Measuring capital requirement for operational risk using the compound Cox process with shot noise intensity: VaR and TCE

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Abstract Due to the new Basel Capital Accord, the financial institutions need to develop a risk management tool for losses arising from all types of operational risk before 2006. In order to quantify the aggregate losses from all lines of operational risk, we employ an actuarial risk models. The Value at Risk (VaR) and tail conditional expectation (TCE, also known as TailVaR) are used to obtain the capital charge for operational risk. As a homogeneous Poisson process is not adequate to deal with irregular loss arrival time as well as a tendency to increase over time in practice, we employ the Cox process with shot noise intensity for loss arrival process from quantifiable operational risk (Embrechts and Samorodnitsky, 2002). We use the asymptotic distribution of the loss intensity to obtain the explicit expression of the Laplace transform of the distribution of aggregate losses. The loss size and primary event jump size distributions are assumed to follow exponential distributions. The calculations of VaR and tail conditional expectation (TCE) are illustrated using Transform Analysis technique from the financial option pricing literature. In order to include losses from non-quantifiable operational risk, we levy a security loading, which is well-known for premium calculation in actuarial science, on the original VaR and TCE calculated only for quantifiable operational risk.

Keywords: The compound Cox process; Shot noise process; Piecewise deterministic Markov process; VaR; Tail conditional expectation (TCE); Operational risk; Transform analysis.

1. Introduction

A capital charge for operational risk is required to the financial institutions (eventually starting in 2006) according to the consultative document on the New Basel Capital Accord, where the Basel Committee for Banking Supervision defined operational risk as follows: "The risk of losses resulting from inadequate or failed internal processes, people and systems or from external events." In Table 1.1 that is adopted from Crouhy et al. (2000), we can find a list of some typical types of operational risks.

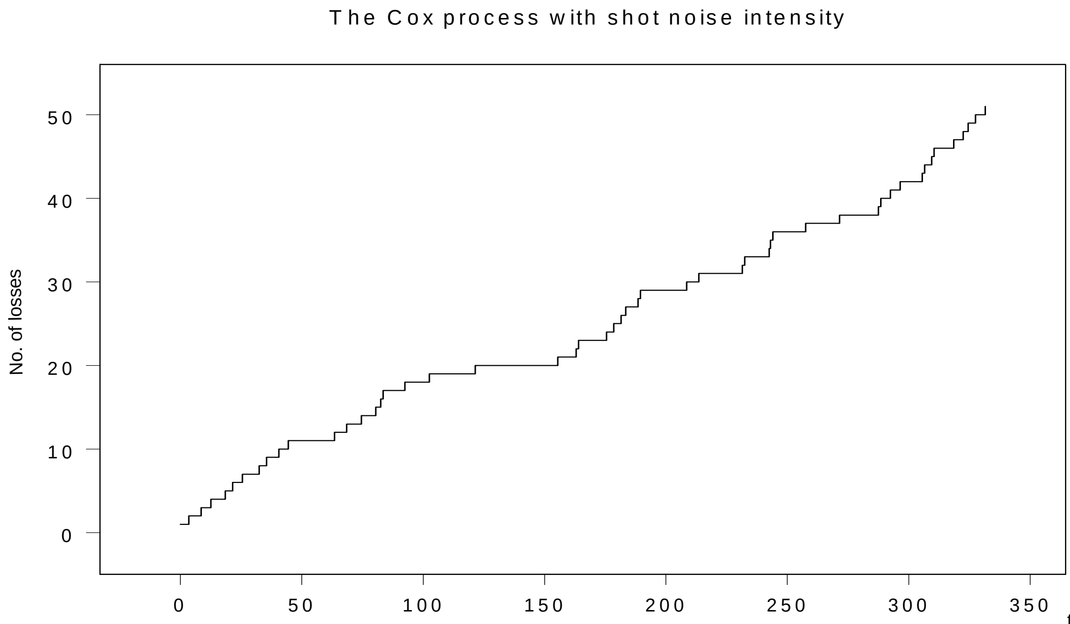
Table 1.1

(i) People risk:	· Incompetency · Fraud
(ii) Process risk:	
- Model risk	· Model/methodology error · Mark-to-model error
- Transaction risk	· Execution error · Product complexity · Booking error · Settlement error · Documentation/contract risk
- Operational control risk	· Exceeding limits · Security risks · Volume risks
(iii) Technology risk:	· System failure · Programming error · Information risk · Telecommnucation failure

As some of risks such as people risk are not easy to quantify so we will focus on quantifying the ordinary operational risks first using an actuarial risk model (Cramér 1930; Bühlmann 1970; Gerber 1979; Grandell 1976, 1991 Beard et al. 1984 and Asmussen 2000). Let X_i, $i = 1, 2, \cdots$, be the loss amounts across all k categories of quantifiable operational risks, which are assumed to be independent and identically distributed with distribution function $H(x)$ $(x > 0)$. Then the total loss up to time t is defined by

$$L_t = \sum_{i=1}^{N_t} X_i \tag{1.1}$$

where N_t is the total number of losses up to time t. As Embrechts and Samorodnitsky (2002) has shown that there exist irregular loss arrival time as well as a tendency to increase over time in practice, it is obvious that the Poisson process, which has deterministic intensity rate, is not adequate to measure loss arrivals from quantifiable operational risk. As an alternative point process for loss arrival process N_t from quantifiable operational risk, we employ the Cox process with shot noise intensity (Cox 1955; Grandell, 1976, 1991 and Brémaud 1981).



Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one used by Dassios and Jang (2003).

Definition 1.1 *Let (Ω, F, P) be a probability space with information structure given by $F = \{\Im_t,\ t \in [0, T]\}$. Let N_t be a point process adapted to F. Let λ_t be a non-negative process adapted to F such that*

$$\int_0^t \lambda_s ds < \infty \quad \textit{almost surely} \quad (\textit{no explosions}).$$

If for all $0 \le t_1 \le t_2$ and $u \in \Re$

$$\mathbb{E}\left\{e^{iu\left(N_{t_2} - N_{t_1}\right)} | \Im_{t_2}^{\lambda}\right\} = \exp\left\{\left(e^{iu} - 1\right) \int_{t_1}^{t_2} \lambda_s ds\right\} \tag{1.2}$$

then N_t is call a $\Im_t$-doubly stochastic Poisson process with intensity λ_t where $\Im_t^{\lambda} = \sigma\{\lambda_s;\ s \le t\}$.

Equation (1.2) gives us

$$\Pr\left\{N_{t_2}-N_{t_1}=k|\lambda_s;t_1\leq s\leq t_2\right\}=\frac{\exp\left(-\int\limits_{t_1}^{t_2}\lambda_s ds\right)\left(\int\limits_{t_1}^{t_2}\lambda_s ds\right)^k}{k!} \tag{1.3}$$

and consider the process $\Lambda_t=\int\limits_0^t\lambda_s ds$ (the aggregated process), then from (1.3) we can easily find that

$$\mathbb{E}\left(\theta^{N_{t_2}-N_{t_1}}\right)=\mathbb{E}\left\{e^{-(1-\theta)\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}\right\}. \tag{1.4}$$

Equation (1.4) suggests that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of Λ_t, the aggregated process. It means that we just have to find the p.g.f. (probability generating function) of N_t to retrieve the m.g.f. (moment generating function) of Λ_t and vice versa.

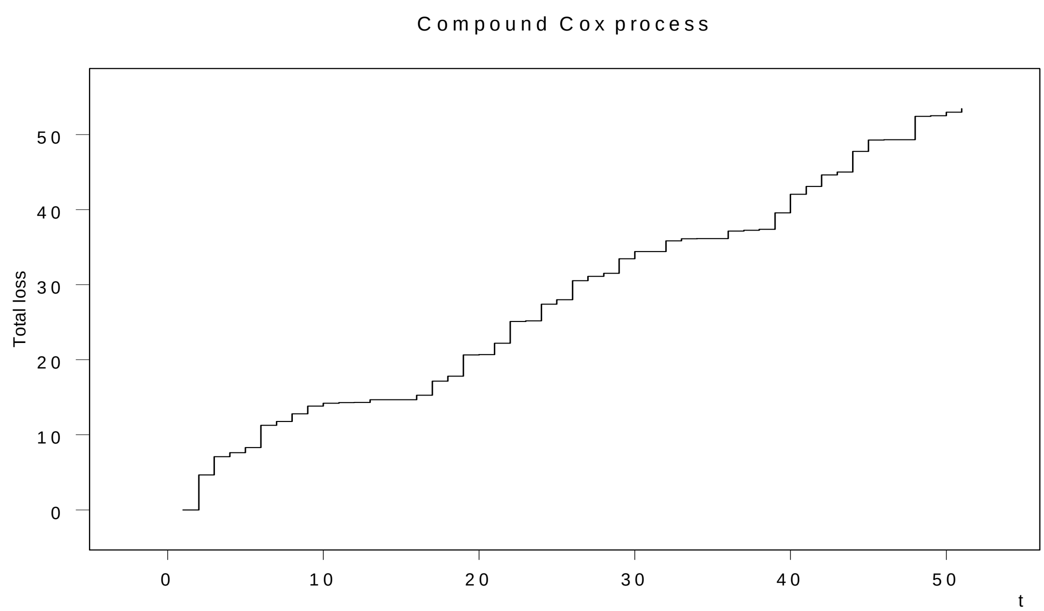
We use the VaR (or the q-quantile) denoted by l_q that is the smallest value satisfying

$$\Pr(L_t\geq l_q)=1-q \tag{1.5}$$

as it was discussed as a risk measure to decide the capital amount required for next t years' operational risk in the consultative document on the New Basel Capital Accord. We also derive the tail conditional expectation (TCE, also known as TailVaR) defined by

$$E(L_t\mid L_t\geq l_q) \tag{1.6}$$

to obtain the capital amount required for next t years' operational risk as a coherent risk measure (Artzner et al. 1999). As we employ the Cox process for loss arrival process of N_t from quantifiable operational risk, we have the compound Cox process of L_t, which is the total loss process from all k categories of quantifiable operational risks.



The paper is organised as follows. In section 2, we introduce the shot noise process as an intensity of the Cox process. We derive the Laplace transform of shot noise process, λ_t and aggregated process, Λ_t by piecewise deterministic Markov processes (PDMP) theory. All proofs are referred to Dassios and Jang (2003) where they used the Cox process with shot noise intensity for the pricing of reinsurance contract. Section 3 deals with finding the Laplace transform of the distribution of the total loss, L_t as it is known that it is not possible for us to obtain its distribution itself. We derive the explicit expression of the Laplace transform of the distribution of total loss using the asymptotic distribution of the loss intensity and exponential jump size distribution for primary events. In section 4, we illustrate the calculation of the VaR and TCE as risk measures of the capital amount required for next year's operational risk using Transform Analysis technique from the financial option pricing literature. Hiring an idea of levying a security loading factor, which is well-known for premium calculation in actuarial science, on the original VaR and TCE calculated only for quantifiable operational risk we evaluate them from all types of operational risk. Section 5 concludes.

2. Shot noise process and aggregated process

In practice, the categories of operational risks are primary events. So the number of losses depends on the frequency and magnitude of these primary events and time period needed to determine the effect of primary events. One of the processes that can be used to measure the impact of primary events is the shot noise process (Cox & Isham 1986; Klüppelberg and Mikosch 1995 and Dassios and Jang 2003). As time passes, the shot noise process decreases as more and more losses are figured out. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of the doubly stochastic Poisson process to measure the number of losses due to primary event, i.e. we will use it as an intensity function to generate the Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i \leq t}} y_i e^{-\delta (t - s_i)}$$

where:
- λ_0 initial value of λ
- y_i jump size of primary event i, where $E(y_i) < \infty$
- s_i time at which primary event i occurs, where $s_i < t < \infty$
- δ exponential decay
- ρ the rate of primary event arrival.

Some works of insurance application using shot noise process can be found in Kluppelberg and Mikosch (1995), Dassios and Jang (2003) and Jang (2003).

The piecewise deterministic Markov processes (PDMP) theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. From now on, we present definitions and important properties of shot noise and aggregated processes with the aid of piecewise deterministic processes theory (Dassios 1987; Dassios and Embrechts 1989 and Dassios and Jang 2003). This theory is used to derive the Laplace transform of shot noise process, λ_t and aggregated process , Λ_t.

The three parameters of the shot noise process described are homogeneous in time. We are now going to generalise the shot noise process by allowing the parameters to depend on time to deal with the problem pointed out by Embrechts and Samorodnitsky (2002), i.e. irregular loss arrival time as well as a tendency to increase over time in practice. The rate of jump arrivals, $\rho(t)$, is bounded on all intervals $[0, t)$ (no explosions). $\delta(t)$ is the rate of decay and the distribution function of jump sizes at any time t is $G(y;t)$ $(y > 0)$ with $E(y;t) = \mu_1(t) = \int_0^\infty y dG(y;t)$. We assume that $\delta(t)$, $\rho(t)$ and $G(y;t)$ are all Riemann integrable functions of t and are all positive.

The generator of the process $(\Lambda_t, N_t, \lambda_t, t)$ acting on a function $f(\Lambda, n, \lambda, t)$ belonging to its domain is given by

$$\begin{aligned} A\ f(\Lambda, n, \lambda, t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} + \lambda \left[f(\Lambda, n+1, \lambda, t) - f(\Lambda, n, \lambda, t) \right] - \delta(t) \lambda \frac{\partial f}{\partial \lambda} \\ &+ \rho(t) \left[\int_0^\infty f(\Lambda, n, \lambda + y, t) dG(y;t) - f(\Lambda, n, \lambda, t) \right]. \end{aligned} \tag{2.4}$$

For $f(\Lambda, n, \lambda, t)$ to belong to the domain of the generator A, it is sufficient that $f(\Lambda, n, \lambda, t)$ is differentiable w.r.t. Λ, λ, t for all Λ, n, λ, t and that

$$\left|\int_0^\infty f(\cdot,\lambda+y,\cdot)\,dG(y;t)-f(\cdot,\lambda,\cdot)\right|<\infty.$$

Let us find a suitable martingale in order to derive the Laplace transforms of the distribution of Λ_t.

Lemma 2.1 *Considering constants k and v such that $k\geq 0$ and $v\geq 0$,*

$$\begin{aligned}&\exp(-v\Lambda_t)\cdot\exp\left[-\left\{ke^{\triangle(t)}-ve^{\triangle(t)}\int_0^t e^{-\triangle(r)}dr\right\}\lambda_t\right]\\ &\times\exp\left[\int_0^t\rho(s)\left[1-\hat{g}\left\{ke^{\triangle(s)}-ve^{\triangle(t)}\int_0^s e^{-\triangle(r)}dr;s\right\}\right]ds\right]\end{aligned}\tag{2.5}$$

is a martingale where $\hat{g}(u;s)=\int_0^\infty e^{-u\,y}dG(y;s)$ and $\triangle(t)=\int_0^t\delta(s)\,ds$.

Proof. See Dassios and Jang (2003). ∎

Let us assume that $\delta(t)=\delta$ throughout the rest of this paper.

Corollary 2.2 *Let $v_1\geq 0$ and $v_2\geq 0$. Then*

$$\begin{aligned}&E\left\{e^{-v_1\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}e^{-v_2\lambda_{t_2}}|\Lambda_{t_1},\lambda_{t_1}\right\}\\ =\;&\exp\left[-\left\{\frac{v_1}{\delta}+\left(v_2-\frac{v_1}{\delta}\right)e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right]\\ &\times\exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{\frac{v_1}{\delta}+\left(v_2-\frac{v_1}{\delta}\right)e^{-\delta(t_2-s)};s\right\}\right]ds\right].\end{aligned}\tag{2.6}$$

Proof. See Dassios and Jang (2003). ∎

Now we can easily obtain the Laplace transforms of the distribution of λ_t, Λ_t.

Corollary 2.3 *The Laplace transforms of the distribution of λ_t and Λ_t are given by*

$$\begin{aligned}E\left\{e^{-v\lambda_{t_2}}|\lambda_{t_1}\right\}&=\exp\left[-ve^{-\delta(t_2-t_1)}\lambda_{t_1}\right]\\ &\quad\times\exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{ve^{-\delta(t_2-s)};s\right\}\right]ds\right],\end{aligned}\tag{2.7}$$

$$\begin{aligned}&E\left\{e^{-v\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}|\lambda_{t_1}\right\}\\ =\;&\exp\left[-\frac{v}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right]\\ &\times\exp\left[-\int_{t_1}^{t_2}\rho(s)\left[1-\hat{g}\left\{\frac{v}{\delta}\left(1-e^{-\delta(t_2-s)}\right);s\right\}\right]ds\right].\end{aligned}\tag{2.8}$$

Proof. See Dassios and Jang (2003). ∎

Let us obtain the asymptotic distributions of λ_t at time t from (2.8), provided that the process started sufficiently far in the past. In this context we interpret it as the limit when $t\to-\infty$. In other words, if we know λ at '$-\infty$' and no information between '$-\infty$' to present time t, '$-\infty$' asymptotic distribution of λ_t can be used as the distribution of λ_t.

Lemma 2.4 *Assume that $\lim_{t\to-\infty}\rho(t)=\rho$ and $\lim_{t\to-\infty}\mu_1(t)=\mu_1$. Then the '$-\infty$' asymptotic distribution of λ_t has Laplace transform*

$$E\left(e^{-v\lambda_{t_1}}\right)=\exp\left[-\int_{-\infty}^{t_1}\rho(s)\left[1-\hat{g}\left\{ve^{-\delta(t_1-s)};s\right\}\right]ds\right].\tag{2.9}$$

Proof. See Dassios and Jang (2003). ■

It will be interesting to find the Laplace transforms of the distribution of λ_t and Λ_t at time t, using a specific jump size distribution of $G(y;t)$ $(y>0)$. We use an exponential jump size distribution, i.e. $g(y;t)=\left(\alpha+\gamma e^{\delta t}\right)e^{-\left(\alpha+\gamma e^{\delta t}\right)y}$, $y>0$, $-\alpha e^{-\delta t}<\gamma\leq 0$. Let us assume that $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$. The reason for this particular assumption is to obtain the explicit expression of the Laplace transform of the distribution of λ_t and Λ_t at time t.

Theorem 2.5 *Let the jump size distribution be exponential, i.e.* $g(y;t)=\left(\alpha+\gamma e^{\delta t}\right)\exp\left\{-\left(\alpha+\gamma e^{\delta t}\right)y\right\}$, $y>0$, $\alpha e^{-\delta t}<\gamma\leq 0$, *and assume that* $\rho(t)=\rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$. *Then*

$$\begin{aligned} E\left\{e^{-v\lambda_{t_1}}|\lambda_{t_0}\right\} &= \exp\left\{-v\lambda_{t_0}e^{-\delta(t_1-t_0)}\right\}\left(\frac{\gamma e^{\delta t_0}+\alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0}+\alpha}\right)^{\frac{\rho}{\delta}} \\ &\quad\times\left(\frac{\gamma e^{\delta t_0}+ve^{-\delta(t_1-t_0)}+\alpha}{\gamma e^{\delta t_0}+(v+\alpha)e^{-\delta(t_1-t_0)}}\right)^{\frac{\rho}{\delta}}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} E\left\{e^{-v\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}|\lambda_{t_1}\right\} &= \exp\left[-\frac{v}{\delta}\left\{1-e^{-\delta(t_2-t_1)}\right\}\lambda_{t_1}\right] \\ &\quad\times\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}} \\ &\quad\times\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}} \end{aligned} \tag{2.11}$$

If λ_t *is '*$-\infty$*' asymptotic,*

$$E\left(e^{-v\lambda_{t_1}}\right)=\left(\frac{\gamma+\alpha e^{-\delta t_1}}{\gamma+(v+\alpha)e^{-\delta t_1}}\right)^{\frac{\rho}{\delta}}, \tag{2.12}$$

$$\begin{aligned} E\left\{e^{-v\left(\Lambda_{t_2}-\Lambda_{t_1}\right)}\right\} &= \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}\right)^{\frac{\rho}{\delta}} \\ &\quad\times\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\left(1-e^{-\delta(t_2-t_1)}\right)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}} \end{aligned} \tag{2.13}$$

Proof. See Dassios and Jang (2003). ■

3. The Laplace transform of the distribution of total loss process

Now let us derive the Laplace transform of total loss process, L_t using a suitable martingale. The generator of the process (N_t, L_t, t) acting on a function $f(n,l,t)$ belonging to its domain is given by

$$A\ f(n,l,t)=\frac{\partial f}{\partial t}+\lambda\left[f(n+1,l+x,t)dH(x)-f(n,l,t)\right]. \tag{3.1}$$

Lemma 3.1 *Considering constants* θ *and* v *such that* $0\leq\theta\leq 1$ *and* $v\geq 0$,

$$\theta^{N_t}\exp\left(-vL_t\right)\exp\left[-\left\{\theta\hat{h}(\nu)-1\right\}\Lambda_t\right] \tag{3.2}$$

is a martingale where $\hat{h}(\nu)=\int\limits_0^\infty e^{-\nu\ x}dH(x)$.

Proof. From (3.1) $f(n,l,t)$ has to satisfy $Af=0$ for it to be a martingale. Setting $f=\theta^n e^{-\nu l}e^{B(t)}$ we get the equation

$$B'(t)+\lambda\left\{\theta\int\limits_0^\infty e^{-\nu\ x}dH(x)-1\right\}=0 \tag{3.3}$$

and solving (3.3) we get

$$B(t) = -\left\{\theta\hat{h}(\nu) - 1\right\}\Lambda_t$$

and the result follows. ■

Now we can easily obtain the Laplace transforms of the distribution of L_t.

Corollary 3.2 *The Laplace transforms of the distribution of L_t is given by*

$$\begin{aligned} E\left(e^{-vL_t}|\lambda_0\right) &= \exp\left[-\frac{\left\{1-\hat{h}(\nu)\right\}}{\delta}\left(1-e^{-\delta t}\right)\lambda_0\right] \\ &\quad\times\exp\left[-\int_0^t \rho(s)\left[1-\hat{g}\left\{\frac{1-\hat{h}(\nu)}{\delta}\left(1-e^{-\delta(t-s)}\right);s\right\}\right]ds\right] \end{aligned} \tag{3.4}$$

Proof. From Lemma 3.1 with $\theta = 1$, we have

$$E\left\{e^{-v\left(L_{t_2}-L_{t_1}\right)}|L_{t_1},\ \lambda_{t_1}\right\} = \exp\left[-\nu L_{t_1}\right]\exp\left[\left\{\hat{h}(\nu)-1\right\}\left(\Lambda_{t_2}-\Lambda_{t_1}\right)\right]$$

and without loss of generality, we have

$$E\left(e^{-vL_t}|\lambda_0\right) = E\left[\exp\left\{-\left(1-\hat{h}(\nu)\right)\Lambda_t\right\}\ |\ \lambda_0\right].$$

Therefore (3.4) follows immediately from (2.8). ■

Theorem 3.3 *Let the jump size distribution be exponential, i.e. $g(y;t) = \left(\alpha+\gamma e^{\delta t}\right)\exp\left\{-\left(\alpha+\gamma e^{\delta t}\right)y\right\}$, $y > 0$, $\alpha e^{-\delta t} < \gamma \leq 0$, and assume that $\rho(t) = \rho\frac{\alpha}{\alpha+\gamma e^{\delta t}}$. If λ_t is '$-\infty$' asymptotic, the Laplace transforms of the distribution of L_t is given by*

$$\left(\frac{\gamma+\alpha e^{-\delta t}}{\gamma+\alpha+\frac{1-\hat{h}(\nu)}{\delta}\left(1-e^{-\delta t}\right)}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma+\alpha+\frac{1-\hat{h}(\nu)}{\delta}\left(1-e^{-\delta t}\right)}{\gamma+\alpha e^{-\delta t}}\right)^{\frac{\alpha\rho}{\delta\alpha+\left\{1-\hat{h}(\nu)\right\}}} \tag{3.5}$$

and if we assume that loss size distribution also follows exponential, i.e. $h(x) = \beta e^{-\beta x}$, $x > 0$, $\beta > 0$, it is given by

$$\left(\frac{\gamma+\alpha e^{-\delta t}}{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)}{\gamma+\alpha e^{-\delta t}}\right)^{\frac{\alpha\rho}{\delta\alpha+\left(\frac{\nu}{\beta+\nu}\right)}} \tag{3.6}$$

Proof. From (2.13), (3.5) follows and (3.6) follows from (3.5). ■

Even though the compound Cox process with shot noise intensity is employed as loss process, in practice, we might need to employ one of the heavy tail distributions for loss sizes such as Pareto, Gumbel and Fréchet, etc. rather than exponential as Embrechts and Samorodnitsky (2002) have shown that loss amounts seemed extremes. However if it is the case, we will not be able to obtain the explicit expression of the Laplace transform of total loss, L_t.

4. Value at Risk (VaR) and tail conditional expectation (TCE) via transform analysis

Let us look at how the Laplace transform derived above can be used to evaluate VaR and TCE as risk measures of the capital amount required for next t years' operational risks. From (1.5), the VaR can be expressed as

$$\mathrm{VaR}(q,\ L_t) = \inf\left\{l_q \in \mathbb{R}:\ \Pr(L_t > l_q) \leq 1-q\right\} \tag{4.1}$$

where $0.9975 \leq q \leq 0.999$, typically for operational risk losses. From (1.6), the TCE can be expressed by

$$
\begin{aligned}
\mathrm{TCE}\,(q,\ L_t) &= E\left\{L_t \mid L_t \geq \mathrm{VaR}\,(q,\ L_t)\right\} \\
&= \frac{E\left[L_t\ I\left\{L_t \geq \mathrm{VaR}\,(q,\ L_t)\right\}\right]}{1-q}
\end{aligned} \tag{4.2}
$$

where $I(\cdot)$ is the indicator function. As it is not possible for us to obtain the distribution of L_t explicitly we employ transform analysis techniques developed by Heston (1993) and Duffie et al. (2000) to calculate the VaR and TCE numerically. We highlight their methodology as applied to our problem below.

We know from the previous section the Laplace transform $\xi(-\nu)$ of L_t

$$
\xi(-\nu) = E\left(e^{-vL_t}\right)
$$

and can consider the function

$$
\widehat{\Psi}(z) = \int_0^\infty e^{izl} d\left(\int_0^l dF_{L_t}(x)\right)
$$

and hence

$$
\begin{aligned}
\widehat{\Psi}(z) &= \int_0^\infty e^{izl} dF_{L_t}(l) \\
&= E\left[e^{izL_t}\right] = \xi(iz)\,.
\end{aligned}
$$

Recall that

$$
\begin{aligned}
\xi(-\nu) &= E\left(e^{-vL_t}\right) \\
&= \left(\frac{\gamma+\alpha e^{-\delta t}}{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)}\right)^{\frac{\rho}{\delta}} \\
&\quad\times\left(\frac{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)}{\gamma+\alpha e^{-\delta t}}\right)^{\frac{\alpha\rho}{\delta\alpha+\left(\frac{\nu}{\beta+\nu}\right)}}
\end{aligned}
$$

and the standard Lévy inversion formula gives

$$
\begin{aligned}
E\left\{I\left(L_t<l\right)\right\} &= P\left(L_t<l\right) = \Psi(l) \\
&= \frac{\widehat{\Psi}(0)}{2} - \frac{1}{\pi}\int_0^\infty \frac{1}{z}\,\mathrm{Im}\left(e^{-izl}\widehat{\Psi}(z)\right)dz
\end{aligned} \tag{4.3}
$$

from which we can easily obtain $P\left(L_t \geq l\right)$.

Consider now another function

$$
\widehat{\Theta}(z) = \int_0^\infty e^{izl} d\left(\int_0^l x dF_{L_t}(x)\right).
$$

Assume $\int\left|\Theta(y)\right| dy < \infty$ and we find that

$$
\begin{aligned}
\widehat{\Theta}(z) &= \int_0^\infty e^{izl} l\ dF_{L_t}(l) \\
&= E\left[L_t\ e^{izL_t}\right],
\end{aligned}
$$

which can be calculated as follows. Differentiating $\xi(-\nu)$ with respect to $-\nu$ gives

$$
\begin{aligned}
& -\frac{\partial}{\partial\nu}\xi(-\nu) = E\left[L_t\, e^{-\nu L_t}\right] \\
= \;& \rho\left(\frac{\beta}{\beta+\nu}\right)^2\left(\frac{\gamma+\alpha e^{-\delta t}}{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)}\right)^{\frac{\rho}{\delta}} \\
& \times\left(\frac{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)}{\gamma+\alpha e^{-\delta t}}\right)^{\frac{\alpha\rho}{\delta\alpha+\left(\frac{\nu}{\beta+\nu}\right)}} \qquad (4.4)\\
& \times\left[\begin{array}{l}\left\{\frac{1-e^{-\delta t}}{\delta\left\{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)\right\}}\right\}\left(\frac{1}{\delta}-\frac{\alpha}{\delta\alpha+\frac{\nu}{\beta+\nu}}\right) \\ +\left\{\frac{\alpha}{\left(\delta\alpha+\frac{\nu}{\beta+\nu}\right)^2}\right\}\ln\left\{\frac{\gamma+\alpha+\frac{\nu}{\delta(\beta+\nu)}\left(1-e^{-\delta t}\right)}{\gamma+\alpha e^{-\delta t}}\right\}\end{array}\right] \\
= \;& \eta(-\nu)
\end{aligned}
$$

and hence

$$\widehat{\Theta}(z) = \eta(iz).$$

Since we now have a closed form formula for $\widehat{\Theta}(y)$ the inversion lemma gives

$$
\begin{aligned}
E\left[L_t\; I\left\{L_t < \mathrm{VaR}(q,\; L_t)\right\}\right] &= \Theta(l) \\
&= \frac{\widehat{\Theta}(0)}{2} - \frac{1}{\pi}\int_0^{\infty}\frac{1}{z}\,\mathrm{Im}\left(e^{-izl}\widehat{\Theta}(z)\right)dz,
\end{aligned}
$$

with

$$\widehat{\Theta}(0) = \eta(0) = E(L_t)$$

which allows us to calculate the numerator of (4.2) as

$$E\left[L_t\; I\left\{L_t \geq \mathrm{VaR}(q,\; L_t)\right\}\right] = E(L_t) - E\left[L_t\; I\left\{L_t < \mathrm{VaR}(q,\; L_t)\right\}\right] \qquad (4.5)$$

where

$$E(L_t) = \frac{\rho}{\alpha\beta\delta^2}\ln\left(\frac{\gamma+\alpha}{\gamma+\alpha e^{-\delta t}}\right). \qquad (4.6)$$

Let us now illustrate the calculations of VaR and TCE using the methods derived above.

Example 4.1

The parameter values used to calculate VaR and TCE using (4.3) and (4.5)are

$$\alpha = 0.1,\; \beta = 0.01,\; \delta = 0.3,\; \gamma = -0.01,\; \rho = 4,\; t = 1.$$

From (4.6), the mean of the total loss arising from quantifiable operational risk in a unit period of time is given by

$$E(L_t) = \frac{\rho}{\alpha\beta\delta^2}\ln\left(\frac{\gamma+\alpha}{\gamma+\alpha e^{-\delta t}}\right) = \$15,096.$$

Using Matlab, the VaR and TCE are shown in Table 4.1.

Table 4.1.

q	VaR	TCE
99.9%	\$31,488	\$33,219
99.75%	\$29,629	\$31,541
95%	\$22,707	\$25,121
90%	\$20,766	\$23,381
50%	\$14,730	\$18,480

Example 4.2

In order to include losses from non-quantifiable operational risk, let us levy a security loading factor $\theta > 0$, which is well-known for premium calculation in actuarial science, on the original VaR and TCE calculated only for quantifiable operational risk., i.e.

$$\mathrm{VaR}^{\mathrm{all}}\left(q,\ L_t\right) = (1+\theta) \times \mathrm{VaR}\left(q,\ L_t\right)$$

and

$$\mathrm{TCE}^{\mathrm{all}}\left(q,\ L_t\right) = (1+\theta) \times \mathrm{TCE}\left(q,\ L_t\right).$$

Assuming that $\theta = 0.3$, where the banks' attitude towards non-quantifiable operational risk determines θ unless the Basel Committee for Banking Supervision predetermines, the VaR and TCE from all types of operational risk are shown in Table 4.2.

Table 4.2.

q	$\mathrm{VaR}^{\mathrm{all}}$	$\mathrm{TCE}^{\mathrm{all}}$
99.9%	\$40, 934	\$43, 185
99.75%	\$38, 518	\$41, 003
95%	\$29, 519	\$32, 657
90%	\$26, 996	\$30, 395
50%	\$19, 149	\$24, 024

5. Conclusion

For total loss arising from quantifiable operational risk, we employed the compound Cox process. We used shot noise process as an intensity of the Cox process as the number of losses arising from operational risk depends on the frequency and magnitude of primary events and time period needed to determine the effect of primary events. In order to deal with the issues raised by Embrechts and Samorodnitsky (2002), i.e. irregular loss arrival time as well as a tendency to increase over time in operational risk, we generalised the shot noise process by allowing the parameters to depend on time. We obtained the explicit expression of the Laplace transform of the distribution of total loss using the asymptotic distribution of the loss intensity and exponential jump size distribution for primary events. Based on this Laplace transform, the VaR and TCE as risk measures of the capital amount required for next year's operational risk were evaluated using Transform Analysis technique from the financial option pricing literature. In order to include losses from non-quantifiable operational risk, we hired an idea of levying a security loading factor, which is well-known for premium calculation in actuarial science, on the original VaR and TCE calculated only for quantifiable operational risk.

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