

# Measuring default premiums using the Cox process with shot noise intensity

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## Abstract

We employ the Cox process with shot noise intensity to model the default time. The survival probability is derived based on the Cox process with shot noise intensity that has doubly stochastic property. As an interest rate process for non-defaultable bond, i.e. a government bond, we use a generalised Cox-Ingersoll-Ross (CIR) model (1985). As Lando (1998) has shown that we can combine the effects of default and of discounting for interest rates, it is used to obtain the default premium between non-defaultable bond and defaultable bond (i.e. a corporate bond). Using an equivalent martingale probability measure obtained via the Esscher transform, risk-neutral default premium formula is derived. The asymptotic distribution of the shot noise intensity is used not to have its initial value. We also assume that the jump size of shot noise intensity follows an exponential distribution to illustrate the calculation of default premium. For simplicity, we ignore the recovery rate.

*Keywords:* The Cox process; shot noise process; survival probability; a generalised CIR model; piecewise deterministic Markov process; the Esscher transform; default premium.

## 1. Introduction

Since Merton (1974), one of the credit risk modelling developed is based on the intensity of a counting process. Jarrow and Turnbull (1995) proposed to use the Poisson process and extended it further employing a discrete state space Markov chain in credit rating with Lando (1997). Lando (1998) examined it deeper introducing the Cox process where its intensity has finite state space. Similar approach was used by Duffie and Singleton (1999), where they considered the fractional reduction in market value that occurs at default with respect to risk neutral probability measure.

In this paper we also employ the Cox process to model the default time (Cox 1955; Grandell, 1976 and Brémaud 1981). Under a doubly stochastic Poisson process, or the Cox process, the intensity function is assumed to be stochastic. The doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process  $\lambda_t$  is used to generate another process

$N_t$  by acting as its intensity. That is,  $N_t$  is a Poisson process conditional on  $\lambda_t$  which itself is a stochastic process (if  $\lambda_t$  is deterministic then  $N_t$  is a Poisson process). Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one used by Dassios and Jang (2003).

**Definition 1.1** Let  $(\Omega, F, P)$  be a probability space with information structure given by  $F = \{\mathfrak{F}_t, t \in [0, T]\}$ . Let  $N_t$  be a point process adapted to  $F$ . Let  $\lambda_t$  be a non-negative process adapted to  $F$  such that

$$\int_0^t \lambda_s ds < \infty \text{ almost surely (no explosions).}$$

If for all  $0 \leq t_1 \leq t_2$  and  $u \in \mathfrak{R}$

$$\mathbb{E} \left\{ e^{iu(N_{t_2} - N_{t_1})} | \mathfrak{F}_{t_2}^\lambda \right\} = \exp \left\{ (e^{iu} - 1) \int_{t_1}^{t_2} \lambda_s ds \right\} \quad (1.1)$$

then  $N_t$  is call a  $\mathfrak{F}_t$ -doubly stochastic Poisson process with intensity  $\lambda_t$  where  $\mathfrak{F}_t^\lambda = \sigma \{ \lambda_s; s \leq t \}$ .

Equation (1.1) gives us

$$\Pr \{ N_{t_2} - N_{t_1} = k | \lambda_s; t_1 \leq s \leq t_2 \} = \frac{\exp \left( - \int_{t_1}^{t_2} \lambda_s ds \right) \left( \int_{t_1}^{t_2} \lambda_s ds \right)^k}{k!}. \quad (1.2)$$

and

$$\begin{aligned} & \Pr \{ \tau_2 > t | \lambda_s; t_1 \leq s \leq t_2 \} \\ &= \Pr \{ N_{t_2} - N_{t_1} = 0 | \lambda_s; t_1 \leq s \leq t_2 \} = \exp \left( - \int_{t_1}^{t_2} \lambda_s ds \right) \end{aligned} \quad (1.3)$$

where  $\tau_k$  denotes the length of the time interval between the  $(k-1)^{\text{th}}$  and  $k^{\text{th}}$  point. Now consider the process  $\Lambda_t = \int_0^t \lambda_s ds$  (*the aggregated process*), then from (1.2) we can easily find that

$$\mathbb{E} \left( \theta^{N_{t_2} - N_{t_1}} \right) = \mathbb{E} \left\{ e^{-(1-\theta)(\Lambda_{t_2} - \Lambda_{t_1})} \right\}. \quad (1.4)$$

Equation (1.4) suggests that the problem of finding the distribution of  $N_t$ , the point process, is equivalent to the problem of finding the distribution of  $\Lambda_t$ , the aggregated process. It means that we just have to find the p.g.f. (probability generating function) of  $N_t$  to retrieve the m.g.f. (moment generating function) of  $\Lambda_t$  and vice versa.

From the Cox process of  $N_t$ , the survival probability is given by

$$\Pr(\tau_1 > t) = \mathbb{E} \left\{ \exp \left( - \int_0^t \lambda_s ds \right) \right\} \quad (1.5)$$

where  $\tau_1 \equiv \inf \{t : N_t = 1 \mid N_0 = 0\}$  is the default arrival time that is equivalent to the first jump time of the Cox process  $N_t$ . If we assume that the interest rate process for a zero-coupon government (risk-free) bond,  $r_t$  follows a generalised Cox-Ingersoll-Ross (CIR) model (1985), i.e.

$$dr_t = c(b - ar_t)dt + \sigma\sqrt{r_t}dB_t \quad (1.6)$$

where  $a > 0$ ,  $b > 0$  and  $c > 0$ , its price paying 1 at time  $t$  is given by

$$\mathbb{E} \left\{ \exp \left( - \int_0^t r_s ds \right) \mid r_0 \right\} \quad (1.7)$$

where  $\mathfrak{S}_t^r = \sigma \{r_s; s \leq t\}$  that is assumed to be independent of  $\mathfrak{S}_t^\lambda = \sigma \{\lambda_s; s \leq t\}$ . For the explicit derivation of (1.7), we refer you CIR (1985). Assuming that  $r_t$  and  $\lambda_t$  are independent, the price of a zero-coupon corporate (defaultable) bond paying  $1_{(\tau_1 > t)}$  at time  $t$  is given by

$$\mathbb{E} \left\{ \exp \left( - \int_0^t r_s ds \right) 1_{(\tau_1 > t)} \mid r_0, \lambda_0 \right\} = \mathbb{E} \left[ \exp \left\{ - \int_0^t (r_s + \lambda_s) ds \right\} \mid r_0, \lambda_0 \right] \quad (1.8)$$

For simplicity, we ignore recovery rate. For fractional recovery, we refer you Duffie and Singleton (1999). The comparison between the classical approach proposed in Black and Scholes (1973) and Merton (1974) and the intensity-based approach can be found in Cooper and Martin (1996). Other works relating to credit risk we refer you Artzner and Delbaen (1995), Madan and Unal (1998), Bielecki and Rutkowski (2000), and Elliot et al. (2000).

From (1.7) and (1.8), the price difference between non-defaultable bond and defaultable bond is given by

$$\begin{aligned} & \mathbb{E} \left\{ \exp \left( - \int_0^t r_s ds \right) \mid r_0 \right\} - \mathbb{E} \left\{ \exp \left( - \int_0^t (r_s + \lambda_s) ds \right) \mid r_0, \lambda_0 \right\} \\ &= \mathbb{E} (e^{-R_t} \mid r_0) - \mathbb{E} (e^{-R_t} e^{-\Lambda_t} \mid r_0, \lambda_0) \\ &= \mathbb{E} (e^{-R_t} \mid r_0) \{1 - \mathbb{E} (e^{-\Lambda_t} \mid \lambda_0)\} \end{aligned} \quad (1.9)$$

where  $R_t = \int_0^t r_s ds$ . Let us define the default premium of zero-coupon corporate bond as

$$1 - \mathbb{E} (e^{-\Lambda_t} \mid \lambda_0)$$

and consider the risk neutrality in the market, i.e.

$$1 - \mathbb{E}^* (e^{-\Lambda_t} \mid \lambda_0) \tag{1.10}$$

where  $E^*$  denotes the expectation with respect to an equivalent martingale probability measure  $P^*$  and it is assumed that  $r_t$  and  $\lambda_t$  are also independent under  $P^*$ . For details on the equivalent martingale probability measure, we refer you Harrison and Kreps (1979) and Harrison and Pliska (1981). The Esscher transform is employed in order to change the probability measure as it provides us with at least one equivalent martingale probability measure when in our incomplete market. We here offer the definition of the Esscher transform that is adopted from Gerber and Shiu (1996).

**Definition 1.2** *Let  $X(t)$  be a stochastic process such that  $e^{h^* X(t)}$  a martingale with  $h^* \in \mathfrak{R}$ . For a measurable function  $f$ , the expectation of the random variable  $f(X(t))$  with respect to the equivalent martingale probability measure is*

$$E^* [f(X(t))] = E \left[ f(X_t) \frac{e^{h^* X(t)}}{E(e^{h^* X(t)})} \right] = \frac{E[f(X(t)) e^{h^* X(t)}]}{E[e^{h^* X(t)}]} \tag{1.11}$$

where  $E(e^{h^* X(t)}) < \infty$ .

The paper is structured as follows. In section 2, we introduce the shot noise process as an intensity of the Cox process. We derive the Laplace transform of shot noise process,  $\lambda_t$  and aggregate process,  $\Lambda_t$  by piecewise deterministic Markov processes (PDMP) theory. All proofs are referred to Dassios and Jang (2003) where they used the Cox process with shot noise intensity for the pricing of reinsurance contract. Section 3 deals with risk neutrality. We examine how the dynamics of process  $\lambda_t$  and  $\Lambda_t$  change after changing probability measure obtained via the Esscher transform. In section 4, we illustrate the calculation of the default premiums using the asymptotic distribution of shot noise process and exponential jump size distribution. Section 5 concludes.

## 2. Shot noise process and aggregated process

In practice, there are primary events such as the governments' fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the rumours of mergers and acquisitions among firms and September 11 WTC catastrophe etc. that affect the value of the firm's risky debt and may lead to the default as the worst case. In other words, the arrival of the default depends on the frequency and magnitude of these primary events and time period needed to go back to the previous (or better) level of intensity immediately after primary events occur. One of the processes that can be used to measure the impact of primary events is the shot noise process (Cox & Isham

1980, 1986; Klüppelberg and Mikosch 1995 and Dassios and Jang 2003). As time passes, the shot noise process decreases as all firms in the market do their best to avoid being in bankruptcy after the arrival of one of primary events. This decrease continues until another event occurs which will result in a positive jump in the shot noise process. Therefore the shot noise process can be used as the parameter of the doubly stochastic Poisson process to measure the time to default due to primary event, i.e. we will use it as an intensity function to generate the Cox process. We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i \leq t}} y_i e^{-\delta(t-s_i)}$$

where:

- $\lambda_0$  initial value of  $\lambda$
- $y_i$  jump size of primary event  $i$ , where  $E(y_i) < \infty$
- $s_i$  time at which primary event  $i$  occurs, where  $s_i < t < \infty$
- $\delta$  exponential decay
- $\rho$  the rate of primary event arrival.

Some works of insurance application using shot noise process can be found in Klüppelberg and Mikosch (1995), Dassios and Jang (2003) and Jang (2003).

The piecewise deterministic Markov processes (PDMP) theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. From now on, we present definitions and important properties of the Cox, shot noise and aggregated processes with the aid of piecewise deterministic processes theory (Dassios 1987; Dassios and Embrechts 1989 and Dassios and Jang 2003). This theory is used to derive the Laplace transform of shot noise process,  $\lambda_t$  and aggregate process,  $\Lambda_t$ . Before doing this let us remind the definition of PDMP.

**Definition 2.1** *PDMP is a Markov process  $X_t$  with two components  $(\eta_t, \xi_t)$  where  $\eta_t$  takes values in a discrete set  $K \in N$  and given  $\eta_t = n \in K$ ,  $\xi_t$  takes values in an open set  $M_n \subset R^{d(n)}$  for some function  $d : K \rightarrow N$ . The state space of  $X_t$  is equal to  $E = \{(n, z) : n \in K, z \in M_n\}$ . We further assume that for every point  $x = (n, z) \in E$ , there is a unique, deterministic integral curve  $\phi_n(t, z) \subset M_n$ , determined by a differential operator  $\chi_n$  on  $R^{d(n)}$ , such that  $z \in \phi_n(t, z)$ . If for some  $t_0 \in R^+$ ,  $X_{t_0} = (n_0, z_0) \in E$ , then  $\xi_t$ , where  $t \geq t_0$  follows  $\phi_{n_0}(t, z_0)$  until either  $t = T_0$ , some random time with hazard rate of function  $\rho$  or until  $\xi_t = \partial M_{n_0}$ , the boundary of  $M_{n_0}$ . In both cases, the process  $X_t$  jumps, according to a Markov transition measure  $Q$  on  $E$ , to a point  $(n_1, z_1) \in E$ .  $\xi_t$  again follows the deterministic path  $\phi_{n_1}$  till a random time  $T_1$  (independent of  $T_0$ ) or till  $\xi_t = \partial M_{n_1}$ , etc. . . . The jump times  $T_i$  are assumed to satisfy the following condition:*

$$\forall t > 0, \mathbb{E} \left( \sum_i \mathbf{1}_{T_i \leq t} \right) < \infty.$$

The stochastic calculus that will enable us to analyse various models rests on the notion of (extended) generator  $A$  of  $X_t$ . Let  $\Gamma$  denotes the set of boundary points of  $E$ ,  $\Gamma = \{(n, z) : n \in K, z \in \partial M_n\}$ , and let  $A$  be an operator acting on measurable functions  $f : E \cup \Gamma \rightarrow R$  satisfying

(i) The function  $t \rightarrow f(n, \phi_n(t, z))$  is absolutely continuous for  $t \in [0, t(n, z)]$ , for all  $(n, z) \in E$ .

(ii) For all  $x \in \Gamma$ ,  $f(x) = \int_E f(y)Q(x; dy)$  (Boundary condition).

(iii) For all  $t \geq 0$ ,  $E \left\{ \sum_{T_i \leq t} |f(X_{T_i}) - f(X_{T_i-})| \right\} < \infty$ .

Hence the set of measurable functions satisfying (i), (ii) and (iii) form a subset of the domain of the extended generator  $A$ , denoted by  $D(A)$ . Now for piecewise deterministic Markov processes, we can explicitly calculate  $A$  by (Davis (1984), Theorem 5.5)

$$\forall f \in \mathcal{D}(\mathbf{A}) : \mathbf{A}f(x) = \chi f(x) + \rho(x) \int_E \{f(y) - f(x)\}Q(x; dy). \quad (2.1)$$

In some cases, it is important to have time  $t$  as an explicit component of the PDMP. In those cases  $A$  can be decomposed as  $\frac{\partial}{\partial t} + A_t$ , where  $A_t$  is given by (2.1) with possibly time-dependent coefficients.

An application of Dynkin's formula provides us with the following important result (Martingales will always be with respect to the natural filtration  $\sigma\{X_s : s \leq t\}$ ):

(a) If for all  $t$ ,  $f(\cdot, t)$  belongs to the domain of  $A_t$  and  $\frac{\partial}{\partial t}f(x, t) + A_t f(x, t) = 0$ , then process  $f(X_t, t)$  is a martingale.

(b) If  $f$  belongs to the domain of  $A$  and  $Af(x) = 0$ , then  $f(X_t)$  is a martingale.

The generator of the process  $X_t$  acting on a function  $f(X_t)$  belonging to its domain as described above is also given by

$$\mathbf{A}f(X_t) = \lim_{h \rightarrow 0} \frac{\mathbb{E}\{f(X_{t+h}) \mid X_t = x\} - f(X_t)}{h}. \quad (2.2)$$

In other words,  $Af(X_t)$  is the expected increment of the process  $X_t$  between  $t$  and  $t + h$ , given the history of  $X_t$  at time  $t$ . From this interpretation the following inversion formula is plausible, i.e.

$$\mathbb{E}[f(X_{t+h}) \mid X_t = x] - f(X_t) = \int_0^h \mathbb{E}\{\mathbf{A}f(X_s)\} ds \quad (2.3)$$

which is Dynkin's formula.

The three parameters of the shot noise process described are homogeneous in time. We are now going to generalise the shot noise process by allowing the parameters to depend on time. The rate of jump arrivals,  $\rho(t)$ , is bounded on all intervals  $[0, t)$  (no explosions).  $\delta(t)$  is the rate of decay and the distribution function of jump sizes at any time  $t$  is  $G(y; t)$  ( $y > 0$ ) with  $E(y; t) = \mu_1(t) =$

$\int_0^\infty y dG(y; t)$ . We assume that  $\delta(t)$ ,  $\rho(t)$  and  $G(y; t)$  are all Riemann integrable functions of  $t$  and are all positive. The generator of the process  $(\Lambda_t, N_t, \lambda_t, t)$  acting on a function  $f(\Lambda, n, \lambda, t)$  belonging to its domain is given by

$$\begin{aligned} A f(\Lambda, n, \lambda, t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} + \lambda [f(\Lambda, n+1, \lambda, t) - f(\Lambda, n, \lambda, t)] - \delta(t) \lambda \frac{\partial f}{\partial \lambda} \\ &\quad + \rho(t) \left[ \int_0^\infty f(\Lambda, n, \lambda + y, t) dG(y; t) - f(\Lambda, n, \lambda, t) \right]. \end{aligned} \quad (2.4)$$

For  $f(\Lambda, n, \lambda, t)$  to belong to the domain of the generator  $A$ , it is sufficient that  $f(\Lambda, n, \lambda, t)$  is differentiable w.r.t.  $\Lambda, \lambda, t$  for all  $\Lambda, n, \lambda, t$  and that

$$\left| \int_0^\infty f(\cdot, \lambda + y, \cdot) dG(y; t) - f(\cdot, \lambda, \cdot) \right| < \infty.$$

Let us find a suitable martingale in order to derive the Laplace transforms of the distribution of  $\Lambda_t$ .

**Lemma 2.2** *Considering constants  $k$  and  $v$  such that  $k \geq 0$  and  $v \geq 0$ ,*

$$\begin{aligned} &\exp(-v\Lambda_t) \cdot \exp \left[ - \left\{ ke^{\Delta(t)} - ve^{\Delta(t)} \int_0^t e^{-\Delta(r)} dr \right\} \lambda_t \right] \\ &\times \exp \left[ \int_0^t \rho(s) \left[ 1 - \hat{g} \left\{ ke^{\Delta(s)} - ve^{\Delta(s)} \int_0^s e^{-\Delta(r)} dr; s \right\} \right] ds \right] \end{aligned} \quad (2.5)$$

is a martingale where  $\hat{g}(u; s) = \int_0^\infty e^{-u y} dG(y; s)$  and  $\Delta(t) = \int_0^t \delta(s) ds$ .

**Proof.** See Dassios and Jang (2003). ■

Let us assume that  $\delta(t) = \delta$  throughout the rest of this paper.

**Corollary 2.3** *Let  $v_1 \geq 0$  and  $v_2 \geq 0$ . Then*

$$\begin{aligned} &E \left\{ e^{-v_1(\Lambda_{t_2} - \Lambda_{t_1})} e^{-v_2 \lambda_{t_2}} | \Lambda_{t_1}, \lambda_{t_1} \right\} \\ &= \exp \left[ - \left\{ \frac{v_1}{\delta} + \left( v_2 - \frac{v_1}{\delta} \right) e^{-\delta(t_2 - t_1)} \right\} \lambda_{t_1} \right] \\ &\quad \times \exp \left[ - \int_{t_1}^{t_2} \rho(s) \left[ 1 - \hat{g} \left\{ \frac{v_1}{\delta} + \left( v_2 - \frac{v_1}{\delta} \right) e^{-\delta(t_2 - s)}; s \right\} \right] ds \right]. \end{aligned} \quad (2.6)$$

**Proof.** See Dassios and Jang (2003). ■

Now we can easily obtain the Laplace transforms of the distribution of  $\lambda_t, \Lambda_t$ .

**Corollary 2.4** *The Laplace transforms of the distribution of  $\lambda_t$  and  $\Lambda_t$  are given by*

$$E \{ e^{-v\lambda_{t_2}} | \lambda_{t_1} \} = \exp \left[ -v e^{-\delta(t_2-t_1)} \lambda_{t_1} \right] \quad (2.7)$$

$$\times \exp \left[ - \int_{t_1}^{t_2} \rho(s) \left[ 1 - \hat{g} \left\{ v e^{-\delta(t_2-s)}; s \right\} \right] ds \right],$$

$$E \left\{ e^{-v(\Lambda_{t_2}-\Lambda_{t_1})} | \lambda_{t_1} \right\} \quad (2.8)$$

$$= \exp \left[ -\frac{v}{\delta} \left\{ 1 - e^{-\delta(t_2-t_1)} \right\} \lambda_{t_1} \right]$$

$$\times \exp \left[ - \int_{t_1}^{t_2} \rho(s) \left[ 1 - \hat{g} \left\{ \frac{v}{\delta} \left( 1 - e^{-\delta(t_2-s)} \right); s \right\} \right] ds \right].$$

**Proof.** See Dassios and Jang (2003). ■

Let us obtain the asymptotic distributions of  $\lambda_t$  at time  $t$  from (2.8), provided that the process started sufficiently far in the past. In this context we interpret it as the limit when  $t \rightarrow -\infty$ . In other words, if we know  $\lambda$  at  $-\infty$  and no information between  $-\infty$  to present time  $t$ ,  $-\infty$  asymptotic distribution of  $\lambda_t$  can be used as the distribution of  $\lambda_t$ .

**Lemma 2.5** *Assume that  $\lim_{t \rightarrow -\infty} \rho(t) = \rho$  and  $\lim_{t \rightarrow -\infty} \mu_1(t) = \mu_1$ . Then the  $-\infty$  asymptotic distribution of  $\lambda_t$  has Laplace transform*

$$E (e^{-v\lambda_{t_1}}) = \exp \left[ - \int_{-\infty}^{t_1} \rho(s) \left[ 1 - \hat{g} \left\{ v e^{-\delta(t_1-s)}; s \right\} \right] ds \right]. \quad (2.9)$$

**Proof.** See Dassios and Jang (2003). ■

It will be interesting to find the Laplace transforms of the distribution of  $\lambda_t$  and  $\Lambda_t$  at time  $t$ , using a specific jump size distribution of  $G(y; t)$  ( $y > 0$ ). We use an exponential jump size distribution, i.e.  $g(y; t) = (\alpha + \gamma e^{\delta t}) e^{-(\alpha + \gamma e^{\delta t})y}$ ,  $y > 0$ ,  $-\alpha e^{-\delta t} < \gamma \leq 0$ . Let us assume that  $\rho(t) = \rho \frac{\alpha}{\alpha + \gamma e^{\delta t}}$ . The reason for this particular assumption will become apparent later when we change the probability measure.

**Theorem 2.6** *Let the jump size distribution be exponential, i.e.  $g(y; t) = (\alpha + \gamma e^{\delta t}) \exp \{ -(\alpha + \gamma e^{\delta t})y \}$ ,  $y > 0$ ,  $\alpha e^{-\delta t} < \gamma \leq 0$ , and assume that  $\rho(t) = \rho \frac{\alpha}{\alpha + \gamma e^{\delta t}}$ . Then*

$$E \{ e^{-v\lambda_{t_1}} | \lambda_{t_0} \} = \exp \left\{ -v \lambda_{t_0} e^{-\delta(t_1-t_0)} \right\} \cdot \left( \frac{\gamma e^{\delta t_0} + \alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0} + \alpha} \right)^{\frac{t_1-t_0}{\delta}}$$

$$\times \left( \frac{\gamma e^{\delta t_0} + v e^{-\delta(t_1-t_0)} + \alpha}{\gamma e^{\delta t_0} + (v + \alpha) e^{-\delta(t_1-t_0)}} \right)^{\frac{t_1-t_0}{\delta}}, \quad (2.10)$$



$$\begin{aligned}
E \left\{ e^{-v(\Lambda_{t_2} - \Lambda_{t_1})} | \lambda_{t_1} \right\} &= \exp \left[ -\frac{v}{\delta} \left\{ 1 - e^{-\delta(t_2 - t_1)} \right\} \lambda_{t_1} \right] \\
&\times \left( \frac{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}}{\gamma e^{\delta t_1} + \alpha} \right)^{\frac{\rho}{\delta}} \\
&\times \left( \frac{\gamma e^{\delta t_1} + \alpha + \frac{v}{\delta} (1 - e^{-\delta(t_2 - t_1)})}{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}} \right)^{\frac{\alpha \rho}{\delta \alpha + v}} \quad (2.11)
\end{aligned}$$

If  $\lambda_t$  is '  $-\infty$ ' asymptotic,

$$E(e^{-v\lambda_{t_1}}) = \left( \frac{\gamma + \alpha e^{-\delta t_1}}{\gamma + (v + \alpha) e^{-\delta t_1}} \right)^{\frac{\rho}{\delta}}, \quad (2.12)$$

$$\begin{aligned}
E \left\{ e^{-v(\Lambda_{t_2} - \Lambda_{t_1})} \right\} &= \left( \frac{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}}{\gamma e^{\delta t_1} + \alpha + \frac{v}{\delta} (1 - e^{-\delta(t_2 - t_1)})} \right)^{\frac{\rho}{\delta}} \\
&\times \left( \frac{\gamma e^{\delta t_1} + \alpha + \frac{v}{\delta} (1 - e^{-\delta(t_2 - t_1)})}{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}} \right)^{\frac{\alpha \rho}{\delta \alpha + v}} \quad (2.13)
\end{aligned}$$

**Proof.** See Dassios and Jang (2003). ■

### 3. No-arbitrage, the Esscher transform and change of probability measure

In general, the Esscher transform is defined as a change of probability measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The parameters involved for the Esscher transform are determined so that the process is a martingale under the new probability measure. We will examine an equivalent martingale probability measure obtained via the Esscher transform (Gerber and Shiu 1996 and Dassios and Jang 2003).

If the market is complete, the fair price/premium of a contingent claim is the expectation with respect to exactly one equivalent martingale probability measure (i.e. by assuming that there is an absence of arbitrage opportunities in the market). For example, when the underlying stochastic process follows geometric Brownian motion or homogeneous Poisson process, we can obtain the fair price/premium with respect to a unique equivalent martingale probability measure. However, as the underlying stochastic process for the counting process is the Cox process with shot noise intensity, we will have infinitely many equivalent martingale probability measures. In other words, we will have several choices of equivalent martingale probability measures to decide default premium as the market is incomplete. It is not the purpose of this paper to decide which is the appropriate one to use. The credit rating agency, bank and trader's attitude towards risk determines which equivalent martingale probability measure should

be used. The attractive thing about the Esscher transform is that it provides us with at least one equivalent martingale probability measure in incomplete market situations.

Let  $M_t$  be the total number of primary events up to time  $t$ . We will assume that default and primary events do not occur at the same time. The generator of the process  $(\Lambda_t, N_t, \lambda_t, M_t, t)$  acting on a function  $f(\Lambda, n, \lambda, m, t)$  belonging to its domain is given by

$$\begin{aligned} & A f(\Lambda, n, \lambda, m, t) \\ = & \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} + \lambda [f(\Lambda, n+1, \lambda, m, t) - f(\Lambda, n, \lambda, m, t)] \\ & - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho \left[ \int_0^\infty f(\Lambda, n, \lambda+y, m+1, t) dG(y) - f(\Lambda, n, \lambda, m, t) \right]. \end{aligned} \quad (3.1)$$

Clearly, for  $f(\Lambda, n, \lambda, m, t)$  to belong to the domain of the generator  $A$ , it is essential that  $f(\Lambda, n, \lambda, m, t)$  is differentiable w.r.t.  $\Lambda, \lambda, t$  for all  $\Lambda, n, \lambda, m, t$  and that

$$\left| \int_0^\infty f(\cdot, \lambda+y, \cdot) dG(y) - f(\cdot, \lambda, \cdot) \right| < \infty.$$

**Lemma 3.1** *Considering constants  $\theta^*$ ,  $\psi^*$  and  $\gamma^*$  such that  $\theta^* \geq 1$ ,  $\psi^* \geq 1$  and  $\gamma^* \leq 0$ ,*

$$\theta^{*N_t} \exp\{-(\theta^* - 1)\Lambda_t\} \psi^{*M_t} \exp(-\gamma^* \lambda_t e^{\delta t}) \exp\left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds\right] \quad (3.2)$$

is a martingale.

**Proof.** From (3.1),  $f(\Lambda, n, \lambda, m, t)$  has to satisfy  $Af = 0$  for  $f(\Lambda_t, N_t, \lambda_t, M_t, t)$  to be a martingale. Trying  $\theta^{*n} e^{\phi^* \Lambda} \psi^{*m} \exp(-\gamma^* \lambda e^{\delta t}) e^{A(t)}$  we get the equation

$$A'(t) + \lambda \phi^* + \lambda \{\theta^* - 1\} + \rho \{\psi^* \hat{g}(\gamma^* e^{\delta t}) - 1\} = 0 \quad (3.3)$$

and solving (3.3) we get

$$\phi^* = -\{\theta^* - 1\} \quad \text{and} \quad A(t) = \rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds$$

and the result follows. ■

Let us examine how the generator  $A^*$  of the process  $(\Lambda_t, N_t, \lambda_t, M_t, t)$  acting on a function  $f(\Lambda, n, \lambda, m, t)$  with respect to the equivalent martingale probability measure can be obtained.

**Lemma 3.2** *Let  $v^*$  be a nonnegative constant. Assuming that  $f(\Lambda, n, \lambda, m, t) = f(\lambda, t)$  for all  $\Lambda, n, m$  and that  $e^{-v^* X_t}$  is a martingale with  $X_t$  an adapted process. The generator  $A^*$  of the process  $(\lambda_t, t)$  acting on a function  $f(\lambda, t)$  with respect to the equivalent martingale measure is given by*

$$A^* f(\lambda, t) = \frac{A \{f(\lambda, t) e^{-v^* X_t}\}}{e^{-v^* X_0}}. \quad (3.4)$$

**Proof.** The generator of the process  $(\lambda_t, t)$  acting on a function  $f(\lambda, t)$  with respect to the equivalent martingale probability measure is

$$A^* f(\lambda, t) = \lim_{dt \downarrow 0} \frac{E^* [f(\lambda_{t+dt}, t+dt) | \lambda_t = \lambda] - f(\lambda, t)}{dt}. \quad (3.5)$$

We will use  $\frac{e^{-v^* X_t}}{E(e^{-v^* X_t})}$  as the Radon-Nikodym derivative to define equivalent martingale probability measure. Hence, the expected value of  $f(\lambda_{t+dt}, t+dt)$  given  $\lambda_t = \lambda$  with respect to the equivalent martingale probability measure is

$$E^* [f(\lambda_{t+dt}, t+dt) | \lambda_t = \lambda] = \frac{E [f(\lambda_{t+dt}, t+dt) e^{-v^* X_{t+dt}} | \lambda_t = \lambda]}{E (e^{-v^* X_{t+dt}} | \lambda_t = \lambda)}. \quad (3.6)$$

Since  $e^{-v^* X_t}$  in (3.6) is a martingale, it becomes

$$\begin{aligned} & E^* [f(\lambda_{t+dt}, t+dt) | \lambda_t = \lambda] \\ &= \frac{f(\lambda, t) e^{-v^* \lambda} + \int_t^{t+dt} E [A f(\lambda_s, s) e^{-v^* X_s} | \lambda_t = \lambda] ds}{e^{-v^* X_0}}. \end{aligned} \quad (3.7)$$

Set (3.7) in (3.5) then

$$A^* f(\lambda, t) = \frac{1}{e^{-v^* X_0}} \lim_{dt \downarrow 0} \int_t^{t+dt} \frac{E [A f(\lambda_s, s) e^{-v^* X_s} | \lambda_t = \lambda] ds}{dt}. \quad (3.8)$$

Therefore, from Dynkin's formula (3.4) follows immediately. ■

Now let us look at how the dynamics of process  $\lambda_t$  and  $\Lambda_t$  change after changing probability measure by obtaining the generator  $\mathbf{A}^*$  of the process  $(\Lambda_t, N_t, \lambda_t, M_t, t)$  acting on a function  $f(\Lambda, n, \lambda, m, t)$  with respect to the equivalent martingale probability measure. This is the key result that we require to establish an arbitrage-free default premium under our equivalent martingale measure. As Dassios and Jang (2003) have shown the change of dynamics of process  $\lambda_t$  and  $\Lambda_t$  with respect to the equivalent martingale probability measure, we offer the theorem adopted from their studies (see theorem 3.5 in section 3).

**Theorem 3.3** *Consider constants  $\theta^*, \psi^*$  and  $\gamma^*$  such that  $\theta^* \geq 1, \psi^* \geq 1$*

and  $\gamma^* \leq 0$ . Suppose that  $\hat{g}(\gamma^* e^{\delta t}) < \infty$ . Then

$$\begin{aligned}
& A^* f(\Lambda, n, \lambda, m, t) \\
= & \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial \Lambda} + \theta^* \lambda \{f(\Lambda, n+1, \lambda, m, t) - f(\Lambda, n, \lambda, m, t)\} \\
& - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho^*(t) \left\{ \int_0^\infty f(\Lambda, n, \lambda + y, m+1, t) dG^*(y; t) - f(\Lambda, n, \lambda, m, t) \right\}
\end{aligned} \tag{3.9}$$

where  $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$  and  $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ .

**Proof.** See Dassios and Jang (2003). ■

Theorem 3.5 yields the following:

- (i) The intensity function  $\lambda_t$  has changed to  $\theta^* \lambda_t$ ;
- (ii) The rate of jump arrival  $\rho$  has changed to  $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$  (it now depends on time);
- (iii) The jump size measure  $dG(y)$  has changed to  $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$  (it now depends on time).

In other words, the risk-neutral Esscher measure is the measure with respect to which  $N_t$  becomes the Cox process with parameter  $\theta^* \lambda_t$  where three parameters of the shot noise process  $\lambda_t$  are  $\delta$ ,  $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$ ,  $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ .

In practice, the credit rating agency such as Moody's and Standard and Poor's, banks and traders need to calculate the default premium using  $\theta^* > 1$ ,  $\psi^* > 1$  and  $\gamma^* < 0$ . This results in the agencies, banks and traders assuming that there will be a higher value of intensity itself, more primary events occurring in a given period of time and a higher value of jump size of intensity. These assumptions are necessary, as the agencies, banks and traders have to consider the risks involved in incomplete market. If  $\theta^* = 1$ ,  $\psi^* = 1$ , and  $\gamma^* = 0$  then net default premium is calculated without considering any risks involved in incomplete market. However, as expected, we have quite a flexible family of equivalent probability measures by the combination of  $\theta^*$ ,  $\psi^*$  and  $\gamma^*$ . It means that the agencies, banks and traders have various ways of obtaining the non-arbitrage default premium (i.e. by changing equivalent martingale probability measures using the combination of  $\theta^*$ ,  $\psi^*$  and  $\gamma^*$ ).

Let us derive the Laplace transform of the ' $-\infty$ ' asymptotic distribution of  $\Lambda_t$  with respect to the equivalent martingale probability measure, i.e.  $\mathbb{E}^*(e^{-\Lambda_t})$ . We will assume that the jump size distribution is exponential, i.e.  $g(y) = \alpha e^{-\alpha y}$ ,  $y > 0$ ,  $\alpha > 0$  and that  $\lambda_t$  is ' $-\infty$ ' asymptotic. Therefore we can obtain that  $g^*(y; t) = (\alpha + \gamma^* e^{\delta t}) \exp\{-(\alpha + \gamma^* e^{\delta t}) y\}$ ,  $y > 0$ ,  $-\alpha e^{-\alpha y} < \gamma^* \leq 0$  and  $t < \frac{1}{\delta} \ln\left(-\frac{\alpha}{\gamma^*}\right)$  since  $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ .

**Corollary 3.4** *Let the jump size distribution be exponential. Consider constants  $\nu$ ,  $\theta^*$ ,  $\psi^*$  and  $\gamma^*$  such that  $\nu \geq 0$ ,  $\theta^* \geq 1$ ,  $\psi^* \geq 1$  and  $\gamma^* \leq 0$ . Furthermore if  $\lambda_t$  is  $-\infty$  asymptotic, then*

$$E^* \left\{ e^{-v(\Lambda_{t_2} - \Lambda_{t_1})} \right\} = \left( \frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}}{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^* \nu}{\delta} (1 - e^{-\delta(t_2 - t_1)})} \right)^{\frac{\psi^* \rho}{\delta}} \\ \times \left( \frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^* \nu}{\delta} (1 - e^{-\delta(t_2 - t_1)})}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2 - t_1)}} \right)^{\frac{\alpha \psi^* \rho}{\delta \alpha + \theta^* \nu}} \quad (3.10)$$

where  $0 < t_1 < t_2 < t$ .

**Proof.** From Theorem 3.3, (1.4) and (2.13), the result follows immediately. ■

#### 4. Measuring default premium using the Cox process with shot noise intensity

Up to now it is assumed that the frequency and magnitude of primary events and time period needed to go back to the previous (or better) level of intensity immediately after primary events occur are the same among all firms. However some of these primary events might not affect at all to a specific firm e.g. ‘AAA’ rating firm. Also even if primary events affect firms’ default intensities, their magnitude should be different to each firms. Time period needed to go back to the previous (or better) level of intensity also need to be discriminated among firms. Therefore an arbitrage-free default premium of zero-coupon corporate bond issued by the firm  $i$  is given by

$$1 - \mathbb{E}^* \left( e^{-\Lambda_t^i} \right) \\ = 1 - \left( \frac{\gamma^* + \alpha^i e^{-\delta^i t}}{\gamma^* + \alpha^i + \frac{\theta^*}{\delta^i} (1 - e^{-\delta^i t})} \right)^{\frac{\psi^* \rho^i}{\delta^i}} \\ \times \left( \frac{\gamma^* + \alpha^i + \frac{\theta^*}{\delta^i} (1 - e^{-\delta^i t})}{\gamma^* + \alpha^i e^{-\delta^i t}} \right)^{\frac{\alpha^i \psi^* \rho^i}{\delta^i \alpha^i + \theta^*}} \quad (4.1)$$

Let us now illustrate the calculation of an arbitrage-free default premium of zero-coupon corporate bond issued by firm  $i$  using (4.1).

**Example 4.1**

The parameter values used to calculate (4.1) are

$$\theta^* = 1.1, \psi^* = 1.1, \gamma^* = -0.01, \alpha^i = 10, \delta^i = 0.5, \rho^i = 4 \text{ and } t = 1.$$

Then an arbitrage-free default premium is as follows:

$$1 - \mathbb{E}^* \left( e^{-\Lambda_t^i} \right) = 1 - 0.396 = 0.604.$$

Using  $\theta^* = 1$ ,  $\psi^* = 1$  and  $\gamma^* = 0$ , the net default premium is given by

$$1 - \mathbb{E} \left( e^{-\Lambda_t^i} \right) = 1 - 0.46409 = 0.53591$$

and the difference between arbitrage-free default premium and net default premium is given by

$$\mathbb{E} \left( e^{-\Lambda_t^i} \right) - \mathbb{E}^* \left( e^{-\Lambda_t^i} \right) = 0.46409 - 0.396 = 0.06809$$

**Example 4.2**

We will now examine the effect on arbitrage-free default premium caused by changes in the value of  $\alpha^i$ ,  $\delta^i$  and  $\rho^i$ . The calculation of arbitrage-free default premiums are shown in Table 4.1 assuming other parameter values are the same as in Example 4.1.

Table 4.1.

	Default premium
$\alpha^i = 0.1$	1
$\alpha^i = 20$	0.37705
$\delta^i = 0.1$	0.98999
$\delta^i = 5$	0.09349
$\rho^i = 0$	0
$\rho^i = 8$	0.84318

**Example 4.3**

We will now examine the effect on arbitrage-free default premium caused by changes in the value of  $\theta^*$ ,  $\psi^*$  and  $\gamma^*$ . The calculation of default premiums are shown in Table 4.2, Table 4.3 and Table 4.4 assuming other parameter values are the same as in Example 4.1.

Table 4.2.

$\theta^*$	$\psi^* = 1.1$ $\gamma^* = -0.01$
1.0	0.57066
1.1	0.604
1.2	0.63453
1.3	0.66249
1.4	0.68812
1.5	0.71163

Table 4.3.

$\psi^*$	$\theta^* = 1.1$ $\gamma^* = -0.01$
1.0	0.56921
1.1	0.604
1.2	0.63598
1.3	0.66538
1.4	0.69241
1.5	0.71725

Table 4.4.

$\gamma^*$	$\theta^* = 1.1$ $\psi^* = 1.1$
0.00	0.60354
-0.01	0.604
-0.02	0.60446
-0.03	0.60492
-0.04	0.60538
-0.05	0.60584

## 5. Conclusion

Using the intensity-based framework, i.e. employing the Cox process with shot noise intensity, we obtained the default premium of zero-coupon corporate bond assuming that risk-free spot interest rate,  $r_t$  and the intensity process,  $\lambda_t$  are independent under the original measure and risk neutral measure. In order to achieve the market neutrality, we examined how the dynamics of the shot noise process  $\lambda_t$  and the aggregate process  $\Lambda_t$  change after changing probability measure obtained via the Esscher transform. We witnessed that there are various ways to quantify the risk involved as the market is incomplete. By discriminating of three parameters of the shot noise intensity, i.e. the frequency and magnitude of primary events and time period needed to go back to the previous (or better) level of intensity immediately after primary events occur for each firm, we illustrated the calculation of arbitrage-free default premiums of zero-coupon corporate bond issued by the firm  $i$ .

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