

STOP LOSS REINSURANCE PRICING IN AN ECONOMIC ENVIRONMENT

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ABSTRACT. We consider the classical Compound Poisson model of insurance risk, with the additional economic assumption of a positive interest rate. Stop-loss reinsurance premiums are priced by enforcing a no-arbitrage condition between the insurance and reinsurance markets. We note a duality result relating the accumulated claims and the shot noise process, and the Esscher transform is used to define a pricing measure. We also illustrate how premiums can be evaluated using Transform Analysis techniques from the financial option pricing literature.

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1. INTRODUCTION

Let $Y(i)$, $i = 1, 2, \dots$, be the claim amounts, which are assumed to be independent and identically distributed with distribution function $G(y)$ ($y > 0$). In classical risk theory one assumes (often implicitly) that interest rates equal zero, and consider the loss process $C(t)$, defined to be

$$C(t) = \sum_{i=1}^{N(t)} Y(i)$$

with $N(t)$ being the number of claims up to time t . Delbaen and Haezendonck (1987) extended the classical risk theory to consider the effect of the introduction of interest rate factors, leading to an explosion of literature in this subject in recent years, see for example Paulsen (1998) for a survey. Most of these papers however deal with the effect of interest rates on the probability of ruin rather than premium setting. More recently authors such as Léveillé and Garrido (2001), and

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Jang (2003) considered the effect of interest on the moments of the accumulated claims process. These aforementioned papers generally consider premium setting by considering classical premium principles. In contrast we will consider premium setting by enforcing the economic concept of no-arbitrage.

More specifically, if we let δ to be the risk free rate of interest, the accumulated value of aggregate claims up to time t , $L(t)$ is given by

$$(1.1) \quad L(t) = \sum_{i=1}^{N(t)} Y(i) e^{\delta(t-s(i))}$$

with $s(i)$ being the time of claim (i) . In this environment, we define the stop loss reinsurance contract with retention b to be a contract that pays an amount equal to $(L(T) - b)^+$ at time T . By imposing the principal of no-arbitrage between the insurance and reinsurance markets (Sonderman 1991) we know that the stop-loss reinsurance premium (for time T) including the effect of the rate of interest is given by

$$(1.2) \quad E^* \left[e^{-\delta T} (L(T) - b)^+ \right]$$

where the expectation is calculated under an appropriate probability measure P^* equivalent to the physical measure P .

We assume that the claim arrival process $N(t)$ follows a Poisson process with claim frequency rate ρ . It is also assumed that is independent of $Y(i)$, $i = 1, 2, \dots$. In order to evaluate the expectation (1.2), we will need to obtain the distribution of the accumulated (or discounted) aggregate claims, $L(T)$. Unfortunately, it is known that it is not possible for us to obtain the distribution of the accumulated aggregate claims explicitly. Hence in this paper we will extend a duality result between the accumulated aggregate claims process and the shot noise process to derive a general form for the Laplace transform of the distribution of the accumulated aggregate claims under an Esscher measure. As in illustration we also calculate the explicit Laplace transform in the case when the claim size is gamma or exponential. Transform analysis techniques from financial option pricing theory (Duffie et al 2001) is then used to derive the stop loss reinsurance premium numerically.

This paper is structured as follows. In Section 2 we setup our reinsurance market and discuss the condition of no-arbitrage. Section 3 illustrates the duality result between the accumulated aggregate claims and the shot noise process. Section 4 considers the effect of a change of probability measure, while Section 5 contains premium calculations via Laplace transforms and Transform Analysis from the financial option pricing literature.

2. REINSURANCE MARKET AND NO-ARBITRAGE

Assume that there exist a liquid reinsurance market, i.e. at any time $t \leq T$, the insurer can decide to sell any part of the risk of $L(u)$, $t \leq u \leq T$, based on the information available at time t defined on (\mathcal{F}_t, F, P) , which is a probability space with information structure given by $\{F = \mathfrak{F}_t, t \in [0, T]\}$. Let $PR(u)$ denote the total value of premiums received and accumulated at the rate of δ up to time u defined on (\mathcal{F}_t, F, P) and define the reinsurance strategy that is adopted from Embrechts & Meister (1995).

Definition 1. *Let $s \in [0, T]$, a reinsurance strategy $\{\phi(u); t \leq u \leq T\}$ is a predictable stochastic process on (\mathcal{F}_t, F, P) with $0 \leq \phi(u) \leq 1$ for all $u \in [t, T]$.*

Let us define the specified process $R(t)$, $0 \leq t \leq T$, given by

$$(2.1) \quad R(t) = PR(t) - L(t) \quad (0 \leq t \leq T)$$

denoting the net surplus from insurance business up to time t . If the insurer choose at time t some reinsurance strategy $\{\phi_u; t \leq u \leq T\} \in H(t)$ where $H(t)$ denotes the set of all reinsurance strategies starting at time t , then the company's final gain at time T is given by

$$G_T(\phi) = \int_t^T \phi(u) dR(u)$$

where it is assumed that the reinsurer receives direct insurer's premiums for his engagement. Following Sonderman (1991) we can define an arbitrage as follows:

Definition 2. (*Arbitrage*) A strategy $\{\phi(u); t \leq u \leq T\}$ allowing for a possible profit without the possibility of a loss is called an arbitrage strategy, i.e. a strategy $\{\phi(u); t \leq u \leq T\}$ satisfying

$$\begin{aligned} G_T(\phi) &\geq 0, P - a.s. \\ E_P[G_T(\phi)] &> 0 \end{aligned}$$

is called an arbitrage strategy.

Therefore, for the reinsurance market (\mathcal{F}, P) , R_t does not allow for arbitrage strategies if there is an equivalent probability measure P^* such that the process R_t is a martingale.

Definition 3. (*Equivalent Martingale Measure*) A probability measure P^* is called an equivalent martingale probability measure if:

- $P^*(A) = 0$ iff $P(A) = 0$, for any $A \in \mathfrak{S}_t$;
- The Radon-Nikodym derivative $\frac{dP^*}{dP}$ belongs to $L^2(\mathcal{F}_t, P)$;
- $e^{-\delta t} R(t)$ is a martingale under P^* , i.e.

$$E^*[e^{-\delta t} R(t) | \mathfrak{S}_s] = e^{-\delta s} R(s),$$

P^* -a.s. for any $0 \leq s \leq t \leq T$, where E^* denotes the expectation with respect to P^* .

The Esscher transform is employed in order to change the probability measure as it provides us with at least one equivalent martingale probability measure when in our incomplete market. We here offer the definition of the Esscher transform that is adopted from Gerber and Shiu (1996).

Definition 4. (*Esscher Transform*) Let $X(t)$ be a stochastic process such that $e^{h^* X(t)}$ a martingale with $h^* \in \mathfrak{R}$. For a measurable function f , the expectation of the random variable $f(X(t))$ with respect to the equivalent martingale probability measure is

$$(2.2) \quad E^*[f(X(t))] = E\left[f(X_t) \frac{e^{h^* X(t)}}{E(e^{h^* X(t)})}\right] = \frac{E[f(X(t)) e^{h^* X(t)}]}{E[e^{h^* X(t)}]}$$

where $E(e^{h^* X(t)}) < \infty$.

If a geometric Brownian motion or a homogeneous Poisson process governs the market, we obtain the Laplace transform of the distribution of the accumulated aggregate claims with respect to a unique equivalent martingale probability measure. However in a Compound Poisson model (even with zero interest rates) there will exist an infinite number of equivalent martingale measures in general as the market is incomplete. It is not the purpose of this paper to decide which is the appropriate one to use. The insurance companies' attitude towards risk determines which equivalent martingale probability measure should be used. The attractive thing about the Esscher transform is that it provides us with at least one equivalent martingale probability measure in incomplete market situations.

3. DUALITY OF THE ACCUMULATED CLAIMS PROCESSES AND SHOT NOISE

The shot noise process can be used in many diverse fields. In particular, it attracts us as it can be applied in financial and insurance field. The shot noise process is particularly useful as it measures the frequency, magnitude and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases until another event occurs which will result in a positive jump in the shot noise process. We will adopt the shot noise process used by Cox & Isham (1980):

$$(3.1) \quad \lambda(t) = \lambda_0 e^{-\kappa t} + \sum_{\substack{\text{all } i \\ s(i) \leq t}} y(i) e^{-\kappa(t-s(i))}$$

where:

- λ_0 initial value of λ
- $y(i)$ jump size of primary event i , where $E(y_i) < \infty$
- $s(i)$ time at which primary event i occurs, where $s_i < t < \infty$
- κ exponential decay
- ρ the rate of primary event arrival.

Some works of insurance application using shot noise process can be found in Klüppelberg & Mikosch (1995), Dassios and Jang (2003), and Jang (2003).

Let us examine how the shot noise process is related to the accumulated aggregate claims process (1.2). If we set $-\delta$ to κ in (3.1), it becomes

$$(3.2) \quad \xi(t) = \xi_0 e^{\delta t} + \sum_{\substack{\text{all } i \\ s(i) \leq t}} y_i e^{\delta(t-s(i))}.$$

Interestingly, we can see that it is equivalent to (1.2) if we substitute ‘ ξ ’ with ‘ L ’ in (4.1) with $\xi_0 = 0$.

The piecewise deterministic Markov processes (PDMP) theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. The shot noise process is an example of a PDMP. Therefore we can present definitions and important properties of the shot noise process with the aid of this theory (Dassios and Embrechts 1989). We will use it to derive the Laplace transform of the distribution of the shot noise process. See also Rolski et al (1999, Chapter11).

The generator of the process $(\lambda(t), N(t), t)$ acting on a function $f(\lambda, n, t)$ belonging to its domain is given by

$$(3.3) \quad Af(\lambda, n, t) = \frac{\partial f}{\partial t} - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho \left[\int_0^\infty f(\lambda + y, n + 1, t) dG(y) - f(\lambda, n, t) \right]$$

where the three parameters of the shot noise process, i.e. δ , ρ and $G(y)$ are constant in time. It is sufficient that $f(\lambda, n, t)$ is differentiable w.r.t. λ, t for all λ, n, t and that for $\left| \int_0^\infty f(\cdot, \lambda + y, \cdot) dG(y) - f(\cdot, \lambda, \cdot) \right| < \infty$ for $f(\lambda, t)$ to belong to the domain of the generator A .

Now let us find a suitable martingale in order to derive the Laplace transform of the distribution of $\lambda(t)$ at time t . We also need it to change the measure applying the Esscher transform, i.e. it can be used to define the Radon-Nikodym derivative $\frac{dP^*}{dP}$ where P is the original probability measure and P^* is the equivalent martingale probability measure with new parameters involved.

Lemma 1. Consider constants ψ^* and γ^* such that $\psi^* \geq 1$ and $\gamma^* \leq 0$. Then

$$(3.4) \quad \psi^{*N(t)} \exp(-\gamma^* \lambda(t) e^{\delta t}) \exp \left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds \right]$$

is a martingale where $\hat{g}(u) = \int_0^\infty e^{-uy} dG(y)$.

Proof. From (3.3), $f(\lambda, n, t)$ has to satisfy $Af = 0$ for $f(X(t), N(t), C(t), \lambda(t), M(t), t)$ to be a martingale. Setting $f(\lambda, n, t) = \psi^{*n} \exp(-\gamma^* \lambda e^{\delta t}) e^{B(t)}$ we get the equation

$$(3.5) \quad -\lambda \delta \gamma^* e^{\delta t} + B'(t) + \delta \lambda \gamma^* e^{\delta t} + \rho \{ \psi^* \hat{g}(\gamma^* e^{\delta t}) - 1 \} = 0$$

and the solution is

$$(3.6) \quad A(t) = \rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds$$

by which the result follows. \square

Using this martingale we can easily obtain the Laplace transform of the distribution of $\lambda(t)$ at time t ,

$$(3.7) \quad E \left\{ e^{-\nu \lambda(t)} | \lambda(0) \right\} = \exp(-\nu e^{-\delta t} \lambda(0)) \exp \left[-\rho \int_0^t \{1 - \hat{g}(\nu e^{-\delta s})\} ds \right]$$

where $\nu \geq 0$.

4. THE ESSCHER TRANSFORM AND CHANGE OF PROBABILITY MEASURE

In general, the Esscher transform is defined as a change of probability measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The parameters involved for the Esscher transform are determined so that the process is a martingale under the new probability measure. We will examine an equivalent martingale probability measure obtained via the Esscher transform (Gerber and Shiu 1996 and Dassios and Jang 2003).

From the duality result in the previous section we see that the underlying stochastic process for accumulated aggregate claims process can be considered as a shot noise process, which is a generalized Lévy process as (3.2) can be expressed as

$$d\lambda(t) = \delta\lambda(t)dt + dK(t)$$

where $K(t) = \sum_{i=1}^{N(t)} Y(i)$ is a pure-jump process (Poisson arrivals of jumps of a given distribution), we will have infinitely many equivalent martingale probability measures. In other words, we will have several choices of equivalent martingale probability measures to derive the Laplace transform of the distribution of the accumulated aggregate claims, as the market is incomplete. The Esscher transform provides one choice of an equivalent martingale probability measure in this setting.

Let us examine how the generator A^* of the process $(\lambda(t), N(t), t)$ acting on a function $f(\lambda, n, t)$ with respect to the equivalent martingale probability measure can be obtained.

Lemma 2. *Let v^* be a nonnegative constant. Assuming that $f(n, \lambda, t) = f(\lambda, t)$ for all n and that $e^{-v^*X(t)}$ is a martingale with $X(t)$ an adapted process. The generator A^* of the process (λ_t, t) acting on a function $f(\lambda, t)$ with respect to the equivalent martingale measure is given by*

$$(4.1) \quad A^* f(\lambda, t) = \frac{A \{f(\lambda, t) e^{-v^*X(t)}\}}{e^{-v^*X(0)}}.$$

Proof. The generator of the process $(\lambda(t), t)$ acting on a function $f(\lambda, t)$ with respect to the equivalent martingale probability measure is

$$(4.2) \quad A^* f(\lambda, t) = \lim_{dt \downarrow 0} \frac{E^* [f(\lambda(t+dt), t+dt) | \lambda(t) = \lambda] - f(\lambda, t)}{dt}.$$

We will use $\frac{e^{-v^*X(t)}}{E(e^{-v^*X(t)})}$ as the Radon-Nikodym derivative to define equivalent martingale probability measure. Hence, the expected value of $f(\lambda_{t+dt}, t+dt)$ given $\lambda_t = \lambda$ with respect to the equivalent martingale probability measure is

$$(4.3) \quad E^* [f(\lambda(t+dt), t+dt) | \lambda_t = \lambda] = \frac{E [f(\lambda(t+dt), t+dt) e^{-v^*X(t+dt)} | \lambda(t) = \lambda]}{E (e^{-v^*X(t+dt)} | \lambda_t = \lambda)}.$$

Since $e^{-v^*\lambda(t)}$ in (4.3) is a martingale, it becomes

$$(4.4) \quad \begin{aligned} & E^* [f(\lambda(t+dt), t+dt) | \lambda(t) = \lambda] \\ &= \frac{f(\lambda, t) e^{-v^*\lambda} + \int_t^{t+dt} E [A f(\lambda_s, s) e^{-v^*X(s)} | \lambda_t = \lambda] ds}{e^{-v^*X(0)}}. \end{aligned}$$

Set (4.4) in (4.2) then

$$(4.5) \quad A^* f(\lambda, t) = \frac{1}{e^{-v^*X(0)}} \lim_{dt \downarrow 0} \int_t^{t+dt} \frac{E [A f(\lambda_s, s) e^{-v^*X(s)} | \lambda(t) = \lambda] ds}{dt}.$$

Therefore, from Dynkin's formula (4.1) follows immediately. \square

Now let us look at how the dynamics of process $\lambda(t)$ and N_t change after changing probability measure by obtaining the generator A^* of the process $(\lambda(t), N(t), t)$ acting on a function $f(\lambda, n, t)$ with respect to the equivalent martingale probability measure. This is the key result that we require to establish the distribution of the accumulated aggregate claims under our equivalent martingale measure.

Theorem 1. *Consider constants ψ^* and γ^* such that $\psi^* \geq 1$ and $\gamma^* \leq 0$. Suppose that $\hat{g}(\gamma^* e^{\delta t}) < \infty$. Then*

$$(4.6) \quad A^* f(\lambda, n, t) = \frac{\partial f}{\partial t} - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho^*(t) \left\{ \int_0^\infty f(\lambda + y, n + 1, t) dG^*(y; t) - f(\lambda, n, t) \right\}$$

where $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

Proof. From Lemma 1 we can use

$$(4.7) \quad \frac{\psi^{*N(t)} \exp(-\gamma^* \lambda(t) e^{\delta t}) \exp \left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds \right]}{E \left[\psi^{*N(t)} \exp(-\gamma^* \lambda(t) e^{\delta t}) \exp \left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds \right] \right]}$$

as the Radon-Nikodym derivative to define an equivalent martingale probability measure. Therefore from lemma 2,

$$A^* f(\lambda(t), N_t, t) = A f(\lambda_t, N_t, t) \times \frac{\psi^{*N(t)} \exp(-\gamma^* \lambda(t) e^{\delta t}) \exp \left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds \right]}{E \left[\psi^{*N(t)} \exp(-\gamma^* \lambda(t) e^{\delta t}) \exp \left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds \right] \right]}.$$

Using the generator with respect to the original probability measure,

$$\begin{aligned} & A f(\lambda(t), N(t), t) \psi^{*N(t)} \exp(-\gamma^* \lambda(t) e^{\delta t}) \exp \left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds \right] \\ &= \left[\begin{array}{c} \frac{\partial f}{\partial t} - \delta \lambda \frac{\partial f}{\partial \lambda} \\ + \rho \left\{ \psi^* \int_0^\infty f(\lambda + y, n + 1, t) \exp(-\gamma^* e^{\delta t} y) dG(y) \right. \\ \left. - \psi^* \hat{g}(\gamma^* e^{\delta t}) f(\lambda, n, t) \right\} \end{array} \right] \\ & \quad \times \psi^{*N(t)} \exp(-\gamma^* \lambda(t) e^{\delta t}) \exp \left[\rho \int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\} ds \right]. \end{aligned}$$

Therefore

$$(4.8) \quad A^* f(\lambda, n, t) = \frac{\partial f}{\partial t} - \delta \lambda \frac{\partial f}{\partial \lambda} + \rho^*(t) \left\{ \int_0^\infty f(\lambda + y, n + 1, t) dG^*(y; t) - f(\lambda, n, t) \right\},$$

where $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. \square

Theorem 1 yields the following:

- (i) The rate of jump arrival ρ has changed to $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$
(it now depends on time);
- (ii) The jump size measure $dG(y)$ has changed to $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$
(it now depends on time);

In other words, the Esscher measure is the measure with respect to which λ_t becomes the shot noise process with three parameters of δ , $\rho^*(t) = \rho \psi^* \hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y; t) = \frac{\exp(-\gamma^* e^{\delta t} y) dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$. To ensure that our model is arbitrage free we then need to ensure that there is consistency between the reinsurance and primary

insurance premiums under the selected Esscher measure. We refer to Sonderman (1991) for more details.

5. REINSURANCE PREMIUM VIA TRANSFORM ANALYSIS

Similar to the previous section, but with time dependent parameters of $\rho(t)$ and $dG(y; t)$ as we have witnessed as results of changing measure, i.e. based on

$$A f(l, t) = \frac{\partial f}{\partial t} + \delta t \frac{\partial f}{\partial l} + \rho(t) \left[\int_0^\infty f(l+y, t) dG(y; t) - f(l, t) \right]$$

we can easily derive the Laplace transform of the process at time t , i.e.

$$(5.1) \quad E \left\{ e^{-vL(t)} | L(0) \right\} = \exp(-ve^{\delta t} L(0)) \exp \left[- \int_0^t \rho(s) \left\{ 1 - \hat{g}(ve^{-\delta(t-s)}; s) \right\} ds \right]$$

where $\rho(t)$ is bounded in all intervals $[0, t)$ (no explosions) and $\rho(t)$ and $G(y; t)$ are Riemann integrable functions of t and all positive. If the jump size distribution is exponential, i.e.

$$g(y; t) = (\alpha + \gamma e^{-\delta t}) e^{-(\alpha + \gamma e^{-\delta t})y},$$

$y > 0$, $-\alpha e^{\delta t} < \gamma \leq 0$, we have

$$(5.2) \quad \exp(-ve^{\delta t} L(0)) \left(\frac{\gamma + \alpha e^{\delta t}}{\gamma + \alpha} \right)^{-\frac{\alpha}{\delta}} \left(\frac{\gamma + \nu e^{\delta t} + \alpha}{\gamma + (\nu + \alpha)e^{\delta t}} \right)^{-\frac{\alpha}{\delta}}.$$

Therefore, assuming that $L(0) = 0$, the Laplace transform of the distribution of the accumulated aggregated claims is given by

$$(5.3) \quad \exp \left[- \int_0^t \rho(s) \left\{ 1 - \hat{g}(ve^{\delta(t-s)}; s) \right\} ds \right]$$

and if the jump size distribution is exponential it is given by

$$(5.4) \quad \left(\frac{\gamma + \alpha e^{\delta t}}{\gamma + \alpha} \right)^{-\frac{\alpha}{\delta}} \left(\frac{\gamma + \nu e^{\delta t} + \alpha}{\gamma + (\nu + \alpha)e^{\delta t}} \right)^{-\frac{\alpha}{\delta}}.$$

Now let us obtain the Laplace transform of distribution of accumulated aggregate claims and its mean with respect to the Esscher measure. From Theorem 1 and (5.3), the Laplace transform of the distribution of the accumulated aggregated claims with respect to the Esscher measure is given by

$$(5.5) \quad \exp \left[- \int_0^t \rho^*(s) \left\{ 1 - \hat{g} \left(ve^{\delta(t-s)}; s \right) \right\} ds \right].$$

If the claim size distribution is exponential, it is given by

$$(5.6) \quad \left(\frac{\gamma^* + \alpha e^{\delta t}}{\gamma^* + \alpha} \right)^{-\frac{\psi^* c}{\delta}} \left(\frac{\gamma^* + \nu e^{\delta t} + \alpha}{\gamma^* + (\nu + \alpha) e^{\delta t}} \right)^{-\frac{\psi^* c}{\delta}}.$$

In practice, the reinsurer will select the Laplace transform of the distribution of the accumulated aggregate claims to calculate the premium for stop-loss reinsurance contract using $\psi^* > 1$ and $\gamma^* < 0$. This results in the reinsurer assuming that there will be a higher value of claim size and more claims occurring in a given period of time. These assumptions are necessary, as the reinsurer wants compensation for the risks involved in operating in incomplete market. The reinsurer also aims to maximize their shareholders' wealth by earning profits rather than operating at breakeven point where premiums are equal to the expected claims that is calculated with respect to the original probability measure.

If $\psi^* = 1$ and $\gamma^* = 0$ then net premium is calculated which should cover the expected losses over the period of contract. Therefore we can consider ψ^* and γ^* as security loading factors by which gross premium, that should be finally charged, will be calculated. However, as expected, we have quite a flexible family of equivalent probability measures by the combination of ψ^* and γ^* . It means that insurance companies need to choose one of the Laplace transform of the distribution of the accumulated aggregated claims in order to calculate its premium that should be finally charged, i.e. an arbitrage-free premium, based on their attitude towards risk. One of the interesting results by changing measure (i.e. by assuming that there is an absence of arbitrage opportunities in the market) is that we can justify

reinsurers' security loading on the net premium for stop-loss reinsurance contract in practice.

The Laplace transforms of the distribution of the accumulated aggregate claims associated with changes in probability measure are shown below. We can easily find that if we set $\psi^* = 1$ and $\gamma^* = 0$, it is the Laplace transform of the distribution of the accumulated/discounted aggregate claims with respect to the original probability measure.

	Laplace transform
$\psi^* > 1$ and $\gamma^* < 0$	$\left(\frac{\gamma^* + \alpha e^{\delta t}}{\gamma^* + \alpha}\right)^{-\frac{\psi^* p}{\delta}} \left(\frac{\gamma^* + \nu e^{\delta t} + \alpha}{\gamma^* + (\nu + \alpha)e^{\delta t}}\right)^{-\frac{\psi^* p}{\delta}}$
$\psi^* > 1$ and $\gamma^* = 0$	$\left(\frac{\nu e^{\delta t} + \alpha}{\nu + \alpha}\right)^{-\frac{\psi^* p}{\delta}}$
$\psi^* = 1$ and $\gamma^* < 0$	$\left(\frac{\gamma^* + \alpha e^{\delta t}}{\gamma^* + \alpha}\right)^{-\frac{p}{\delta}} \left(\frac{\gamma^* + \nu e^{\delta t} + \alpha}{\gamma^* + (\nu + \alpha)e^{\delta t}}\right)^{-\frac{p}{\delta}}$
$\psi^* = 1$ and $\gamma^* = 0$	$\left(\frac{\nu e^{\delta t} + \alpha}{\nu + \alpha}\right)^{-\frac{p}{\delta}}$

Let us look at how the Laplace Transform can be used to evaluate the premium for a stop-loss reinsurance contract. First notice that

$$\begin{aligned} E^* \left[e^{-\delta T} (L(T) - b)^+ \right] &= e^{-\delta T} \int_b^\infty (x - b) dF_{L(T)}(x) \\ &= e^{-\delta T} E \left[L(T) 1_{L(T) > b} \right] - e^{-\delta T} b E \left[1_{L(T) > b} \right] \end{aligned}$$

where $F_{L(T)}(x)$ is the distribution function of the accumulated aggregate claims $L(T)$ with respect to an equivalent martingale probability measure. However as mentioned earlier in introduction, it is not possible for us to obtain its distribution explicitly. So we calculate the stop-loss reinsurance premium numerically.

By analogy with option pricing problems in finance we can first notice that the shot noise process is 'affine'. This suggests that transform analysis techniques developed by Heston (1993) and Duffie et al (2000) might prove useful. We highlight their methodology as applied to our problem below.

We know from the previous section the Laplace transform $\varsigma(-\nu)$ of $L(T)$.

$$\varsigma(-\nu) = E \left[e^{-\nu L(T)} \right]$$

and can consider the function

$$\widehat{G}(z) = \int_{-\infty}^{\infty} e^{izy} d \left(\int_0^y dF_{L(T)}(x) \right),$$

and hence

$$\begin{aligned}\widehat{G}(z) &= \int_{-\infty}^{\infty} e^{izy} dF_{L_T}(y) \\ &= E[e^{izL_T}] = \varsigma(iz).\end{aligned}$$

Recall that

$$\begin{aligned}\varsigma(-\nu) &= E[e^{-\nu L(T)}] \\ &= \left(\frac{\gamma^* + \alpha e^{\delta T}}{\gamma^* + \alpha}\right)^{-\frac{\psi^* \rho}{\delta}} \left(\frac{\gamma^* + \nu e^{\delta T} + \alpha}{\gamma^* + (\nu + \alpha)e^{\delta T}}\right)^{-\frac{\psi^* \rho}{\delta}},\end{aligned}$$

and the standard Lévy inversion formula gives

$$\begin{aligned}E[1_{L(T) \leq y}] &= G(y) \\ &= \frac{\widehat{G}(0)}{2} - \frac{1}{\pi} \int_0^{\infty} \frac{1}{z} \operatorname{Im}\left(e^{-izy} \widehat{G}(z)\right) dz.\end{aligned}$$

Consider now the function

$$\widehat{H}(z) = \int_{-\infty}^{\infty} e^{izy} d\left(\int_0^y x dF_{L(T)}(x)\right).$$

Assume $\int |H(y)| dy < \infty$, and we find that

$$\begin{aligned}\widehat{H}(z) &= \int_{-\infty}^{\infty} e^{izy} y dF_{L(T)}(y) \\ &= E[L(T) e^{izL(T)}],\end{aligned}$$

which can be calculated as follows. Differentiating $\varsigma(-\nu)$ with respect to $-\nu$ gives

$$\begin{aligned}& -\frac{\partial}{\partial \nu} \varsigma(-\nu) \\ &= E[L(T) e^{-\nu L(T)}] \\ &= \left(\frac{\gamma^* + \alpha e^{\delta T}}{\gamma^* + \alpha}\right)^{-\frac{\psi^* \rho}{\delta}} \left(\frac{\gamma^* + \nu e^{\delta T} + \alpha}{\gamma^* + (\nu + \alpha)e^{\delta T}}\right)^{-\frac{\psi^* \rho}{\delta}} \\ &\quad \times \left(\frac{(e^{\delta T} - 1) \psi^* \rho \alpha e^{\delta T}}{\delta (\gamma^* + \nu e^{\delta T} + \alpha) (\gamma^* + (\nu + \alpha)e^{\delta T})}\right) \\ &= \eta(-\nu),\end{aligned}$$

and hence

$$\widehat{H}(z) = \eta(iz).$$

Since we now have a closed form formula for $\widehat{H}(y)$ the inversion lemma gives

$$\begin{aligned} E[L(T)1_{L(T)\leq y}] &= H(y) \\ &= \frac{\widehat{H}(0)}{2} - \frac{1}{\pi} \int_0^\infty \frac{1}{z} \operatorname{Im} \left(e^{-izy} \widehat{H}(z) \right) dz, \end{aligned}$$

with

$$\widehat{H}(0) = \varphi(0) = E[L(T)],$$

which allows us to calculate the stop loss premium as

$$E[L(T)1_{L(T)>b}] = E[L(T)] - E[L(T)1_{L(T)\leq b}].$$

As an illustration, of our methodology the following stop-loss premiums have been calculated corresponding to different retention level b and parameters $\alpha = 0.01, \delta = 0.05, \rho = 50, T = 1$:

Retention b	$\psi^* = 1.2$	$\psi^* = 1$	$\psi^* = 1.2$	$\psi^* = 1$
	$\gamma = -0.002$	$\gamma = -0.002$	$\gamma = 0$	$\gamma = -0.002$
0	9034.3	7528.6	5852.5	4877.1
4877.1	4395.2	2892.7	1269.5	514.4
5852.5	3649.0	1993.4	578.6	147.7
7528.6	1931.3	735.4	67.8	6.6
9034.3	829.3	179.0	4.1	0.1
10000	388.9	5.45	0.4	0
11000	145.6	1.26	0	0

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