 |
| 0.10 | 4,758.1 | 4,916.7 | 4,982.8 |

**TABLE 3**
The Standard Deviations of the Discounted Aggregate Claims at Each Value of the Instantaneous Rate of Interest

| $\delta$ | Exponential | Mixture of Two Exponential ($\beta_1 = 0.7$ and $\beta_2 = 0.3$) | Mixture of Three Exponential ($\beta_1 = 0.7$, $\beta_2 = 0.2$, and $\beta_3 = 0.1$) |
|---|---|---|---|
| 0.05 | 975.51 | 1,009.3 | 1,024.6 |
| 0.06 | 970.74 | 1,004.3 | 1,019.6 |
| 0.07 | 966.00 | 999.4 | 1,014.6 |
| 0.08 | 961.30 | 994.6 | 1,009.7 |
| 0.09 | 956.64 | 989.7 | 1,004.8 |
| 0.10 | 952.02 | 985.0 | 999.9 |

**TABLE 4**
The Actuarial Net Premiums and Variances of the Discounted Aggregate Claims With Infinite Time Horizon

| | Mean | Standard Deviation | Variance |
|---|---|---|---|
| Exponential | 100,000 | 3,162.3 | 10,000,000 |
| Mixture of two exponential ($\beta_1 = 0.7$ and $\beta_2 = 0.3$) | 103,330 | 3,271.7 | 10,704,000 |
| Mixture of three exponential ($\beta_1 = 0.7$, $\beta_2 = 0.2$, and $\beta_3 = 0.1$) | 104,720 | 3,321.4 | 11,032,000 |

**Example 2:** We will now examine the effect on actuarial net premiums and variances of the discounted aggregate claims caused by changes in the value of the instantaneous rate of interest $\delta$. The calculations of the actuarial net premiums at each value of $\delta$ are shown in Table 2 and the calculations of the standard deviations of the discounted aggregate claims at each value of $\delta$ are shown in Table 3.

**Example 3:** The calculations of the actuarial net premiums and variances of the discounted aggregate claims when the time horizon goes to infinity are shown in Table 4.

## CONCLUSION

We derived the Laplace transform of the distribution of the shot noise process applying the piecewise deterministic Markov processes theory and the martingale. Using the relationship between the shot noise process and accumulated/discounted aggregate claims process, the Laplace transform of the distribution of the accumulated aggregate claims was obtained.

Assuming that the claim arrival process follows the Poisson process and claim sizes are assumed to be exponential and mixture of exponential, we derived the explicit formulae of the means (i.e., actuarial net premiums) and variances of the discounted aggregate claims, which are expressed in terms of the annuities paid continuously. The numerical values of the actuarial net premiums and variances of the discounted aggregate claims were also given with examples.

As other heavy-tail distributions such as gamma, log-normal, and Pareto, etc., can also be applied for claim size distribution to more realistic cases in practice, we presented the Laplace transform of the distribution of the accumulated aggregate claims with gamma claim size distribution. As it was not possible for us to obtain the explicit expression of the Laplace transform to derive the actuarial net premium and variance of the discounted aggregate claims, a numerical method needs to be employed to calculate them if the claim size follows gamma distribution. Also as the rate of interests are stochastic in the real world, it is of interest to obtain the Laplace transform of the distribution of the accumulated/discounted aggregate claims using a stochastic rate of interest.

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