Non-Expected Utility Risk Measures and Implications for Asset Allocation

John van der Hoek
Department of Applied Mathematics
University of Adelaide, Adelaide, S.A. 5005 AUSTRALIA
Tel: +61 (0)8 8303 5903, Fax: +61 (0)8 8303 3696
E-mail: jvanderh@maths.adelaide.edu.au

Michael Sherris
Faculty of Commerce and Economics
University of NSW, SYDNEY, NSW, 2052, AUSTRALIA
Tel: +61 (0)2 9385 2333, Fax: +61 (0)2 9385 1883
E-mail: m.sherris@unsw.edu.au

May 24, 1999

Abstract

The main aim of this paper is to develop an application to asset allocation of a risk measure recently proposed for insurance pricing. This paper reviews the distortion function approach developed in the actuarial literature for insurance risk. The proportional hazards transform is a particular case. The relationship between this approach to risk and other approaches including the dual theory of choice under risk is discussed. The paper considers the portfolio selection problem using expected utility and the distortion function approach. A new risk measure with suitable properties for asset allocation based on the distortion function approach to insurance risk is presented. This measure treats upside and downside risk differently. Examples are provided using Gaussian and log-normal return assumptions.

1 Introduction

The appropriate measure to use to quantify risk for portfolio selection continues to be a subject for debate. Balzer (1994) [2] outlines a range of specific risk measures, discusses the advantages and disadvantages of them and comes to the conclusion that semi-variance is the best measure of investment risk. Markowitz (1952) [12] originally developed a portfolio selection methodology based on mean-variance analysis using the variance of returns as a risk measure although he preferred a semi-variance measure. Markowitz (1959) [13] demonstrated the relationship between a number of different risk measures and utility functions. Lipman [9] provides a review of expected utility and considers a utility approach to investment strategy incorporating benchmarks.

Expected utility is a standard approach to uncertainty in financial and economic theory. Investor preferences over uncertain investment outcomes are assumed to have an expected utility property. Investors select from alternative investment portfolios by maximising expected utility. Von Neumann and Morgenstern (1944) [16] develop a set of axioms under which investor preferences given by a utility function will have the expected utility property. Expected utility theory has been criticised because observed risk taking behavior does not always exhibit the expected utility property (Allais, 1953 [1] and Kahneman and Tversky, 1979 [1]) and experiments do not conform to the independence axiom.

Recently Wang (1995) [17] has proposed an approach to insurance pricing using the proportional hazards (PH) transform. This approach to insurance risk is related to the dual theory of choice under uncertainty of Yaari (1987) [22]. Insurance and investment risks are closely related, so it is of interest to consider this approach to insurance risk in the investment context. These alternative approaches are examples of non-expected utility risk measures. We do not cover all of the alternative non-expected utility risk measures in this paper. Instead we concentrate on the distortion function approach and the dual theory.

This paper begins by outlining the proportional hazards (PH) transform approach to insurance risk proposed by Wang (1995) [17] and the more general distortion function approach. The dual theory of choice under uncertainty is then outlined and the links between this theory and expected utility theory and the PH transform discussed. Wang and Young (1997) [21] provide a more comprehensive discussion of utility theory and Yaari's dual theory of risk than is covered here.

The portfolio selection problem is then reviewed for the expected utility case and for the distortion function as a measure of risk. We consider the single-period case. More details on the expected utility approach can be found in books such as Ingersoll [7] and Pliska [14].

We then propose a new risk measure for portfolio selection based on the dual theory concepts but with desirable properties required for an asset allocation model including risk aversion and diversification. Examples based on normal and log-normal return distributions are given.

2 Proportional Hazards Transform

Wang (1995, 1996a) [17],[18] considers a non-negative loss random variable $X > 0$ with distribution function $F_X(t)$, decumulative distribution function

$$S_X(t) = \Pr\{X > t\} = 1 - F_X(t) = \overline{F}_X(t)$$

and density function $f_X(t)$.

The proportional hazards transform is defined as

$$g(x) = x^r$$

where $0 < r \leq 1$. The risk-adjusted premium is then

$$H_g(X) = \int_0^\infty g\left[S_X(t)\right] dt = \int_0^1 S_X^{-1}(q) dg(q)$$

For instance if $r = 1$ then $H_g(X) = \int_0^\infty \left[S_X(t)\right] dt = E\left[X\right]$ and we have risk-neutrality. Wang (1996b) [19] also interprets the parameter r as a measure of ambiguity aversion. $H_g(X) - E\left[X\right]$ can be interpreted as an insurance risk premium.

For insurance pricing, we require g to be concave and increasing with $g(0) = 0$, $g(1) = 1$, $g(x) \geq x$ and $0 \leq x \leq 1$. The function g is referred to as a distortion function.

The use of the proportional hazards transform approach to insurance pricing appears to have been motivated by assuming equivalence between a loss distribution and a survival distribution. The expected loss is treated as equivalent to the expected lifetime.

If $S_T(t)$ is the survival distribution function, $F_T(t)$ the decumulative distribution function, and $f_T(t)$ the density function, then the hazard (or failure) rate function $\lambda(t)$ is defined as

$$\lambda(t) = \frac{f_T(t)}{1 - F_T(t)} = \frac{-\frac{d}{dt}S_T(t)}{S_T(t)} = -\frac{d}{dt}\ln\left[S_T(t)\right]$$

so that

$$S_T(t) = \exp\left(-\int_0^t \lambda(s)ds\right).$$

The expected life time is

$$E\left[T\right] = \int_0^\infty t f_T(t)dt = \int_0^\infty \left[S_T(t)\right]dt$$

Now assume a proportional change to the hazard function so that

$$\lambda^*(t) = r\lambda(t)$$

The survival distribution function under the new hazard rate is given by

$$-\frac{d}{dt}\ln\left[S_{T^*}(t)\right] = r\lambda(t)$$

so that

$$S_{T^*}(t) = \exp\left(-\int_0^t r\lambda(s)ds\right) = \left[S_T(t)\right]^r.$$

The expected lifetime is now

$$E\left[T^*\right] = \int_0^\infty \left[S_T(t)\right]^r dt.$$

We note that the proportional hazards transform approach to insurance pricing considers the loss distribution in a similar manner to the survival distribution. The proportional hazards transform distorts the probability of a claim occurring and then uses the distorted probabilities to calculate the expected value.

For X and Y non negative random variables we have the following properties of the proportional hazards transform with $g(x) = x^r$ where $0 < r \leq 1$ (Wang, 1996a) [18],

1. $E[X] \leq H_g[X] \leq \max(X)$
2. $H_g(aX + b) = aH_g(X) + b,\ a \geq 0,\ b \geq 0$
3. $H_g(X + Y) \leq H_g(X) + H_g(Y)$.

This third property is referred to as sub-additivity. $H_g(X)$ will be additive in the special case of comonotonic risks. Risks X_1 and X_2 are comonotonic if there exists a risk Z and nondecreasing real-valued functions f and h such that $X_1 = f(Z)$ and $X_2 = h(Z)$ (Wang and Young, 1997, [21]). The concept of comonotonic risks is an extension of perfect correlation.

In general we have the following properties using the distortion function approach

1. If $g(p) = p$, for all $p \in [0, 1]$, then $H_g[X] = E[X]$.
2. If $g(p) \geq p$, for all $p \in [0, 1]$, then $H_g[X] \geq E[X]$.
3. $H_g(aX + b) = aH_g(X) + b,\ a \geq 0,\ b \geq 0$.
4. For X and Y comonotonic, $H_g(X + Y) = H_g(X) + H_g(Y)$.
5. For concave g, $H_g[X] \geq E[X]$, and $H_g(X + Y) \leq H_g(X) + H_g(Y)$.
6. For convex g, $H_g[X] \leq E[X]$, and $H_g(X + Y) \geq H_g(X) + H_g(Y)$.

3 Dual Theory of Choice

Consider preference over risks. The symbol $\succ$ will denote preference or riskiness, so that $X \succ Y$ indicates X is preferred to Y. The use of expected utility as a risk measure is derived from five axioms. These are

1. If risks X_1 and X_2 have the same cumulative distribution function, then X_1 and X_2 are equally risky.

2. $\succ$ is reflexive, transitive and connected (weak order).

3. $\succ$ is continuous in the topology of weak convergence.

4. If $S_X \leq S_Y$, then $X \succ Y$.

5. If $X \succ Y$ and Z is any risk then

$$\{(\alpha, X), (1-\alpha, Z)\} \succ \{(\alpha, Y), (1-\alpha, Z)\}$$

for all α such that $0 \leq \alpha \leq 1$ where $\{(\alpha, X), (1-\alpha, Z)\}$ is the probabilistic mixture with

$$F_{\{(\alpha,X),(1-\alpha,Z)\}}(x) = \alpha F_X(x) + (1-\alpha)F_Z(x)$$

or equivalently

$$\overline{F}_{\{(\alpha,X),(1-\alpha,Z)\}}(x) = \alpha \overline{F}_X(x) + (1-\alpha)\overline{F}_Z(x).$$

This last axiom is referred to as the independence axiom. We can write this as

$$\alpha \overline{F}_X(x) + (1-\alpha)\overline{F}_Z(x) \succ \alpha \overline{F}_Y(x) + (1-\alpha)\overline{F}_Z(x).$$

The independence axiom states that if X is preferred to Y, then a lottery that pays X with probability α and Z with probability $(1-\alpha)$ will be preferred to a lottery that pays Y with probability α and Z with probability $(1-\alpha)$. Thus there is independence with respect to probability mixtures of uncertain outcomes.

If investors conform to these axioms then they will prefer strategies that have higher expected utilities. For a nonnegative random variable X, the expected utility is given by

$$E\left[U(X)\right] = \int_0^\infty S_X(t)dU(t) = \int_0^1 U\left[S_X^{-1}(q)\right]dq$$

Consider two random payments X and Y. Under the expected utility property, X is preferred to Y if

$$E[U(X)] > E[U(Y)]$$

where U is assumed to be a continuous non-decreasing function. U is a concave function for a risk averse individual and is unique up to a positive affine transformation.

Yaari (1987) [22] develops a dual theory of choice where this independence axiom is replaced with the dual independence axiom which states that if X is preferred to Y, then $\left[pX^{-1} + (1-p)Z^{-1}\right]^{-1}$ is preferred to $\left[pY^{-1} + (1-p)Z^{-1}\right]^{-1}$. Equivalently, if $X \succ Y$ and Z is any risk, then

$$pS_X^{-1} + (1-p)S_Z^{-1} \succ pS_Y^{-1} + (1-p)S_Z^{-1}$$

In this case X is preferred to Y if

$$H_g(X) = \int_0^\infty g\left[S_X(t)\right] dt > \int_0^\infty g\left[S_Y(t)\right] dt = H_g(Y)$$

where g is a continuous and non-decreasing function defined on the unit interval with $g(0) = 0$ and $g(1) = 1$.

An important implication of the independence axiom and expected utility is that the preference function is linear in the probabilities. Experimental evidence has suggested that decision making behaviour does not conform with the independence axiom (Machina, 1982, [10], 1987, [11]). The most famous of these violations of the independence axiom is probably the *Allais Paradox*. This is a particular example of what is referred to as the *common consequence effect*. An alternative approach is to use preference functions which are not linear in the probabilities such as Yaari's dual theory. Other nonlinear functional forms have also been suggested as referenced in Machina (1987) [11].

4 Investment Selection

4.1 Expected Utility

The expected utility-based portfolio theory approach to investment selection is to assume that investors maximise expected utility subject to constraints.

Panjer et al (Chapter 8, 1998) [15] gives more details. It will be assumed that the investor has a single period investment horizon and that an appropriate utility function exists. Assume that there are $N+1$ assets. The return on the ith asset, $i = 0, 1, 2, \ldots N$, is given by R_i. The investor with initial wealth W_0 selects a portfolio $x^T = (x_0, x_1, \ldots, x_N)$ and this selection provides wealth of $W_1 = W_0(1 + R_x)$ at the end of the period where $R_x = \sum_{i=0}^N x_i R_i$. The problem is then

$$\max E\left[U\left(W_0\left[1 + R_x\right]\right)\right]$$

subject to $\sum_{i=0}^N x_i = 1$ and possibly other constraints such as non-negativity and perhaps a shortfall constraint. Substituting $x_0 = 1 - \sum_{i=1}^N x_i$ into the objective to be maximised we obtain

$$\max E\left[U\left(W_0\left[1 + R_0 + \sum_{i=1}^N x_i\left(R_i - R_0\right)\right]\right)\right]$$

Differentiating with respect to each x_i gives the first order conditions for an optimum as

$$E\left[\left(R_i - R_0\right)\frac{\partial}{\partial W_1}U\left(W_1\right)\right] = 0 \quad \text{for } i = 1 \text{ to } N$$

We assume that U is strictly increasing $\left(\frac{\partial}{\partial W_1}U > 0\right)$ and risk aversion so that U is concave $\left(\frac{\partial^2}{\partial W_1^2}U < 0\right)$. The first order conditions are necessary and sufficient for a maximum. We also have that

$$E\left[U\left(W_0\left[1 + R_x\right]\right)\right] \leq U\left(E\left[W_0\left[1 + R_x\right]\right]\right)$$

and therefore

$$U^{-1}E\left[U\left(W_0\left[1 + R_x\right]\right)\right] \leq E\left[W_0\left[1 + R_x\right]\right]$$

where $U^{-1}E\left[U\left(W_0\left[1 + R_x\right]\right)\right]$ is the certainty equivalent of the random end of period wealth.

4.1.1 Quadratic utility

A common assumption underlying mean-variance portfolio selection models is that the utility function is quadratic. In this case

$$U(W) = W - \frac{1}{2b}W^2, \text{ with } b > 0 \text{ and } W < b$$

$$\frac{\partial}{\partial W}U(W) = 1 - \frac{1}{b}W$$

and the first order conditions are

$$E\left[(R_i - R_0)\left(1 - \frac{1}{b}W_0\left[1 + R_0 + \sum_{j=1}^{N} x_j (R_j - R_0)\right]\right)\right] = 0$$
$$\text{for } i = 1 \text{ to } N$$

which simplifies to

$$\left[\begin{array}{c} \left[1 - \frac{1}{b}W_0(1 + R_0)\right] E[R_i - R_0] - \\ \frac{1}{b}W_0 \sum_{j=1}^{N} x_j E[(R_j - R_0)(R_i - R_0)] \end{array}\right] = 0 \quad \text{for } i = 1 \text{ to } N$$

If we let

$$\mathbf{C} = \begin{pmatrix} E[(R_1 - R_0)(R_1 - R_0)] & \cdots & E[(R_N - R_0)(R_1 - R_0)] \\ E[(R_1 - R_0)(R_2 - R_0)] & \cdots & E[(R_N - R_0)(R_2 - R_0)] \\ \vdots & \ddots & \vdots \\ E[(R_1 - R_0)(R_N - R_0)] & \cdots & E[(R_N - R_0)(R_N - R_0)] \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} E(R_1 - R_0) \\ E(R_2 - R_0) \\ \vdots \\ E(R_1 - R_0) \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

then the solution to the first order conditions can be written as

$$\mathbf{x} = k\mathbf{C}^{-1}\mathbf{E}$$

where

$$k = \frac{b}{W_0} - (1 + R_0)$$

Note that the constraint $\sum_{i=0}^{N} x_i = 1$ is automatically satisfied.

4.1.2 Gaussian returns and exponential utility

In this case we assume that the returns $R_0, R_1, \ldots, R_N$ have a multivariate normal distribution. Therefore $R_x = \sum_{i=0}^{N} x_i R_i$ has a normal distribution as does $W_1 = W_0(1 + R_x)$. Assume the vector of expected returns is $\mu^T = (\mu_0, \mu_1, \mu_2, \ldots, \mu_N)$ with $E[R_i] = \mu_i$ and the covariance matrix is $\Sigma = (\sigma_{ij})$ with $\sigma_{ij} = Cov(R_i, R_j)$.

With $x^T = (x_0, x_1, \ldots x_N)$, where x_i is the proportion in asset i such that $\sum_{i=1}^{N} x_i = 1$, we then have

$$\begin{aligned} E[W_1] &= E\left[W_0\left(1 + \sum_{i=0}^{N} x_i R_i\right)\right] \\ &= W_0\left(1 + \mu^T x\right) \end{aligned}$$

and

$$Var[W_1] = W_0^2 Var(R_x) = W_0^2 \sum_{i=1}^{N}\sum_{j=1}^{N} x_i \sigma_{ij} x_j = W_0^2 x^T \Sigma x$$

We assume that

$$U(W) = -\exp(-bW), \text{ with } b \geq 0 \text{ and } W > 0$$

so that

$$\frac{\partial}{\partial W}U(W) = b\exp\left(-bW\right) > 0$$

The optimisation problem is

$$\max E\left[-\exp\left(-bW_0\left[1+R_x\right]\right)\right]$$

subject to $\sum_{i=0}^{N} x_i = 1$.

Using the moment generating function of the normal distribution we then have the objective

$$\max\left[-\exp\left(-bW_0\left(1+\mu^T x\right)+\frac{1}{2}b^2W_0^2x^T\Sigma x\right)\right]$$

which simplifies to

$$\max\left\{\frac{2}{bW_0}\mu^T x - x^T\Sigma x\right\}$$
$$\text{subject to } \sum_{i=0}^{N} x_i = 1$$

Note that for the multivariate normal distribution assumption

$$\Pr\left(W_0\left[1+R_x\right]\leq 0\right) > 0$$

so that unlimited liability is assumed in this case.

In Panjer et al [15] the objective for mean-variance optimisation is expressed as

$$\max\left\{2\tau\mu^T x - x^T\Sigma x\right\}$$

The objective for exponential utility with Gaussian returns is therefore the same as for the standard mean-variance problem with $\tau = \frac{1}{bW_0}$. Note that as initial wealth W_0 increases the risk tolerance parameter decreases. This is often considered to be an undesirable feature in determining asset allocations since tolerance to risky investments decreases with increasing initial wealth.

4.1.3 Log-normal returns and log-utility

Although standard mean-variance investment selection assumes that returns are multi-variate normal, the empirical evidence does not support this assumption. Over longer time horizons the normal distribution assumption will be better than for shorter horizons but it is still a relatively poor assumption based on empirical evidence. A more realistic assumption is that continuous compounding returns have a normal distribution so that the returns have a log-normal distribution. Since we are interested in the return on a portfolio, the assumption that individual security returns are log-normal is inconsistent with the assumption that the return on a portfolio will have a log-normal distribution since the weighted sum of log-normals is not log-normal. In this section we make the assumption that the portfolio return has a log-normal distribution.

Thus we assume that

$$Y_x = (1 + R_x) = \left(1 + \sum_{i=0}^{N} x_i R_i\right)$$

has a log-normal distribution with

$$E[Y_x] = 1 + \mu^T x$$

and

$$Var(Y_x) = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i \sigma_{ij} x_j = x^T \Sigma x$$

The density function for Y_x is

$$\begin{aligned} f(y_x) &= \left([Var(\ln Y_x)]^{\frac{1}{2}} \sqrt{2\pi} y_x\right)^{-1} \exp\left\{-\frac{1}{2}\left(\frac{\ln y_x - E[\ln Y_x]}{[Var(\ln Y_x)]^{\frac{1}{2}}}\right)^2\right\} \\ \text{for } y_x &> 0 \\ &= 0 \text{ otherwise} \end{aligned}$$

where

$$E[\ln Y_x] = \ln\left(1 + \mu^T x\right) - \frac{1}{2}\ln\left[1 + \frac{x^T \Sigma x}{(1 + \mu^T x)^2}\right]$$

and

$$Var\left(\ln Y_x\right) = \ln\left[1 + \frac{x^T \Sigma x}{\left(1 + \mu^T x\right)^2}\right]$$

If we assume log utility then

$$U\left(W\right) = \ln\left(W\right), \text{ with } W > 0$$

The optimisation problem becomes

$$\begin{aligned} & \max E\left[\ln\left[Y_x\right]\right] \\ = & \max\left\{\ln\left(1 + \mu^T x\right) - \frac{1}{2}\ln\left[1 + \frac{x^T \Sigma x}{\left(1 + \mu^T x\right)^2}\right]\right\} \end{aligned}$$

subject to $\sum_{i=0}^{N} x_i = 1$.

Therefore in the log utility case, assuming a log-normal distribution for the portfolio return, the optimum portfolio is the one that maximises the expected continuous compounding growth rate of the portfolio.

4.2 Dual Theory and Distortion Functions

When the distortion function approach is applied to the portfolio selection problem there are some issues which need to be treated with care. Note that $H_g\left(W_0(1 + R_x)\right)$ is the certainty equivalent of the random end of period wealth. For risk aversion, to be consistent with expected utility, we require

$$H_g\left(W_0(1 + R_x)\right) \leq E\left[W_0\left[1 + R_x\right]\right]$$

so that g must be convex. This contrasts with the insurance case, where the random variable under consideration is a non-negative loss variable i.e. a negative change in wealth. For the insurance non-negative loss variable the function g is concave.

Some special cases with convex g are:

The proportional hazards transform

$$g(x) = x^r,\ r \geq 1$$

The dual-power transform

$$g(x) = 1 - [1 - x]^{\frac{1}{r}}, \;\; r \geq 1$$

Under the dual theory the investor is assumed to solve the following problem

$$\max H_g\left(W_0(1 + R_x)\right)$$

subject to the same constraints as for the standard portfolio selection problem. Assuming that $H_g(R_x)$ is defined, which will only be the case for $R_x \geq 0$, then from the properties of the distortion function

$$H_g\left(W_0(1 + R_x)\right) = W_0 + W_0 H_g(R_x)$$

To apply this risk measure to asset allocation we will need to define $H_g(R_x)$ over all possible values of R_x since asset returns can in general be negative as well as positive. Thus we cannot apply the distortion function approach directly to investment returns for asset allocation purposes without a new definition of a risk measure to allow for both positive and negative returns.

4.2.1 Gaussian returns

As covered earlier, a standard approach to portfolio selection is to assume a multivariate normal distribution for returns. This is the basis for the mean-variance models so often used in practice. If we assume that the returns $R_0, R_1, \ldots, R_N$ have a multivariate normal distribution then $R_x = \sum_{i=0}^{N} x_i R_i$ has a normal distribution with

$$\mu_{W_x} \equiv E\left[W_0(1 + R_x)\right] = W_0\left(1 + \mu^T x\right)$$

and

$$\sigma_{W_x}^2 \equiv Var(W_0(1 + R_x)) = W_0^2 x^T \Sigma x$$

The distortion function only applies to positive values of $W_0(1 + R_x)$ by definition. Assuming a multivariate normal distribution of returns will mean that

$$\Pr\left(W_0\left[1 + R_x\right] \leq 0\right) > 0$$

and the distortion function is not defined for values of $W_0 [1 + R_x] \leq 0$.

We should confine our consideration to distributions of end-of-period wealth with non-negative values. One way of doing this is to constrain the holdings in the asset classes. In this case it is then possible to maximise $H_g (W_0(1 + R_x))$ subject to the constraint that asset class holdings are such that $W_0 [1 + R_x] \geq 0$ and where $W_0(1 + R_x)$ has a Gaussian distribution. In this case we have the optimisation problem

$$\begin{aligned} \max H_g (W_0(1 + R_x)) &= \int_0^{\infty} g \left\{ N \left[\frac{\mu_{W_x} - t}{\sigma_{W_x}} \right] \right\} dt \\ \text{subject to } \sum_{i=0}^{N} x_i &= 1 \end{aligned}$$

If we set

$$u = \frac{\mu_{W_x} - t}{\sigma_{W_x}}$$

then

$$H_g (W_0(1 + R_x)) = \sigma_{W_x} \int_{-\infty}^{\frac{\mu_{W_x}}{\sigma_{W_x}}} g \{N [u]\} du$$

where

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp(-\frac{z^2}{2}) dz.$$

Note that maximising this objective is not necessarily the same as maximising σ_{W_x} for a given μ_{W_x} since the integral decreases as σ_{W_x} increases for a fixed μ_{W_x}.

For the distortion function we have

$$H_g (W_0(1 + R_x)) = \int_0^{\infty} \left\{ N \left[\frac{\mu_{W_x} - t}{\sigma_{W_x}} \right] \right\}^r dt \quad r \geq 1$$

and for the dual distortion function

$$H_g\left(W_0(1+R_x)\right) = \int_0^\infty 1 - \left\{1 - N\left[\frac{\mu_{W_x} - t}{\sigma_{W_x}}\right]\right\}^{\frac{1}{r}} dt \quad r \geq 1$$

4.2.2 Log-normal returns

Assume that $\log(1+R_x)$ is normal then $W_0(1+R_x)$ is log-normal and

$$H_g\left(W_0(1+R_x)\right) = \int_0^\infty g\left\{N\left[\frac{\mu_{\ln W_x} - \log t}{\sigma_{\ln W_x}}\right]\right\} dt$$

where $N(.)$ is the cumulative normal distribution function with

$$\mu_{\ln W_x} \equiv E\left[\ln W_0(1+R_x)\right]$$

$$\sigma^2_{\ln W_x} \equiv Var\left(\ln W_0(1+R_x)\right)$$

The optimisation problem is then to select the portfolio weights to maximise these functions subject to the relevant constraints such as positive holdings in all assets. Numerical experiments with the log-normal return assumption using sample data encountered difficulties in evaluating the objective function.

4.2.3 The Two-Asset Case

As an example of the optimisation for the distortion function we consider the two asset case with asset 0 a riskless asset and asset 1 a risky asset. The objective is

$$\begin{aligned} & \max H_g\left[W_0(1 + x_0 R_0 + (1 - x_0) R_1)\right] \\ = \; & \max W_0\left(1 + H_g(R_1)\right) + x_0 W_0\left(R_0 - H_g(R_1)\right) \end{aligned}$$

Now if $R_0 > H_g(R_1)$ then the solution is to place as much as possible in the riskless asset and to short the risky asset. If short selling is not allowed then the whole of the investor's wealth would be placed in the risk free asset.

Similarly if $R_0 < H_g(R_1)$ then the riskless asset is shorted and a long position taken in the risky asset.

$H_g(R_1)$ can be interpreted as the certainty equivalent of the return. Thus if the risk-adjusted return on the risky asset, $H_g(R_1)$, exceeds the riskless asset return, then as much as possible would be placed in the risky asset. Yaari (1987) [22] refers to this as "plunging" since the investor will place all their wealth into the risky asset once the risk adjusted return exceeds the risk free rate of return.

For expected utility, an investor always places some wealth into the risky asset provided its expected return is positive. Thus there is always some diversification with expected utility but this is not the case with the dual theory where the asset allocation is an "all-or-nothing" strategy.

5 A New Risk Measure

5.1 The Risk Measure

A problem with the distortion function approach in portfolio optimisation is that the measure is defined only for positive random variables. Investment alternatives will usually have both positive and negative outcomes. In order to apply a risk measure to the portfolio problem it is necessary to define the measure over positive and negative outcomes. There are also other problems with applying the distortion function approach to asset allocation identified in the previous section of this paper. This includes the "plunging" result or lack of diversification.

The concept of "regret" as a risk measure has also found applications in asset allocation. Dembo and Freeman, (1998) [3] describe a risk measure based on the concept of "regret" where the downside of a prospect is treated differently to the upside. Downside outcomes are given a value based on what would be paid to insure against this outcome.

Our approach to adapting the distortion function approach to asset allocation is to consider a point α in the wealth outcomes where we treat outcomes above and below this point differently. Thus α could be the return on a benchmark portfolio or the minimum return required to meet a liability on a future date. We take the certainty equivalent of downside outcomes using the distortion function approach treating the downside as a (positive) loss random variable. The certainty equivalent for upside outcomes is also determined using the distortion function approach suitably adapted to investment, rather than loss, random variables.

Recall that a function g is a distortion function if g is concave and increasing with $g(0) = 0$ and $g(1) = 1$. It then follows that $g(x) \geq x$ for $0 \leq x \leq 1$. In fact for such x, $g(x) = g(x.1 + (1-x).0) \geq x.g(1) + (1-x).g(0) = x$. Corresponding to g we define another distortion function h which is convex, with $h(x) \leq x$, increasing and $h(0) = 0$ and $h(1) = 1$. For instance $h(x) = 1 - g(1-x)$ would be one possible choice.

If g is a distortion function and X is a non-negative random variable, then as above, we set

$$H_g(X) = \int_0^\infty g[S_X(t)]dt.$$

We do not write down conditions under which $H_g(X)$ is finite for this non-negative random variable. Without some restrictions it is possible that $H_g(X) = \infty$.

For each choice of $\alpha \in \mathcal{R}$, and g and h distortion functions as defined above, set

$$H_{VHS}(X) \equiv H_{\alpha,g,h}(X) = \alpha + H_h((X-\alpha)^+) - H_g((\alpha - X)^+),$$

where $a^+ = \max[0, a]$. Note that $H_{VHS}(X)$ is now defined for random variables X of arbitrary sign. We do not write down conditions under which $H_{VHS}(X)$ is finite for this random variable X. Without some restrictions it is possible that $H_{VHS}(X)$ could be unbounded. From now on we will write simply $H_{VHS}(X)$, but of course it implies that a choice was made for α, g, h. The paper by Wang (1996a) [18] gives an extensive list of possible types of distortion functions that can be employed in the definition of H. We will refer to this risk measure as the **Van der Hoek - Sherris Risk Measure (VHS - Risk Measure)**.

5.2 Properties of the Risk Measure

Properties that are satisfied by $H_{VHS}(X)$ will follow from properties of H_g which are listed in various places, for example in Wang, Panjer and Young (1996) [20], and proved in Denneberg (1994) [4]. These are listed earlier in this paper.

1. For $X = C$, a constant, we have

$$H_{VHS}(C) = C$$

This follows because

$$\begin{aligned} H_{VHS}(C) &= \alpha + H_h((C-\alpha)^+) - H_g((\alpha - C)^+) \\ &= \alpha + (C-\alpha)^+ - ((\alpha - C)^+ \\ &= C \end{aligned}$$

for any real constant C.

2. For any random variable

$$H_{VHS}(X) \leq E[X]$$

This follows because

$$\begin{aligned} H_{VHS}(X) &= \alpha + H_h((X-\alpha)^+) - H_g((\alpha - X)^+) \\ &\leq \alpha + E[(X-\alpha)^+] - E[(\alpha - X)^+] \\ &= E[\alpha + (X-\alpha)^+ - (\alpha - X)^+] \\ &= E[X] \end{aligned}$$

This argument still holds even if some of the expressions are infinite!

3. H is concave, so that for X and Y random variables and $0 \leq \lambda \leq 1$, then

$$H_{VHS}(\lambda X + (1-\lambda)Y) \geq \lambda H_{VHS}(X) + (1-\lambda)H_{VHS}(Y)$$

The proof uses

(a)

$$\begin{aligned} & H_{VHS}(\lambda X + (1-\lambda)Y) \\ = \ & \alpha + H_h((\lambda X + (1-\lambda)Y - \alpha)^+) - H_g((\alpha - \lambda X + (1-\lambda)Y)^+)\} \end{aligned}$$

(b)

$$(\lambda X + (1-\lambda)Y - \alpha)^+ \leq \lambda (X-\alpha)^+ + (1-\lambda)(Y-\alpha)^+$$

(c)

$$\begin{aligned} & H_h((\lambda X + (1-\lambda)Y - \alpha)^+) \\ \geq \; & H_h(\lambda(X-\alpha)^+ + (1-\lambda)(Y-\alpha)^+) \\ \geq \; & H_h(\lambda(X-\alpha)^+) + H_h((1-\lambda)(Y-\alpha)^+) \\ \geq \; & \lambda H_h((X-\alpha)^+) + (1-\lambda)H_h((Y-\alpha)^+) \end{aligned}$$

(d)

$$\begin{aligned} & H_g((\lambda X + (1-\lambda)Y - \alpha)^+) \\ \leq \; & H_g(\lambda(X-\alpha)^+ + (1-\lambda)(Y-\alpha)^+) \\ \leq \; & H_g(\lambda(X-\alpha)^+) + H_g((1-\lambda)(Y-\alpha)^+) \\ \leq \; & \lambda H_g((X-\alpha)^+) + (1-\lambda)H_g((Y-\alpha)^+) \end{aligned}$$

(e)

$$H_h((\alpha - \lambda X + (1-\lambda)Y)^+) \geq \lambda H_h((\alpha - X)^+) + (1-\lambda)(\alpha - Y)^+)$$

(f)

$$H_g((\alpha - \lambda X + (1-\lambda)Y)^+) \leq \lambda H_g((\alpha - X)^+) + (1-\lambda)(\alpha - Y)^+)$$

(g)

$$H_g(X) \leq H_g(Y)$$

whenever $X \geq Y \geq 0$ with probability 1. This follows because under these conditions $S_X(t) \geq S_Y(t)$ for all $t \geq 0$ and g is an non-decreasing function.

(h) The subadditivity (superadditivity) of H_g for concave (convex) g.

We can now regard the mapping

$$X \to H_{VHS}(X)$$

as an alternative to an expected utility, and we can define the orderings on random variables by $X \succeq Y$ (X is weakly-preferred to Y) if $H_{VHS}(X) \geq H_{VHS}(Y)$ and $X \succ Y$ (X is strictly-preferred to Y) if $H_{VHS}(X) > H_{VHS}(Y)$.

We can now use these orderings in portfolio selection. It is a consequence of the concavity of H_{VHS}, that such a risk measure will imply diversification for portfolio theory. Diversification is not a property of the distortion function risk measure used in insurance pricing if it is applied directly to asset allocation without modification.

5.3 Examples

We now provide some examples of evaluations of $H_{VHS}(X)$ for various distributions for X and explicitly show that for some choices of α, g, h that the ordering so defined above does not satisfy the substitution (or independence) axiom for expected utilities.

5.3.1 Normally distributed wealth

If X is normally distributed $N(\mu, \sigma^2)$, then

$$\begin{aligned} H_{VHS}(X) &= \alpha + \sigma \int_{-\infty}^{\frac{\mu-\alpha}{\sigma}} h(N(u))du - \sigma \int_{-\infty}^{\frac{\alpha-\mu}{\sigma}} g(N(u))du \\ &= \alpha + \sigma \left[\int_{-\infty}^{\frac{\mu-\alpha}{\sigma}} h(N(u))du - \int_{-\infty}^{\frac{\alpha-\mu}{\sigma}} g(N(u))du \right] \end{aligned}$$

where

$$N(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp(-\frac{z^2}{2})dz.$$

VHS - proportional hazards transform and Gaussian Returns The risk measure proposed can use any of the distortion functions used in the insurance literature. We will use the proportional hazards transform and Gaussian returns to illustrate its implications for asset allocations. The proportional hazards transform is

$$g(x) = x^r,\ 0 < r \leq 1$$

and we use

$$h(x) = 1 - (1-x)^r,\ 0 < r \leq 1$$

so that in this case the risk measure becomes

$$H_{VHS}(X) = \alpha + \sigma \left[\int_{-\infty}^{\frac{\mu-\alpha}{\sigma}} 1 - \left[1 - (N(u))\right]^r du - \int_{-\infty}^{\frac{\alpha-\mu}{\sigma}} (N(u))^r du \right]$$

where the change of variable

$$u = \frac{\mu - \alpha - t}{\sigma}$$

is used.

The value used for r can be used as a risk aversion parameter. A parameter value of $r = 0$ would be the risk neutral case and higher values of r would reflect higher values of risk aversion.

For the asset allocation optimisation we define as before

$$\mu_{W_x} \equiv E\left[W_0\left(1 + R_x\right)\right] = W_0\left(1 + \mu^T x\right)$$

$$\sigma_{W_x}^2 \equiv Var(W_0\left(1 + R_x\right)) = W_0^2 x^T \Sigma x$$

Then

$$H_{VHS}\left(W_1\right) = \alpha + \sigma_{W_x} \left[\int_{-\infty}^{\frac{\mu_{W_x}-\alpha}{\sigma_{W_x}}} 1 - \left\{1 - N\left[u\right]\right\}^r du - \int_{-\infty}^{\frac{\alpha-\mu_{W_x}}{\sigma_{W_x}}} \left\{N\left[u\right]\right\}^r du \right]$$

where $W_1 = W_0\left(1 + R_x\right)$, $R_x = \sum_{i=0}^N x_i R_i$. The problem is then to maximise $H_{VHS}(W_1)$ subject to $\sum_{i=0}^N x_i = 1$, and any other constraints on the sign or size of each x_i. The value of α would be selected to reflect a return where the risk function would be different for returns above and below this point. For instance this could be taken as the wealth arising from a risk free investment strategy.

This maximization problem for general distortion functions and Gaussian returns can be solved using Lagrange multipliers to show that the optimal asset allocation $\mathbf{x}$, satisfies

$$\mathbf{x} = \mathbf{a} - \frac{v f'(z)}{\left[-f(z) + z f'(z)\right]} \mathbf{b}$$

where

$$f(z) \equiv \int_{-\infty}^{z} h(N(u))du - \int_{-\infty}^{-z} g(N(u))du$$

$$\mathbf{a} = \frac{\Sigma^{-1}\mathbf{e}}{\mathbf{e}^T\Sigma^{-1}\mathbf{e}}$$

$$\mathbf{b} = \frac{\mathbf{e}^T\Sigma^{-1}\boldsymbol{\mu}}{\mathbf{e}^T\Sigma^{-1}\mathbf{e}}\Sigma^{-1}\mathbf{e} - \Sigma^{-1}\boldsymbol{\mu}$$

$$v^2 = x^T\Sigma x$$

$$z = \frac{\mu_{W_x} - \alpha}{\sigma_{W_x}}$$

and

$$\mathbf{e}^T = (1, 1, \dots, 1)$$

We note that the allocations $\mathbf{x}$ are of the form

$$\mathbf{x} = \mathbf{a} - \eta\mathbf{b}$$

where η satisfies a non-linear equation

$$\eta = \frac{vf'(z)}{[-f(z) + zf'(z)]}$$

where the right hand can be expressed in terms of η by substituting $\mathbf{x} = \mathbf{a} - \eta\mathbf{b}$ in all expressions depending on $\mathbf{x}$.

5.3.2 Binomial Model

In practice asset models are often implemented using discrete distributions of asset returns. In option pricing, the binomial model is one of the most popular models used in practice. For asset allocation the use of similar models allows for numerical optimisation of different investment criteria and makes available the techniques used in option pricing for asset allocation. Pliska (1997) [14] provides coverage of the use of discrete models in asset allocation.

For these models we are interested in the case when X has a Bernoulli type distribution. That is, X takes a value a with probability p and a value b with probability $q = 1 - p$. We can clearly assume without loss of generality that $a \geq b$. In this situation we can compute $H_{VHS}(X)$ explicitly as follows:

$$\begin{aligned} H_{VHS}(X) &= \alpha + (b-\alpha)^+ - (\alpha - a)^+ + h(p)\left[(a-\alpha)^+ - (b-\alpha)^+\right] \\ &\quad - g(q)\left[(\alpha - b)^+ - (\alpha - a)^+\right] \end{aligned}$$

5.4 Non-expected utility property

As mentioned earlier, the risk behaviour of individuals often does not satisfy the axioms of the expected utility theory. In order to address this, a risk measure consistent with actual risk behaviour will need to also satisfy different axioms to those of expected utility. The distortion function approach to pricing satisfies a different set of axioms to those of expected utility. In particular, the independence axiom is not satisfied.

Our risk measure need not satisfy the axioms of expected utility. This example, based on similar examples found in Kahneman and Tversky (1979) [8], demonstrates that it is possible to make a choice of (α, g, h) so that the corresponding H_{VHS} does not obey the **substitution axiom** or **independence axiom** of expected utility (see Huang and Litzenberger (1988) [6]).

We choose throughout this example $\alpha = 0$, h arbitrary, and g a piecewise linear function generated by

$$\begin{aligned} g(0) &= 0, \\ g(0.2) &= 0.40, \\ g(0.25) &= 0.49, \\ g(0.8) &= 0.87 \\ \text{and}\quad g(1.0) &= 1.0. \end{aligned}$$

The slopes for g are as follows:

$$\begin{array}{c}\text{on } [0, 0.2] \text{ it is } 2,\\ \text{on } [0.2, 0.25] \text{ it is } 1.8,\\ \text{on } [0.25, 0.8] \text{ it is } 0.6909\ldots\text{,and}\\ \text{on } [0.8, 1] \text{ it is } 0.65.\end{array}$$

We conclude that g is concave and a legitimate distortion function.
With this choice of (α, g, h), we see that

- $H_{VHS}(X_1) = -5220$, if $X_1 = -6000$ with probability 0.8 and $X_1 = 0$ with probability 0.2;
- $H_{VHS}(Y_1) = -5000$ if $Y_1 = -5000$ with probability 1.

Thus $Y_1 \succ X_1$.
Also

- $H_{VHS}(X_2) = -2400$, if $X_2 = -6000$ with probability 0.2 and $X_2 = 0$ with probability 0.8;
- $H_{VHS}(Y_2) = -2450$ if $Y_2 = -5000$ with probability 0.25 and $Y_2 = 0$ with probability 0.75.

Thus $X_2 \succ Y_2$.

But these conclusions cannot follow from an expected utility ordering. For if so, then $E[U(Y_1)] > E[U(X_1)]$ implies $U(-5000) > 0.8U(-6000)$, for some utility function with $U(0) = 0$, while $E[U(X_2)] > E[U(Y_2)]$ implies $0.2U(-6000) > 0.25U(-5000)$ or $U(-5000) < 0.8U(-6000)$, a contradiction.

We can conclude that the new measure $H_{VHS}(X)$ does not in general give rise to ordering of risks equivalent to that arising from an expected utility ordering. Thus the risk measure is not expected utility maximising and can be consistent with violations of the expected utility axioms.

6 Numerical Examples

In this section of the paper we present the optimal portfolios obtained using the different risk measures covered in this paper. We use hypothetical data for various asset classes. The following tables give the data used in the examples.

The asset classes are US Cash, US Bonds, Japanese Bonds, Euro-bonds, US Equity, Japanese Equity and Euro-equity.

The assumption is made that asset returns are multivariate Gaussian. The following data was used for the return distributions. Expected returns and standard deviations are shown from left to right for these asset classes in the above order. The correlation matrix also shows the correlations for these asset classes in the same order.

Expected Returns

$$\mu = \begin{pmatrix} 6.0 & 6.5 & 6.0 & 5.5 & 9.5 & 9.5 & 9.5 \end{pmatrix}$$

Standard Deviations

$$\sigma = \begin{pmatrix} 0.50 & 5.23 & 14.66 & 12.52 & 14.45 & 26.19 & 17.06 \end{pmatrix}$$

Correlation coefficients

$$\rho = \begin{pmatrix} 1.00 & 0.20 & -0.04 & 0.12 & 0.08 & -0.02 & 0.12 \\ 0.20 & 1.00 & 0.22 & 0.34 & 0.33 & 0.13 & 0.30 \\ -0.04 & 0.22 & 1.00 & 0.68 & -0.07 & 0.53 & 0.30 \\ 0.12 & 0.34 & 0.68 & 1.00 & -0.04 & 0.38 & 0.51 \\ 0.08 & 0.33 & -0.07 & -0.04 & 1.00 & 0.21 & 0.61 \\ -0.02 & 0.13 & 0.53 & 0.38 & 0.21 & 1.00 & 0.49 \\ 0.12 & 0.30 & 0.30 & 0.51 & 0.61 & 0.49 & 1.00 \end{pmatrix}$$

Asset allocations were determined for quadratic and exponential expected utility risk measures assuming a multivariate Gaussian return distribution with these expected returns, standard deviations and correlations.

Both unconstrained allocations and constrained allocations with positive asset holdings were determined for various risk parameters. As noted earlier, the exponential and quadratic utility functions with Gaussian returns produce the same optimal asset allocations as for the mean-variance criteria provided the risk tolerance parameter is equivalent in each case. The mean-variance optimal portfolios were determined but the results are not shown since they correspond to the expected utility results for the cases shown. The equivalent tau for the mean-variance case was determined. The log-optimal portfolio was also determined for a log-normal wealth distribution and is given for comparison purposes.

The asset allocations for normally distributed wealth and the proportional hazards transform example using the new risk measure proposed in this paper were also determined for comparison. A fixed return of 6% was used as the benchmark.

The results are given in the Appendices. Appendix 1 covers quadratic utility, Appendix 2 covers exponential utility, Appendix 3 gives the log-optimal portfolio and Appendix 4 covers the VHS risk measure using a proportional hazards distortion function.

The Tables and Graphs in the Appendices confirm the equivalence of the mean-variance and expected utility optimisation as expected. For the constrained case, once the standard deviation of the optimal portfolio reaches that of the log-optimal portfolio then, in the expected utility case, the optimal portfolio is the log-optimal portfolio.

The VHS risk measure with proportional hazards distortion function produces results almost identical to that for expected utility, at least for the unconstrained asset allocations shown. More realistic return distributions and other distortion functions for the new risk measure are the subject of further research.

It is interesting to note that, for Gaussian return distributions, the asset allocations are effectively the same as those derived from mean-variance optimisation for the risk measures considered.

7 Conclusions

In this paper we propose a new risk measure for use in asset allocation. The risk measure incorporates concepts developed and applied recently to insurance pricing. The measure has properties of risk aversion and diversification. It treats the upside and downside of possible outcomes differently in a similar manner to the concept of "regret". The measure does not satisfy the axioms of expected utility and is therefore not an expected utility risk measure. This means that the measure can handle cases where risk taking behaviour does not conform to the expected utility axioms. These issues will be addressed in extensions of this research.

8 Acknowledgments

The authors would like to acknowledge the research assistance of Paul Park and the financial support of an Australian Research Council Grant and support from an Institute of Actuaries of Australia Research Grant.

References

[1] Allais, M., (1953), "Le Comportement de l'Homme Rationnel devant le Risque", *Econometrica*, 21, 503-546.

[2] Balzer, L., (1994), "Measuring Investment Risk, A Review", *Journal of Investing*, Fall, 47-58.

[3] Dembo, R. S., and A. Freeman, (1998), Seeing Tomorrow: Rewriting the Rules of Risk, John Wiley and Sons, New York.

[4] D.Denneberg 'Non-Additive Measure and Integral' Kluwer Academic Publishers, Boston 1994.

[5] Gerber, H. U., and G. Pafumi, 1998, Utility Functions: From Risk Theory to Finance, *North American Actuarial Journal*, 2 (3), 74-90.

[6] Huang, C. and R. H. Litzenberger, (1988), *Foundations for Financial Economics*, Elsevier Science Publishing, New York.

[7] Ingersoll, J. E., (1987), *The Theory of Financial Decision Making*, Rowman and Littlefield.

[8] Kahneman, D. and A. Tversky, (1979), "Prospect Theory: An Analysis of Decision under Risk", *Econometrica*, 47, 263-291.

[9] Lipman, R. A. (1990), Utility, Benchmarks and Investment Objectives, *Transactions of The Institute of Actuaries of Australia*, 842-870.

[10] Machina, M. J., (1982), "Expected Utility" Analysis without the Independence Axiom, *Econometrica*, Vol 50, No 2, 277-323.

[11] Machina, M.J., (1987), Choice Under Uncertainty: Problems Solved and Unsolved, *Economic Perspectives*, Vol 1, No 1, 121-154.

[12] Markowitz, H. (1952), Portfolio Selection, *Journal of Finance*, 7: 77-91.

[13] Markowitz, H. (1959), *Portfolio Selection*, Yale, Blackwell.

[14] Pliska, S., (1997), *Introduction to Mathematical Finance: Discrete Time Models*, Oxford, Blackwell.

[15] Panjer, H. H. (Editor), P.P. Boyle, S. H. Cox, D Dufresne, H. U. Gerber, H. H. Mueller, H. W. Pedersen. S. R. Pliska, M. Sherris, E. S. Shiu, K. S. Tan, 1998, *Financial Economics with Applications to Investments, Insurance and Pensions*, The Actuarial Foundation, Schaumburg, Illinois.

[16] Von Neumann, J and O. Morgenstern, (1944), *Theory of Games and Economic Behavior*, Princeton, N.J., Princeton University Press.

[17] Wang, S. (1995), "Insurance pricing and increased limits ratemaking by proportional hazard transforms", *Insurance: Mathematics and Economics*, 17, 43-54.

[18] Wang, S. (1996a), "Premium Calculation by Transforming the Layer Premium Density", *Astin Bulletin*, 26, 71-92.

[19] Wang, S. (1996b), Ambiguity-Aversion and the Economics of Insurance, Institute of Insurance and Pension Research, University of Waterloo, *Research Report 96-04*.

[20] Wang, S., H.Panjer and V.R.Young, 'Axiomatic Characterization of Insurance Prices' Working Paper 96-14, University of Waterloo.

[21] Wang, S and Young, V. R., (1997), Ordering Risks: Utility Theory versus Yaari's Dual Theory of Risk, *IIPR Research Report 97-08*, University of Waterloo.

[22] Yaari, M. E., (1987), The Dual Theory of Choice Under Risk, *Econometrica*, 55, 95-115.

Appendix

Quadratic Utility with Gaussian Returns

Exponential Utility with Gaussian Returns

Van der Hoek - Sherris Risk Measure (VHS Risk Measure)