Measuring capital charge for a credit loan portfolio: VaR and TCE

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Abstract. Due to the new Basel Capital Accord, the banks need to develop a risk management tool for losses arising from credit loan portfolios before 2006. In order to quantify the aggregate loss from a credit loan portfolio, we employ an actuarial risk model. The Value at Risk (VaR) and tail conditional expectation (TCE, also known as TailVaR) are used to obtain the capital charge for a credit loan portfolio. It is assumed that the default arrival process follows the Poisson process and the loss sizes are assumed to be independent and identical truncated exponential. Applying the piecewise deterministic Markov processes theory, we obtain the explicit expression of the Laplace transform of the distribution of the aggregate losses. Transform analysis techniques from financial option pricing theory is used to derive the VaR and TCE numerically based on the Laplace transform of the distribution of the aggregate losses.

JEL classfication: C16; C24; G21; G28

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1. Introduction

Traditionally, banks have managed their credit risks at origination where credit risk is the risk that an obligor does not honour his payment obligations. Once the risk was launched, it remained on the books until the loan was paid off or the obligor defaulted. When the obligor defaulted, some banks could not have the capacity to meet their obligations to pay principal and interest to obligees, which brought about social unrest. In order to make it sure for banks to pay obligees back all or part of principal and interest, a capital amount for credit risk is required to the banks (eventually starting in 2006) according to the consultative document on the New Basel Capital Accord.

Focusing on the capital amount for a credit loan portfolio such as credit card loan or mortgage loan portfolio, we employ an actuarial risk model (Bühlmann 1970; Gerber 1979; and Beard et al. 1984). Let X_i, $i = 1, 2, \cdots$, be the loss amount, i.e. the amount of money that bank suffers when the i-th borrower defaults, which are assumed to be independent and identically distributed with a distribution function $H(x)$ $(0 < x < \infty)$. Ignoring the rate of interest, the total loss up to time t from this portfolio is given by

$$L_t = \sum_{i=1}^{N_t} X_i \tag{1.1}$$

where N_t is the total number of losses up to time t.

We use the VaR (or the q-quantile) denoted by l_q that is the smallest value satisfying

$$\Pr(L_t \geq l_q) = 1 - q \tag{1.2}$$

as it was discussed as a risk measure to decide the capital amount required for next t years' credit risk in the consultative document on the New Basel Capital Accord. We can think of the VaR as the amount of extra-capital that a bank needs to hold in order to reduce the probability of going bankrupt to $1 - q$. We also derive the tail conditional expectation (TCE, also known as TailVaR) defined by

$$E(L_t \mid L_t \geq l_q) \tag{1.3}$$

to obtain the capital amount required for next t years' credit risk as a coherent risk measure that satisfies the following definition (Artzner et. al., 1999).

Definition *A coherent risk measure is a function* $\rho : L_t \to \mathbb{R}$ such that

(i) *Subadditivity - For all random losses* L_t^1 *and* L_t^2,

$$\rho(L_t^1 + L_t^2) \leq \rho(L_t^1) + \rho(L_t^2);$$

(ii) *Monotonicity - If* $L_t^1 \leq L_t^2$ *almost surely, then*

$$\rho(L_t^1) \leq \rho(L_t^2);$$

(iii) *Positive Homogeneity - For all* $a \geq 0$,

$$\rho(aL_t^1) = a\ \rho(L_t^1);$$

(iv) *Translation Invariance - For any constant* b,

$$\rho(L_t^1 + b) = \rho(L_t^1) + b.$$

The VaR is not a coherent risk measure as it does not hold the property of subadditivity. In other words, it violates a natural requirement that the diversification of portfolios provides us with less risk. It can be easily found that (1.3) can be rewritten as

$$E\left(L_t \mid L_t > l_q\right) = l_q + E\left(L_t - l_q \mid L_t > l_q\right)$$

and it obviously shows that the TCE is more conservative risk measure than the VaR as

$$E\left(L_t \mid L_t > l_q\right) \geq l_q.$$

The TCE tells us how great the losses are as it takes an average over the worst cases and therefore takes into account the tail distribution of the losses. However, the VaR only looks at a quantile and it does not tell us how great losses are. In practice, the regulator determines which risk measure should be used for the financial institutions to decide the capital amount needed against credit risk. Hence it is not the purpose of this paper to decide which is the appropriate one to use.

Now let us consider the maximum amount of loan, denoted by $U < \infty$, to which the bank allows individual borrowers to have their credit loans. In other words, the bank can classifies a credit loan portfolio where its maximum amount to lend is identical (or very close) one another. So we need to use a truncated loss size distribution denoted by $H_U(x)$ $(0 < x < U)$. Hence (1.1) is expressed by

$$L_t^U = \sum_{i=1}^{N_t} X_i^U \tag{1.4}$$

where L_t^U denotes the total loss up to time t in which individual loss amounts are capped at U and (1.2) where the q-quantile is denoted by m_q, becomes

$$\Pr(L_t^U \geq m_q) = 1 - q \tag{1.5}$$

and (1.3) becomes

$$E(L_t^U \mid L_t^U \geq m_q) \tag{1.6}$$

Notice that L_t^U is not necessarily bounded above, even though individual loss amounts are capped at U, as total number of losses up to time t, N_t can be ∞.

We assume that the claim arrival process N_t follows a Poisson process with loss frequency rate λ. In order to evaluate (1.5) and (1.6), we will need to obtain the distribution of the aggregate loss, L_t^U. Unfortunately, it is known that it is not possible for us to obtain the distribution itself of the aggregate losses explicitly. So we derive the general form of the Laplace transform of the distribution of the aggregate loss, applying the piecewise deterministic Markov processes theory. Based on this general form of the Laplace transform, we obtain the explicit Laplace transform in the case when the loss size is truncated exponential as the loan is capped at U. Transform analysis techniques from financial option pricing theory (Duffie et al. 2000) is then used to derive the VaR and TailVaR numerically as a capital charge for a credit loan portfolio.

The paper is organised as follows. In section 2, we derive the general form of the Laplace transform of the distribution of aggregate loss process of L_t^U by using piecewise deterministic Markov processes (PDMP) theory. Section 3 we derive the explicit expression of the Laplace transform of the distribution of total loss using truncated exponential distribution for loss sizes. In section 4, we illustrate the calculation of the VaR and TCE as risk measures of the capital amount required for next year's credit risk using Transform Analysis technique from the financial option pricing literature. Section 5 concludes.

2. **The Laplace transform of the distribution of aggregate loss**

The total loss up to time t where individual loss amounts are capped at U, L_t^U is a piecewise deterministic Markov process (also a Lévy process). For details for Lévy processes, we refer you Bertoin (1998). The piecewise deterministic Markov processes theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. From now on, we present definitions and important properties of L_t^U with the aid of piecewise deterministic processes theory (Dassios and Embrechts 1989; Rolski et al. 1998 and Dassios and Jang 2003). This theory is used to derive the general form of the Laplace transform of the distribution of aggregate loss process of L_t^U.

The generator of the process $\left(L_t^U, t\right)$ acting on a function $f\left(l^U, t\right)$ belonging to its domain is given by

$$A\ f\left(l^U, t\right) = \frac{\partial f}{\partial t} + \lambda \left[\int_0^U f\left(l^U + x, t\right) dH_U\left(x\right) - f\left(l^U, t\right) \right] \tag{2.1}$$

where $f : (0, \infty) \times \Re^+ \rightarrow (0, \infty)$. For $f\left(l^U, t\right)$ to belong to the domain of the generator A, it is sufficient that $f\left(l^U, t\right)$ is differentiable w.r.t. l^U, t for all l^U, t and that

$$\left| \int_0^U f\left(l^U + x, t\right) dH_U\left(x\right) - f\left(l^U, t\right) \right| < \infty.$$

Let us find a suitable martingale in order to derive the Laplace transforms of the distribution of L_t^U.

Lemma *Considering a constant* $v \geq 0$,

$$\exp\left(-\nu L_t^U\right) \exp\left[\lambda \int_0^t \left\{1 - \hat{h}_U\left(\nu\right)\right\} ds\right] \tag{2.2}$$

is a martingale where $\hat{h}_U\left(\nu\right) = \int_0^U e^{-\nu x} dH_U(x)$.

Proof. From (2.1), $f\left(l^U, t\right)$ has to satisfy $Af = 0$ for $f(L_t^U, t)$ to be a martingale. Setting $f\left(l^U, t\right) = \exp\left(-\nu l^U\right) e^{B(t)}$ we get the equation

$$B'(t) + \lambda \left\{\hat{h}_U\left(\nu\right) - 1\right\} = 0 \tag{2.3}$$

and the solution is

$$B\left(t\right) = \lambda \int_0^t \left\{1 - \hat{h}_U\left(\nu\right)\right\} ds \tag{2.4}$$

by which the result follows. ■

Using the martingale obtained in Lemma, we can easily obtain the general form of the Laplace transform of the distribution of L_t^U at time t,

$$E\left\{e^{-\nu L_t^U} | L_0^U\right\} = \exp\left(-\nu L_0^U\right) \exp\left[-\lambda \int_0^t \left\{1 - \hat{h}_U\left(\nu\right)\right\} ds\right]. \tag{2.5}$$

For simplicity, we assume that L_0^U, that can be considered as the total loss up to present time 0, is 0 then (2.5) is given by

$$E\left(e^{-\nu L_t^U}\right) = \exp\left[-\lambda \int_0^t \left\{1 - \hat{h}_U(\nu)\right\} ds\right]. \tag{2.6}$$

In order to obtain the explicit expression of the Laplace transform of the distribution of L_t^U at time t, let us use truncated exponential loss size distribution, i.e.

$$h_U(x) = \left(\frac{1}{1 - e^{-\alpha U}}\right) \alpha e^{-\alpha x},\ 0 < x < U,\ \alpha > 0 \tag{2.7}$$

then the Laplace transform of the distribution of L_t^U with capped loss size U is given by

$$\exp\left\{-\lambda \left(\frac{\nu + \left(e^{-\nu U} - \alpha - \nu\right) e^{-\alpha U}}{(\alpha + \nu)\left(1 - e^{-\alpha U}\right)}\right) t\right\}. \tag{2.8}$$

It will be interesting to find more Laplace transforms of the distribution of L_t^U at time t, using other loss size distributions of $H_U(x)$ $(0 < x < U)$ such as log-normal, gamma and Pareto. However it is obvious that we will not be able to derive explicit forms of the Laplace transform of the distribution of L_t^U if we apply heavy-tailed distributions. So in order to obtain the moments of L_t^U, i.e. mean and variance and risk measures, i.e. VaR and TCE, we need to apply numerical framework using analytical forms of the Laplace transform of the distribution of L_t^U.

If we differentiate (2.8) with respect to ν and put $\nu = 0$, we can obtain the mean of the aggregate loss, where individual loss amounts are capped at U, i.e.

$$E(L_t^U) = \lambda \left(\frac{1 - (1 + \alpha U) e^{-\alpha U}}{\alpha\left(1 - e^{-\alpha U}\right)}\right) t \tag{2.9}$$

and higher moments can be obtained by differentiating it further, i.e.

$$Var(L_t^U) = 2\lambda \left(\frac{1 - (1 + \alpha U) e^{-\alpha U}}{\alpha^2\left(1 - e^{-\alpha U}\right)}\right) t - \lambda \frac{U^2 e^{-\alpha U}}{\left(1 - e^{-\alpha U}\right)} t. \tag{2.10}$$

3. Value at Risk (VaR) and tail conditional expectation (TCE) via transform analysis

Let us look at how the Laplace transform derived above can be used to evaluate VaR and TCE as risk measures of the capital amount required for a credit loan portfolio for next t years. From (1.5), the VaR can be expressed as

$$\mathrm{VaR}\left(q,\ L_t^U\right) = \inf\left\{m_q \in \mathbb{R}:\ \Pr(L_t^U > m_q) \le 1 - q\right\} \tag{3.1}$$

where q is predetermined by the regulator. From (1.6), the TCE can be expressed by

$$\begin{aligned} \mathrm{TCE}\left(q,\ L_t^U\right) &= E\left\{L_t^U \mid L_t^U \ge \mathrm{VaR}\left(q,\ L_t^U\right)\right\} \\ &= \frac{E\left[L_t^U\ I\left\{L_t^U \ge \mathrm{VaR}\left(q,\ L_t^U\right)\right\}\right]}{1 - q} \end{aligned} \tag{3.2}$$

where $I(\cdot)$ is the indicator function.

Now as it is not possible for us to obtain the distribution itself of L_t^U explicitly, we employ transform analysis techniques developed by Heston (1993) and Duffie et al. (2000) to approximate the VaR and TCE numerically. We highlight their methodology as applied to our problem below.

We know from the previous section the Laplace transform of L_t^U, denoted by $\xi(-\nu)$,

$$\xi(-\nu) = E\left(e^{-\nu L_t^U}\right)$$

and can consider the function

$$\widehat{\Psi}(z)=\int_0^\infty e^{izl}d\left(\int_0^l dF_{L_t^U}(x)\right)$$

and hence

$$\begin{aligned}\widehat{\Psi}(z) &= \int_0^\infty e^{izl}dF_{L_t^U}(l)\\ &= E\left[e^{izL_t^U}\right]=\xi(iz).\end{aligned}$$

Recall that

$$\begin{aligned}\xi(-\nu) &= E\left(e^{-\nu L_t^U}\right)\\ &= \exp\left\{-\lambda\left(\frac{\nu+\left(e^{-\nu U}-\alpha-\nu\right)e^{-\alpha U}}{(\alpha+\nu)\left(1-e^{-\alpha U}\right)}\right)t\right\}\end{aligned}$$

and the standard Lévy inversion formula gives

$$\begin{aligned}E\left\{I\left(L_t^U<l\right)\right\} &= P\left(L_t^U<l\right)=\Psi(l)\\ &= \frac{\widehat{\Psi}(0)}{2}-\frac{1}{\pi}\int_0^\infty\frac{1}{z}\operatorname{Im}\left(e^{-izl}\widehat{\Psi}(z)\right)dz\end{aligned}$$

from which we can easily obtain $P\left(L_t^U\geq l\right)$.

Consider now another function

$$\widehat{\Theta}(z)=\int_0^\infty e^{izl}d\left(\int_0^l x dF_{L_t^U}(x)\right).$$

Assume $\int|\Theta(y)|\,dy<\infty$ and we find that

$$\begin{aligned}\widehat{\Theta}(z) &= \int_0^\infty e^{izl}l\;dF_{L_t^U}(l)\\ &= E\left[L_t^U\;e^{izL_t^U}\right],\end{aligned}$$

which can be calculated as follows. Differentiating $\xi(-\nu)$ with respect to $-\nu$ gives

$$\begin{aligned}-\frac{\partial}{\partial\nu}\xi(-\nu) &= E\left[L_t^U\;e^{-\nu L_t^U}\right]\\ &= \exp\left\{-\lambda\left(\frac{\left(\alpha e^{-\nu U}-\alpha-\nu\right)e^{-\alpha U}+\nu}{(\alpha+\nu)\left(1-e^{-\alpha U}\right)}\right)t\right\}\\ &\quad\times\left\{\frac{1-\left(1+\alpha U e^{-\nu U}\right)e^{-\alpha U}}{(\alpha+\nu)\left(1-e^{-\alpha U}\right)}-\frac{\left(\alpha e^{-\nu U}-\alpha-\nu\right)e^{-\alpha U}+\nu}{(\alpha+\nu)^2\left(1-e^{-\alpha U}\right)}\right\}\lambda t\\ &= \eta(-\nu)\end{aligned}$$

and hence

$$\widehat{\Theta}(z)=\eta(iz).$$

Since we now have a closed form formula for $\widehat{\Theta}(y)$ the inversion lemma gives

$$\begin{aligned}E\left[L_t^U\;I\left\{L_t^U<\mathrm{VaR}\left(q,\;L_t^U\right)\right\}\right] &= \Theta(l)\\ &= \frac{\widehat{\Theta}(0)}{2}-\frac{1}{\pi}\int_0^\infty\frac{1}{z}\operatorname{Im}\left(e^{-izl}\widehat{\Theta}(z)\right)dz,\end{aligned}$$

with

$$\widehat{\Theta}(0) = \eta(0) = E\left(L_t^U\right)$$

which allows us to calculate the numerator of (4.2) as

$$E\left[L_t^U \; I\left\{L_t^U \geq \text{VaR}\left(q,\; L_t^U\right)\right\}\right] = E\left(L_t^U\right) - E\left[L_t^U \; I\left\{L_t^U < \text{VaR}\left(q,\; L_t^U\right)\right\}\right]. \tag{3.3}$$

Let us now illustrate the calculations of VaR and TCE as risk measures of the capital amount required for a credit loan portfolio for next t years using the method above.

<u>**Example**</u>

The parameter values used to approximate VaR and TCE using (3.1) and (3.2) are

$$\alpha = 0.00001,\; U = 1,000,000,\; \lambda = 10,\; t = 1.$$

From (2.9) and (2.10), the mean and standard deviation of the total loss arising from a credit loan portfolio in a unit period of time are given by

$$E(L_t^U) = 999,500$$

and

$$\sqrt{Var(L_t^U)} = 446,600.$$

Using *Matlab*, the calculation of VaR and TCE are shown in Table below.

Table

q	VaR	TCE
99%	\$2,248,200	\$2,534,600
95%	\$1,811,300	\$2,092,700
90%	\$1,597,600	\$1,894,000
50%	\$949,330	\$1,353,300

4. Conclusion

For total loss arising from a credit loan portfolio such as credit card loan or mortgage loan portfolio, we employed an actuarial risk model. In order to measure the capital charge required for next t years' credit loan risk, the Value at Risk (VaR) and tail conditional expectation (TCE, also known as TailVaR) were used. Applying piecewise deterministic Markov processes (PDMP) theory, we obtained the explicit expression of the Laplace transform of the distribution of total loss using independent identical truncated exponential distribution for individual loss sizes, as the maximum amount of credit loan that individual borrowers can have should be bounded in practice. Based on this Laplace transform, the VaR and TCE were evaluated as risk measures of the capital amount required for next year's credit loan risk. In order to invert the Laplace transform of the distribution of aggregate loss to calculate the VaR and TCE numerically, we employed Transform Analysis technique from the financial option pricing literature.

References

Artzner, P., Delbaen, F., Eber, J. M. and Heath, D. (1999) : *Coherent measures of risk*, Mathematical Finance, 9/3, 203-228.

Basel Committee on Banking Supervision (2001) : Consultative Document. Overview of the New Basel Capital Accord.

Beard, R.E., Pentikainen, T. and Pesonen, E. (1984) : Risk Theory, 3rd Edition, Chapman & Hall, London.

Bertoin, J. (1996) : Lévy processes, Cambridge University Press, Cambridge.

Bühlmann, H. (1970) : Mathematical Methods in Risk Theory, Springer-Verlag, Berlin-Heidelberg.

Crouhy, C., Galai, D. and Mark, R. (2000) : Risk Management, McGraw-Hill, New York.

Dassios, A. and Embrechts, P. (1989) : *Martingales and insurance risk*, Commun. Stat.-Stochastic Models, 5(2), 181-217.

Dassios, A. and Jang, J. (2003) : *Pricing of catastrophe reinsurance & derivatives using the Cox process with shot noise intensity*, Finance & Stochastics, 7/1, 73-95.

Davis, M. H. A. (1984) : *Piecewise deterministic Markov processes: A general class of non diffusion stochastic models*, J. R. Stat. Soc. B, 46, 353-388.

Duffie, D., J. Pan, and K.Singleton. (2000) : *Transform analysis and asset pricing for affine jump-diffusions*, Econometrica, 68, 1343-1376.

Gerber, H. U. (1979) : An Introduction to Mathematical Risk Theory, S. S. Huebner Foundation for Insurance Education, Philadelphia.

Heston, S. (1993): *A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, Review of Financial Studies, 6: 327-344.

Jang, J. (2004) : *Martingale approach for moments of discounted aggregate claims*, Journal of Risk and Insurance, 71/2, 201-211.

Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. L. (1998) : Stochastic Processes for insurance and Finance, John Wiley & Sons, UK.