

# CAPM and option pricing with elliptically contoured distributions

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## Abstract

This paper offers an alternative proof of the Capital Asset Pricing Model (CAPM) when asset returns follow a multivariate elliptical distribution. Empirical studies continue to demonstrate the inappropriateness of the normality assumption for modelling asset returns. The class of elliptically contoured distributions, which includes the more familiar Normal distribution, provides flexibility in modelling the thickness of tails associated with the possibility that asset returns take extreme values with non-negligible probabilities. As summarized in this paper, this class preserves several properties of the Normal distribution. Within this framework, we prove a new version of Stein's lemma for this class of distributions and use this result to derive the CAPM when returns are elliptical. Furthermore, using the probability distortion function approach based on the dual utility theory of choice under uncertainty, we also derive an explicit form solution to call option prices when the underlying is log-elliptically distributed. The Black-Scholes call option price is a special case of this general result when the underlying is log-normally distributed.

## 1 Introduction

This paper considers the general class of symmetric distributions in extending familiar results of Capital Asset Pricing Model (CAPM) and the theory of asset pricing. This class, called the class of elliptically contoured (or simply elliptical) distributions, includes the familiar Normal distribution and shares many of its familiar properties. However, this family of distributions provides greater flexibility in modelling tails or extremes that are becoming commonly important in financial economics. Apart from this flexibility, it preserves several well-known properties of the Normal distribution allowing one to derive attractive explicit solution forms. Its usefulness is increasingly appearing in the finance and insurance literature. In the financial literature, Bingham and Kiesel (2002) propose a semi-parametric model for stock prices and asset returns based on elliptical distributions because as the authors observed, Gaussian or normal models provide mathematical tractability but are inconsistent with empirical data. Embrechts, McNeil, and Straumann (2001) also demonstrated that within this class of distributions, the property of the Markowitz minimum variance portfolio is preserved. Landsman and Valdez (2003) provided explicit forms of the tail conditional expectation, an increasingly popular risk measure, for this class. Applications in capital allocation for financial institutions have also been proposed in Valdez and Chernih (2003).

In this paper, we further exploit the attractive properties of this family of distributions in asset pricing. In particular, as an illustration, the classical CAPM result expressed as

$$\mathbb{E}(R_k) = R_F + \beta [\mathbb{E}(R_M) - R_F] \quad (1)$$

gives the expected return on an asset  $k$  as a linear function of the risk-free rate  $R_F$  and the expected return on the market. This result can be derived by assuming asset returns are multivariate normally distributed. See, for example, ?, Lintner (1965), and Mossin (1966). It has been demonstrated in Owen and Rabinovitch (1983) and again, in Ingersoll (1987), that relaxing this normality assumption into the wider class of elliptical distributions preserves the result in (1). This paper re-investigates this CAPM result under the general class of elliptical distributions by first offering a proof of a version of Stein's Lemma for elliptical distributions, and then using this result to allow us to further demonstrate the preservation of the CAPM formula under this general class. For Normal distributions, the Stein's lemma states that for a bivariate normal random variable  $(X, Y)$ , we have

$$Cov(X, h(Y)) = \mathbb{E}[h'(Y)] \cdot Cov(X, Y) \quad (2)$$

for any differentiable  $h$  satisfying certain regularity conditions. See Stein (1973, 1981). This paper extends the Stein's lemma in the case where we have a bivariate elliptical random vector  $(X, Y)$ . The result states that we can express the same covariance in (2) as

$$Cov[h(X), Y] = \frac{c}{\tilde{c}} \cdot \mathbb{E}[h'(\tilde{X})] \cdot Cov(X, Y)$$

where  $c$  is the normalizing constant in the density of  $(X, Y)$ , and  $\tilde{X}$  is another elliptical random variable with normalizing constant  $\tilde{c}$ .

Furthermore, this time in a continuous-time setup, this paper considers option pricing when the underlying security price has an elliptical distribution. Using probability distortion functions based on the dual theory of choice under uncertainty Yaari (1987), we are able to derive an explicit form of the solution to the fair premium of a European call option. In particular, suppose that the underlying security price process  $\{S_t\}$  follows a log-elliptical (analogously defined for log-normal) distribution. Then, it is shown in this paper that the fair premium for a European call option with this underlying security has the following form

$$I(X, \alpha) = e^{\mu + \sigma \alpha} \psi(-\sigma^2) F_{U^*} \left( \frac{\mu + \sigma \alpha - \log K}{\sigma} \right) - K F_U \left( \frac{\mu + \sigma \alpha - \log K}{\sigma} \right).$$

The value of this call option collapses to the familiar Black-Scholes option price in the special case when the elliptical distribution is Normal.

This paper is organized as follows. In Section 2, we introduce elliptical distributions, as in Fang, Kotz, and Ng (1987). We develop and repeat some results that will be used in later sections. Most results proved elsewhere are simply stated, but some basic useful results are also proved. In Section 3, we state and prove the Stein's lemma for elliptical distributions. Section 4 provides a re-derivation of the CAPM assuming multivariate elliptical distribution of returns in a discrete one period model. Section 5 derives European call option price in a continuous-time setup when the underlying is log-elliptically distributed. We conclude in Section 6.

## 2 Elliptical distributions and their properties

The class of elliptical distribution models provides a generalization of the class of normal models. In the following, we describe this class of models in the multivariate dimension. This class has been introduced in the statistical literature by Kelker (1970) and widely discussed in Fang, Kotz, and Ng (1987) and Gupta and Varga (1993).

### 2.1 Characteristic function and fensity

Recall that a random vector  $\mathbf{X} = (X_1, \dots, X_n)^T$  is said to have an  $n$ -dimensional Normal distribution if its characteristic function is given by

$$\mathbb{E} [\exp(it^T \mathbf{X})] = \exp(it^T \boldsymbol{\mu}) \exp(-\frac{1}{2} \mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n). \quad (3)$$

for some fixed vector  $\boldsymbol{\mu}(n \times 1)$  and some fixed matrix  $\boldsymbol{\Sigma}(n \times n)$ . We use the notation  $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  for random vectors belonging to the class of multivariate Normal distributions with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$ . It is well-known that the vector  $\boldsymbol{\mu}$  is the mean vector and that the matrix  $\boldsymbol{\Sigma}$  is the variance-covariance matrix. Note that we can also write

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \mathbf{A} \mathbf{Z},$$

where  $\mathbf{Z} = (Z_1, \dots, Z_m)^T$  is a random vector consisting of  $m$  mutually independent standard Normal random variables,  $\mathbf{A}$  is an  $n \times m$  matrix,  $\boldsymbol{\mu}$  is a  $n \times 1$  vector and  $\stackrel{d}{=}$  stands for "equality in distribution". The relationship between  $\boldsymbol{\Sigma}$  and  $\mathbf{A}$  is given by  $\boldsymbol{\Sigma} = \mathbf{A} \mathbf{A}^T$ .

The class of multivariate elliptical distributions is a natural extension to the class of multivariate Normal distributions.

**Definition 1** *The random vector  $\mathbf{X} = (X_1, \dots, X_n)^T$  is said to have an elliptical distribution with parameters  $\boldsymbol{\mu}(n \times 1)$  and  $\boldsymbol{\Sigma}(n \times n)$  if its characteristic function can be written as*

$$\mathbb{E} [\exp(it^T \mathbf{X})] = \exp(it^T \boldsymbol{\mu}) \psi(\frac{1}{2} \mathbf{t}^T \boldsymbol{\Sigma} \mathbf{t}), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n), \quad (4)$$

for some scalar function  $\psi$  and where  $\boldsymbol{\Sigma}$  is given by

$$\boldsymbol{\Sigma} = \mathbf{A} \mathbf{A}^T \quad (5)$$

for some matrix  $\mathbf{A}(n \times m)$ .

We can conveniently write  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$  and say that  $\mathbf{X}$  is elliptical, if the random vector  $\mathbf{X}$  has the elliptical distribution as defined above. The function  $\psi$  is called the *characteristic generator* of  $\mathbf{X}$ , and clearly, the case of the multivariate Normal distribution gives  $\psi(u) = \exp(-u)$ . It is well-known that the characteristic function of a random vector always exists and there exists a one-to-one

correspondence between distribution functions and characteristic functions. However, not every function  $\psi$  may be used to construct a characteristic function of an elliptical distribution. First, the function  $\psi$  must meet the requirement that  $\psi(0) = 1$ . A necessary and sufficient condition for the function  $\psi$  to be a characteristic generator of an  $n$ -dimensional elliptical distribution is given in Theorem 2.2 of Fang, Kotz, and Ng (1987).

In the sequel, we shall denote the elements of  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  by  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$  and  $\boldsymbol{\Sigma} = (\sigma_{kl})$  for  $k, l = 1, 2, \dots, n$ , respectively. Note that (5) guarantees that the matrix  $\boldsymbol{\Sigma}$  is symmetric, positive definite and has positive elements on the first diagonal. Hence, for any  $k$  and  $l$ , one has that  $\sigma_{kl} = \sigma_{lk}$ , whereas  $\sigma_{kk} \geq 0$  which will often be denoted by  $\sigma_k^2$ .

It is interesting to note that in the one-dimensional case, the class of elliptical distributions consists mainly of the class of symmetric distributions which include well-known distributions like normal and student  $t$ .

The random vector  $\mathbf{X}$  does not, in general, possess a density  $f_{\mathbf{X}}(\mathbf{x})$ , but if it does, it will have the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\boldsymbol{\Sigma}|}} g_n \left[ \frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right], \quad (6)$$

for some non-negative function  $g_n(\cdot)$  called the *density generator* and for some constant  $c_n$  called the normalizing constant. This density generator is subscripted with an  $n$  to emphasize that it may depend on the dimension of the vector. We shall drop this  $n$ , and simply write  $g$ , in the univariate case. It was demonstrated in Landsman and Valdez (2003) that the normalizing constant  $c_n$  in (6) can be explicitly determined using

$$c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty x^{n/2-1} g_n(x) dx \right]^{-1} \quad (7)$$

The condition

$$0 < \int_0^\infty x^{n/2-1} g_n(x) dx < \infty \quad (8)$$

guarantees  $g_n(x)$  to be density generator (see Fang, Kotz, and Ng (1987) and therefore the existence of the density of  $\mathbf{X}$ . Alternatively, we may introduce the elliptical distribution via the density generator and we then write  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ .

## 2.2 Sums and linear combinations of elliptical

The class of elliptical distributions possesses the linearity property which is quite useful for portfolio theory. Indeed, an investment portfolio is usually a linear combination of several assets. The linearity property can be briefly summarized as follows: If the returns on assets are assumed to have elliptical distributions, then the return on a portfolio of these assets will also have an elliptical distribution.

From (4), it follows that if  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$  and  $A$  is some  $m \times n$  matrix of rank  $m \leq n$  and  $b$  some  $m$ -dimensional column vector, then

$$A\mathbf{X} + b \sim E_m(A\boldsymbol{\mu} + b, A\boldsymbol{\Sigma}A^T, g_m). \quad (9)$$

In other words, any linear combination of elliptical distributions is another elliptical distribution with the same characteristic generator  $\psi$  or from the same sequence of density generators  $g_1, \dots, g_n$ , corresponding to  $\psi$ . Therefore, any marginal distribution of  $\mathbf{X}$  is also elliptical with the same characteristic generator. In particular, for  $k = 1, 2, \dots, n$ ,  $X_k \sim E_1(\mu_k, \sigma_k^2, g_1)$  so that its density can be written as

$$f_{X_k}(x) = \frac{c_1}{\sigma_k} g_1 \left[ \frac{1}{2} \left( \frac{x - \mu_k}{\sigma_k} \right)^2 \right]. \quad (10)$$

If we define the sum  $S = X_1 + X_2 + \cdots + X_n = \mathbf{e}^T \mathbf{X}$ , where  $\mathbf{e}$  is a column vector of ones with dimension  $n$ , then it immediately follows that

$$S \sim E_n(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e}, g_1). \quad (11)$$

An  $n$ -dimensional random vector  $\mathbf{X} = (X_1, \dots, X_n)^T$  is Normal with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  if and only if for any vector  $\mathbf{b}$ , the linear combination  $\mathbf{b}^T \mathbf{X}$  of the marginals  $X_k$  has a univariate Normal distribution with mean  $\mathbf{b}^T \boldsymbol{\mu}$  and variance  $\mathbf{b}^T \boldsymbol{\Sigma} \mathbf{b}$ . It is straightforward to generalize this result into the case of multivariate elliptical distributions.

**Theorem 1** *The  $n$ -dimensional random vector  $\mathbf{X}$  is elliptical with parameters  $\boldsymbol{\mu}(n \times 1)$  and  $\boldsymbol{\Sigma}(n \times n)$ , that is,  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ , if and only if for any vector  $\mathbf{b}(n \times 1)$ , one has that*

$$\mathbf{b}^T \mathbf{X} \sim E_1(\mathbf{b}^T \boldsymbol{\mu}, \mathbf{b}^T \boldsymbol{\Sigma} \mathbf{b}, \psi).$$

**Proof.** See Fang, Kotz, and Ng (1987). ■

From Theorem 1, we find in particular that for  $k = 1, 2, \dots, n$ ,

$$X_k \sim E_1(\mu_k, \sigma_k^2, \psi). \quad (12)$$

Hence, the marginal components of a multivariate elliptical distribution have an elliptical distribution with the same characteristic generator.

As

$$S = \sum_{k=1}^n X_k = \mathbf{e}^T \mathbf{X}$$

where  $\mathbf{e}(n \times 1) = (1, 1, \dots, 1)^T$ , it follows that

$$S \sim E_1(\mathbf{e}^T \boldsymbol{\mu}, \mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e}, \psi) \quad (13)$$

where  $\mathbf{e}^T \boldsymbol{\mu} = \sum_{k=1}^n \mu_k$  and  $\mathbf{e}^T \boldsymbol{\Sigma} \mathbf{e} = \sum_{k=1}^n \sum_{l=1}^n \sigma_{kl}$ .

In the following theorem, it is stated that any random vector with components that are linear combinations of the components of an elliptical distribution is again an elliptical distribution with the same characteristic generator.

**Theorem 2** *For any matrix  $\mathbf{B}(m \times n)$ , any vector  $\mathbf{c}(m \times 1)$  and any random vector  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ , we have that*

$$\mathbf{B}\mathbf{X} + \mathbf{c} \sim E_m(\mathbf{B}\boldsymbol{\mu} + \mathbf{c}, \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}^T, \psi). \quad (14)$$

**Proof.** See Fang, Kotz, and Ng (1987). ■

### 2.3 Mean and covariance property

As pointed out by Embrechts, McNeil, and Straumann (2001), the linear correlation measure provides a canonical scalar measure of dependencies for elliptical distributions. Observe that the condition (8) does not require the existence of the mean and covariance of vector  $\mathbf{X}$ . However, if the mean vector exists, it will be  $\mathbb{E}(\mathbf{X}) = \boldsymbol{\mu}$ , and if the covariance matrix exists, it will be

$$\text{Cov}(\mathbf{X}) = -\psi'(0) \boldsymbol{\Sigma}, \quad (15)$$

where  $\psi'$  denotes the first derivative of the characteristic function. See Fang, Kotz, and Ng (1987). The characteristic generator can be chosen such that  $\psi'(0) = -1$  leaving us with the variance matrix  $Cov(\mathbf{X}) = \mathbf{\Sigma}$ . A necessary condition for this covariance matrix to exist is

$$|\psi'(0)| < \infty,$$

see Cambanis, Huang, and Simons (1981). We shall denote the elements of  $\boldsymbol{\mu}$  and  $\mathbf{\Sigma}$  respectively by

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)^T$$

and

$$\mathbf{\Sigma} = (\sigma_{ij}) \text{ for } i, j = 1, 2, \dots, n.$$

The diagonals of  $\mathbf{\Sigma}$  are often written as  $\sigma_{kk} = \sigma_k^2$ . Observe that the matrix  $\mathbf{\Sigma}$  coincides with the covariance matrix up to a constant. However, this is not quite true for the correlation, because if we take any pairs  $(X_i, X_j)$ , we have its correlation expressed as

$$\rho(X_i, X_j) = \frac{Cov(X_i, X_j)}{\sqrt{Var(X_i) \cdot Var(X_j)}} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}} \cdot \sqrt{\sigma_{jj}}}. \quad (16)$$

In the special case where  $\boldsymbol{\mu} = (0, \dots, 0)^T$ , the zero vector, and  $\mathbf{\Sigma} = \mathbf{I}_n$ , the identity matrix, we have the standard elliptical, oftentimes called spherical, random vector, and in which case, we shall denote it by  $\mathbf{Z}$ .

It is easily seen that Theorem 2 is a generalization of Theorem 1. The moments of  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \mathbf{\Sigma}, \psi)$  do not necessarily exist. However, it can be shown that if  $\mathbb{E}(X_k)$  exists, then it will be given by

$$\mathbb{E}(X_k) = \mu_k \quad (17)$$

so that  $\mathbb{E}(\mathbf{X}) = \boldsymbol{\mu}$ , if the mean vector exists.

## 2.4 Spherical distributions

An  $n$ -dimensional random vector  $\mathbf{Z} = (Z_1, \dots, Z_n)^T$  is said to have a multivariate standard Normal distribution if all the  $Z_i$ 's are mutually independent and standard Normally distributed. We will write this as  $\mathbf{Z} \sim N_n(\mathbf{0}_n, \mathbf{I}_n)$ , where  $\mathbf{0}_n$  is the  $n$ -vector with  $i$ -th element  $\mathbb{E}(Z_i) = 0$  and  $\mathbf{I}_n$  is the  $n \times n$  covariance matrix which equals the identity matrix in this case. The characteristic function of  $\mathbf{Z}$  is given by

$$\mathbb{E}[\exp(it^T \mathbf{Z})] = \exp\left(-\frac{1}{2} \mathbf{t}^T \mathbf{t}\right), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n). \quad (18)$$

Hence from (18), we find that the characteristic generator of  $N_n(\mathbf{0}_n, \mathbf{I}_n)$  is given by  $\psi(u) = \exp(-u)$ . The class of multivariate spherical distributions is an extension of the class of standard multivariate Normal distributions.

**Definition 2** A random vector  $\mathbf{Z} = (Z_1, \dots, Z_n)^T$  is said to have an  $n$ -dimensional spherical distribution with characteristic generator  $\psi$  if  $\mathbf{Z} \sim E_n(\mathbf{0}_n, \mathbf{I}_n, \psi)$ .

We will often use the notation  $S_n(\psi)$  for  $E_n(\mathbf{0}_n, \mathbf{I}_n, \psi)$  in the case of spherical distributions. From the definition above, we find the following corollary.

**Corollary 1** The random vector  $\mathbf{Z}$  has a spherical distribution and we write  $\mathbf{Z} \sim S_n(\psi)$  if and only if

$$\mathbb{E}[\exp(it^T \mathbf{Z})] = \psi\left(\frac{1}{2} \mathbf{t}^T \mathbf{t}\right), \quad \mathbf{t}^T = (t_1, t_2, \dots, t_n) \dots \quad (19)$$

Consider an  $m$ -dimensional random vector  $\mathbf{X}$  such that

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + \mathbf{AZ}, \quad (20)$$

for some vector  $\boldsymbol{\mu}(n \times 1)$ , some matrix  $\mathbf{A}(n \times m)$  and some  $m$ -dimensional elliptical random vector  $\mathbf{Z} \sim S_m(\psi)$ . Then it is straightforward to prove that  $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ , where the variance matrix is given by  $\boldsymbol{\Sigma} = \mathbf{AA}^T$ .

From equation (19), it immediately follows that the correlation matrices of members of the class of spherical distributions are identical. That is, if we let  $\text{Corr}(\mathbf{Z}) = (r_{ij})$  be the correlation matrix, then

$$r_{ij} = \frac{\text{Cov}(Z_i, Z_j)}{\sqrt{\text{Var}(Z_i)\text{Var}(Z_j)}} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}.$$

The factor  $-\psi'(0)$  in the covariance matrix enters into the covariance structure but cancels in the correlation matrix. Note that although the correlations between different components of a spherical distributed random variable are 0, this does not imply that these components are mutually independent. As a matter of fact, it can only be independent if they belong to the family of multivariate normal distributions.

Observe that from the characteristic functions of  $\mathbf{Z}$  and  $\mathbf{a}^T\mathbf{Z}$ , one immediately finds the following result.

**Lemma 1**  $\mathbf{Z} \sim S_n(\phi)$  if and only if for any  $n$ -dimensional vector  $\mathbf{a}$ , one has that

$$\frac{\mathbf{a}^T\mathbf{Z}}{\sqrt{\mathbf{a}^T\mathbf{a}}} \sim S_1(\phi). \quad (21)$$

As a special case of this result, we find that any component  $Z_i$  of  $\mathbf{Z}$  has a  $S_1(\psi)$  distribution. Furthermore, from the results concerning elliptical distributions, we find that if a spherical random vector  $\mathbf{Z} \sim S_n(\psi)$  possess a density  $f_{\mathbf{Z}}(\mathbf{z})$ , then it will have the form

$$f_{\mathbf{Z}}(\mathbf{z}) = cg\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right), \quad (22)$$

where for some density generator  $g$  and a normalizing constant  $c$ . Furthermore, the opposite also holds: any non-negative function  $g(\cdot)$  satisfying the condition (8) can be used to define an  $n$ -dimensional density  $cg\left(\frac{1}{2}\mathbf{z}^T\mathbf{z}\right)$  of a spherical distribution with a normalizing constant  $c$ . One often writes  $S_n(g)$  for the  $n$ -dimensional spherical distribution generated from the density generator  $g(\cdot)$ .

## 2.5 Log-elliptical distributions

Multivariate log-elliptical distributions are natural generalizations of multivariate log-normal distributions. For any  $n$ -dimensional vector  $\mathbf{x} = (x_1, \dots, x_n)^T$  with positive components  $x_i$ , we define

$$\log \mathbf{x} = (\log x_1, \log x_2, \dots, \log x_n)^T.$$

Recall that an  $n$ -dimensional random vector has a multivariate log-normal distribution if  $\log \mathbf{X}$  has a multivariate normal distribution. In this case we have that  $\log \mathbf{X} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ .

**Definition 3** The random vector  $\mathbf{X}$  is said to have a multivariate log-elliptical distribution with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  if  $\log \mathbf{X}$  has an elliptical distribution:

$$\log \mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi). \quad (23)$$

In the sequel, we shall denote  $\log \mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$  as  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ . When  $\boldsymbol{\mu} = \mathbf{0}_n$  and  $\boldsymbol{\Sigma} = \mathbf{I}_n$ , we shall write  $\mathbf{X} \sim LS_n(\psi)$ . Clearly, if  $\mathbf{Y} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$  and  $\mathbf{X} = \exp(\mathbf{Y})$ , then  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ .

If the density of  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$  exists, then the density of  $\mathbf{Y} = \log \mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$  also exists. From (6), it follows that the density of  $\mathbf{X}$  must be equal to

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c}{\sqrt{|\boldsymbol{\Sigma}|}} \left( \prod_{k=1}^n x_k^{-1} \right) g \left[ \frac{1}{2} (\log \mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\log \mathbf{x} - \boldsymbol{\mu}) \right], \quad (24)$$

see Fang et al. (1990). The density of the multivariate log-normal distribution with parameters  $\boldsymbol{\mu}$  and  $\boldsymbol{\Sigma}$  follows from (24) with  $g(u) = \exp(-u)$ . Furthermore, any marginal distribution of a log-elliptical distribution is again log-elliptical. This immediately follows from the properties of elliptical distributions.

**Theorem 3** *Let  $\mathbf{X} \sim LE_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \psi)$ . If the mean of  $X_k$  exists, then it is given by*

$$\mathbb{E}(X_k) = e^{\mu_k} \cdot \psi\left(-\frac{1}{2}\sigma_k^2\right).$$

*Provided the covariances exist, they are given by*

$$\text{Cov}(X_k, X_l) = \exp(\mu_k + \mu_l) \cdot \left\{ \psi\left[-\frac{1}{2}(\sigma_k + \sigma_l)^2\right] - \psi\left(-\frac{1}{2}\sigma_k^2\right) \cdot \psi\left(-\frac{1}{2}\sigma_l^2\right) \right\}.$$

**Proof.** See Dhaene and Valdez (2003). ■

### 3 Stein's lemma for elliptical distributions

Charles Stein (1973, 1981) used the property of the exponential function inherent in Normal distributions and integration by parts to prove the following result: If the random pair  $(X, Y)$  has a bivariate Normal distribution and  $h$  is a differentiable function satisfying the condition that

$$\mathbb{E}[h'(X)] < \infty,$$

then

$$\text{Cov}[h(X), Y] = \mathbb{E}[h'(X)] \cdot \text{Cov}(X, Y).$$

In this section, we extend Stein's lemma for elliptical distributions. Besides the advantage gained by proving a new result, this has also applications in proving the Capital Asset Pricing Model when the asset returns are multivariate elliptical. First, we prove the following lemma.

**Lemma 2** *Suppose  $X \sim E_1(\mu_X, \sigma_X^2, g)$  with density generator  $g$  and a normalizing constant  $c$ . For any differentiable function  $h$  satisfying  $\mathbb{E}[|h'(X)|] < \infty$ , we have*

$$\sigma_X^2 \mathbb{E}[h'(X)] = \frac{c}{c^*} \cdot \mathbb{E}[h(X^*)(X^* - \mu)] \quad (25)$$

where the random variable  $X^* \sim E_1(\mu_X, \sigma_X^2, -g')$  with density generator  $-g'$  and  $c^*$  as the normalizing constant.

**Proof.** We have

$$\mathbb{E}[h'(X)] = \int_{-\infty}^{\infty} h'(x) \frac{c}{\sigma_X} g \left[ \frac{1}{2} \left( \frac{x - \mu}{\sigma_X} \right)^2 \right] dx$$

where the normalizing constant  $c$  is given in (7) with  $n = 1$ . Applying integration by parts with

$$u = \frac{c}{\sigma_X} g \left[ \frac{1}{2} \left( \frac{x - \mu}{\sigma_X} \right)^2 \right] \quad du = \frac{c}{\sigma_X} g' \left[ \frac{1}{2} \left( \frac{x - \mu}{\sigma_X} \right)^2 \right] \frac{1}{\sigma_X} (x - \mu) dx,$$

$$dv = h'(x) dx \quad v = h(x)$$

we obtain

$$\mathbb{E} [h'(X)] = \frac{c}{\sigma_X} g \left[ \frac{1}{2} \left( \frac{x - \mu}{\sigma_X} \right)^2 \right] h(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} h(x) \frac{c}{\sigma_X^2} g' \left[ \frac{1}{2} \left( \frac{x - \mu}{\sigma_X} \right)^2 \right] (x - \mu) dx.$$

The first term of the above equality vanishes due to the condition imposed on  $h$  and also the property of the density generator  $g$ . Thus, it follows that

$$\sigma_X^2 \mathbb{E} [h'(X)] = \int_{-\infty}^{\infty} h(x) (x - \mu) \frac{c}{\sigma_X} \left( -g' \left[ \frac{1}{2} \left( \frac{x - \mu}{\sigma_X} \right)^2 \right] \right) dx.$$

Now, define the random variable  $X^* \sim E_1(\mu, \sigma_X^2, -g')$  where the density generator of  $X^*$  is the negative derivative of the density generator of  $X$ . By defining  $c^*$  to be normalizing constant, then the equation above becomes

$$\sigma_X^2 \mathbb{E} [h'(X)] = \frac{c}{c^*} \int_{-\infty}^{\infty} h(x) (x - \mu) \frac{c^*}{\sigma_X} \left( -g' \left[ \frac{1}{2} \left( \frac{x - \mu}{\sigma_X} \right)^2 \right] \right) dx.$$

Recall that

$$c = \frac{\Gamma(1/2)}{\sqrt{2\pi}} \left[ \int_0^{\infty} x^{-\frac{1}{2}} g(x) dx \right]^{-1} \quad \text{whereas} \quad c^* = \frac{\Gamma(1/2)}{\sqrt{2\pi}} \left[ \int_0^{\infty} x^{-\frac{1}{2}} (-g'(x)) dx \right]^{-1}$$

so that

$$\frac{c}{c^*} = \frac{\int_0^{\infty} x^{-\frac{1}{2}} (-g'(x)) dx}{\int_0^{\infty} x^{-\frac{1}{2}} g(x) dx}.$$

Therefore, the result immediately follows that

$$\sigma_X^2 \mathbb{E} [h'(X)] = \frac{c}{c^*} \mathbb{E} [h(X^*)(X^* - \mu)].$$

■

Note that in the case of the Normal distribution, the density generator is given by  $g(x) = e^{-x}$  so that  $g'(x) = -e^{-x}$ . Hence  $-g'(x) = -g(x)$  and therefore

$$\sigma_X^2 \mathbb{E} [h'(X)] = \mathbb{E} [h(X)(X - \mu)].$$

In this case, we have  $X^* \stackrel{d}{=} X$ . This same result has been used in Casella and Berger (2002).

Notice that equation (25) in the lemma can be re-stated equivalently as

$$\sigma_X^2 \mathbb{E} [h'(\tilde{X})] = \frac{\tilde{c}}{c} \cdot \mathbb{E} [h(X)(X - \mu)] \tag{26}$$

where the random variable  $\tilde{X} \sim E_1(\mu, \sigma_X^2, -\int g)$  with  $\tilde{c}$  as its normalizing constant. This follows immediately by taking the density function of  $X$  to be the negative primitive,  $-\int g$ , of  $g$ .

**Lemma 3 Stein's Lemma for Elliptical.** Let the bivariate vector  $(X, Y) \sim E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)$  with density generator denoted by  $g_2$  and

$$\boldsymbol{\mu} = \begin{pmatrix} \mathbb{E}(X) \\ \mathbb{E}(Y) \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}.$$

For any differentiable function  $h$  satisfying  $\mathbb{E}[|h'(X)|] < \infty$ , we have

$$\text{Cov}[h(X), Y] = \frac{c}{\tilde{c}} \cdot \text{Cov}(X, Y) \cdot \mathbb{E}[h'(\tilde{X})] \quad (27)$$

where  $\tilde{X} \sim E_1(\mu, \sigma_X^2, -\int g)$ .

**Proof.** Note that

$$\begin{aligned} \text{Cov}[h(X), Y] &= \mathbb{E}[(h(X) - \mathbb{E}[h(X)])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}[(h(X) - \mathbb{E}[h(X)])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}_X[\mathbb{E}_Y[(h(X) - \mathbb{E}[h(X)])(Y - \mathbb{E}[Y]|X)] \\ &= \mathbb{E}_X \left[ (h(X) - \mathbb{E}[h(X)]) \underbrace{\mathbb{E}_Y[Y - \mathbb{E}[Y]|X]}_{\text{Conditional variance}} \right]. \end{aligned}$$

It can be shown (see Dhaene and Valdez (2003)) that

$$\mathbb{E}_Y[Y - \mathbb{E}[Y]|X] = \frac{\text{Cov}(X, Y)}{\sigma_X^2} (X - \mathbb{E}[X]).$$

Thus,

$$\begin{aligned} \text{Cov}[h(X), Y] &= \frac{\text{Cov}(X, Y)}{\sigma_X^2} \mathbb{E}[(h(X) - \mathbb{E}[h(X)])(X - \mathbb{E}[X])] \\ &= \frac{\text{Cov}(X, Y)}{\sigma_X^2} \mathbb{E}[h(X)(X - \mathbb{E}[X])]. \end{aligned}$$

Using the equation (26) resulting from the previous lemma, we can write

$$\mathbb{E}[h(X)(X - \mathbb{E}[X])] = \frac{c}{\tilde{c}} \cdot \sigma_X^2 \cdot \mathbb{E}[h'(\tilde{X})]$$

where  $\tilde{X} \sim E_1(\mu_X, \sigma_X^2, -\int g)$ . Thus, we have

$$\begin{aligned} \text{Cov}[h(X), Y] &= \frac{\text{Cov}(X, Y)}{\sigma_X^2} \cdot \frac{c}{\tilde{c}} \cdot \sigma_X^2 \cdot \mathbb{E}[h'(\tilde{X})] \\ &= \frac{c}{\tilde{c}} \cdot \text{Cov}(X, Y) \cdot \mathbb{E}[h'(\tilde{X})]. \end{aligned}$$

■

Note that in the case of the Normal distribution, we have  $\tilde{X} \stackrel{d}{=} X$  so that  $\frac{c}{\tilde{c}} = 1$  and therefore

$$\text{Cov}[h(X), Y] = \text{Cov}(X, Y) \cdot \mathbb{E}[h'(X)]$$

which gives the familiar Stein's lemma for Normal distribution. We shall  $\tilde{X}$  the integrated elliptical random variable associated with  $X$ .

## 4 C.A.P.M. with elliptical distributions

Much of the current theory of capital asset pricing is based on the assumption that asset prices (or returns) are multivariate Normal random variables. Several empirical studies have indicated violation of this fundamental assumption. The class of elliptical distributions offers a more flexible framework for modelling asset prices or returns. Similar to the Normal distribution, the dependence structure in an elliptical distribution can be summarized in terms of the variance-covariance matrix, but with also the characteristic generator. Because many of the properties of the Normal distribution extend to this larger class, existing results on asset pricing relying on the Normal distribution assumption may be preserved. This induces us to examine the validity of CAPM by relaxing the normality assumption and generalizing it to elliptical distributions. Owen and Rabinovitch (1983) derive the Tobin's separation and the Ross's mutual fund separation theorems in the case when the underlying returns are elliptical. Ingersoll (1987) derives the CAPM and portfolio allocation in this case as well.

However, in this section, we offer yet an alternative, and simple, proof of CAPM using the Stein's lemma for elliptical distributions proved in the previous section.

### 4.1 Set-up

In this section, we adopt the "equilibrium pricing approach" used in both Panjer (1998). Consider a one-period economy where  $\omega$  denotes the state of nature at the end of the period. Assume there are  $I$  agents each with time-additive utility function

$$u_{i0}(c_{i0}) + u_{i1}(C_{i1}(\omega)), \quad \text{for } i = 1, 2, \dots, I.$$

Expected utility is thus

$$u_{i0}(c_{i0}) + \sum_{\omega} p_i(\omega) u_{i1}(C_{i1}(\omega)).$$

Agents are expected utility maximizers. Assume there are Arrow-Debreu securities which pay 1 for each state  $\omega$  and none for all other states. These Arrow-Debreu prices are denoted by  $\Psi_{\omega}$ . Optimal consumption at equilibrium exists and are to be denoted by  $c_{i0}^*$  and  $C_{i1}^*(\omega)$ .

Now, consider a particular state, say  $\omega_a$ , and suppose the agent buys additional  $\alpha$  units at time 0 so that consumptions are  $c_{i0}^* - \alpha\Psi_{\omega_a}$  at time 0 and  $C_{i1}^*(\omega_a) + \alpha$  at time 1 in state  $\omega_a$ . Expected utility becomes

$$u_{i0}(c_{i0}^* - \alpha\Psi_{\omega_a}) + \sum_{\omega \neq \omega_a} p_i(\omega) u_{i1}(C_{i1}^*(\omega)) + p_i(\omega_a) u_{i1}(C_{i1}^*(\omega_a) + \alpha)$$

and taking the first derivative with respect to  $\alpha$ , we get

$$-\Psi_{\omega_a} u'_{i0}(c_{i0}^* - \alpha\Psi_{\omega_a}) + p_i(\omega_a) u'_{i1}(C_{i1}^*(\omega_a) + \alpha)$$

which must be equal to 0 (since already optimal) at  $\alpha = 0$ . It follows immediately that

$$\Psi_{\omega} = p_i(\omega) \frac{u'_{i1}(C_{i1}^*(\omega))}{u'_{i0}(c_{i0}^*)}$$

where we have dropped the subscript  $a$  without ambiguity. These are called the state prices.

Now using these state prices to price any other security, consider for example a security that pays 1 unit at time 1 in each state. This is precisely a unit discount bond that pays 1 unit at time 1, regardless of the state. We must then have

$$\sum_{\omega} \Psi_{\omega} = \sum_{\omega} p_i(\omega) \frac{u'_{i1}(C_{i1}^*(\omega))}{u'_{i0}(c_{i0}^*)} = \frac{1}{1 + R_F}$$

where  $R_F$  is the risk-free interest rate. As yet another example, consider a security that pays  $X(\omega)$  in state  $\omega$ . Suppose  $\pi(x)$  denotes the price for this security. Then, clearly it must be equal to

$$\pi(x) = \sum_{\omega} p_i(\omega) \frac{u'_{i1}(C_{i1}^*(\omega))}{u'_{i0}(c_{i0}^*)} X(\omega) = \mathbb{E}(ZX)$$

where  $Z(\omega)$  is equal to  $\frac{u'_{i1}(C_{i1}^*(\omega))}{u'_{i0}(c_{i0}^*)}$ , sometimes called the price density or pricing kernel.

Note that the pricing formula above depends on the preferences and consumption allocation of a particular agent. To derive the pricing formula at equilibrium, we would have to maximize each agent's utility and then let market clear. Alternatively, if the subjective probabilities are the same across agents, we can simplify this procedure by maximizing a representative agent and then letting market clear by assuming this representative agent has all the aggregate consumption and aggregate endowment. The representative agent's utility function is thus  $v_0(c_0^a) = \sum_{i=1}^I k_i u_{i0}(c_{i0})$  and  $v_1(C_1^a) = \sum_{i=1}^I k_i u_{i1}(C_{i1})$  where  $c_0$  and  $C_1$  are the aggregate consumptions and  $\sum_{i=1}^I k_i = 1$ . Therefore, the state prices are

$$\Psi_{\omega} = p(\omega) \frac{v'_1(C_1^a(\omega))}{v'_0(c_0^a)}$$

and

$$Z = \frac{v'_1(C_1^a)}{v'_0(c_0^a)}. \quad (28)$$

## 4.2 Deriving the C.A.P.M.

Using the equilibrium approach, we derive the CAPM. Consider a security  $j$  that pays an amount of  $X_j(\omega)$  at time 1 in state  $\omega$ . Let  $\pi_j$  be the current price of the security. By arbitrage (two portfolios with equal payoffs have the same value), we have

$$\pi_j = \sum_{\omega} \Psi_{\omega} X_j(\omega) = \sum_{\omega} p(\omega) \frac{v'_1(C_1^a(\omega))}{v'_0(c_0^a)} X_j(\omega) = \mathbb{E}(ZX_j). \quad (29)$$

Denote by  $R_j(\omega)$  the rate of return in state  $\omega$  so that

$$R_j(\omega) = \frac{X_j(\omega) - \pi_j}{\pi_j}. \quad (30)$$

From equation (29), we have

$$\mathbb{E}\left(Z \frac{X_j}{\pi_j}\right) = 1$$

from equation (30), we get

$$\begin{aligned} \mathbb{E}\left(Z \frac{X_j}{\pi_j}\right) &= \mathbb{E}[Z(1 + R_j)] = \mathbb{E}(Z) + \mathbb{E}(ZR_j) \\ &= \mathbb{E}(Z) + Cov(R_j, Z) + \mathbb{E}(Z)\mathbb{E}(R_j) \\ &= \mathbb{E}(Z)[1 + \mathbb{E}(R_j)] + Cov(R_j, Z). \end{aligned} \quad (31)$$

For the one-period bond, we have

$$\mathbb{E}(Z) = \sum_{\omega} \Psi_{\omega} = \frac{1}{1 + R_F}. \quad (32)$$

where  $R_F$  denotes the one-period risk-free rate. Replacing (32) in (31), we obtain

$$1 = \frac{1}{1 + R_F} [1 + \mathbb{E}(R_j)] + Cov(R_j, Z).$$

Thus, we have

$$\mathbb{E}(R_j) - R_F = -(1 + R_F) Cov(R_j, Z). \quad (33)$$

Because at equilibrium the total consumption will equal to the total wealth in the economy, the market rate of return can be expressed as

$$1 + R_m(\omega) = \frac{C_1^a(\omega)}{c_0^a}$$

so that this return also satisfies the same form of equation

$$\mathbb{E}(R_m) - R_F = -(1 + R_F) Cov(R_m, Z). \quad (34)$$

Dividing the equation (33) by equation (34), we have

$$\frac{\mathbb{E}(R_j) - R_F}{\mathbb{E}(R_m) - R_F} = \frac{Cov(R_j, Z)}{Cov(R_m, Z)}$$

A re-arrangement leads us to the following CAPM formula:  $\mathbb{E}(R_j) = R_F + \frac{Cov(R_j, Z)}{Cov(R_m, Z)} \cdot [\mathbb{E}(R_m) - R_F] = R_F + \beta_j \cdot [\mathbb{E}(R_m) - R_F]$  where  $\beta_j = \frac{Cov(R_j, Z)}{Cov(R_m, Z)}$ . The problem with this equation is that the “beta” is unobservable. However, we can simplify this by imposing assumption of elliptical distributions on the returns.

**Proposition 1 CAPM with Multivariate Elliptical Returns** *Assume a market with  $n$  securities and that all securities have returns that are jointly distributed as multivariate elliptical. Then, the expected rate of return for security  $j$  can be expressed as*

$$\mathbb{E}(R_j) = R_F + \beta_j \cdot [\mathbb{E}(R_m) - R_F], \quad \text{for } j = 1, 2, \dots, n$$

where  $R_F$  is the risk-free rate,  $R_m$  is the market rate of return, and

$$\beta_j = \frac{Cov(R_j, R_m)}{Var(R_m)}.$$

**Proof.** The proof follows a similar reasoning as that of Panjer, et al. (1998) for the case of Normal distribution. From the property of elliptical, each  $R_j$  has an elliptical distribution. The rate of return in the market  $R_m$  is a linear combination of rates of return of all securities. Hence, it follows that  $R_m$  has also an elliptical distribution. Furthermore, each random pair  $(R_j, R_m)$  will have a bivariate elliptical distribution. Using equation (28) to evaluate the covariances, we have

$$\begin{aligned} \frac{Cov(R_j, Z)}{Cov(R_m, Z)} &= \frac{Cov\left(R_j, \frac{v_1'(C_1^a)}{v_0'(c_0^a)}\right)}{Cov\left(R_m, \frac{v_1'(C_1^a)}{v_0'(c_0^a)}\right)} \\ &= \frac{Cov(R_j, v_1'(C_1^a))}{Cov(R_m, v_1'(C_1^a))} \\ &= \frac{Cov(R_j, v_1'(c_0^a(1 + R_m)))}{Cov(R_m, v_1'(c_0^a(1 + R_m)))}. \end{aligned}$$

Applying Stein's lemma for elliptical distribution, we simplify this to

$$\frac{Cov(R_j, Z)}{Cov(R_m, Z)} = \frac{(c/\tilde{c}) \cdot Cov(R_j, R_m) \cdot \mathbb{E} \left[ v_1'' \left( c_0^a (1 + \tilde{R}_m) \right) \right]}{(c/\tilde{c}) \cdot Cov(R_m, R_m) \cdot \mathbb{E} \left[ v_1'' \left( c_0^a (1 + \tilde{R}_m) \right) \right]} = \frac{Cov(R_j, R_m)}{Var(R_m)}$$

where  $\tilde{R}_m$  is the integrated elliptical random variable associated with  $R_m$ , and  $c$  and  $\tilde{c}$  are the normalizing constants corresponding to  $R_m$  and  $\tilde{R}_m$  respectively. ■

This concludes the section about CAPM in one period model, where the returns are elliptical. It is worth noting that CAPM, as a capital asset pricing model, is an equilibrium model to price financial assets of any kind, even if standard implementation is usually limited to common stocks. Black and Scholes (1973) show the relationship between continuous-time CAPM and option pricing. Indeed, in their seminal paper, they presented an alternative derivation of the partial differential equation governing option prices by applying CAPM to one stock and one option.

It is not in the scope of this paper to investigate such relationship in the case of elliptical distribution. This can be the subject of future research, where an equivalent continuous-time CAPM could be derived.

The following section considers option pricing where the underlying is log-elliptically distributed. We consider a complete market, continuous-time setup.

## 5 Option pricing using probability distortion functions

The standard approach in option pricing starts by considering a dynamics of the underlying, usually described through a stochastic differential equation under a physical probability measure. Then, an equivalent martingale measure is derived, under which, the discounted underlying prices are martingale. Once this is obtained, then, pricing any contingent claim, in particular options, is equal to the expectation of the discounted terminal payoff under the martingale measure.

In this section, however, we consider a different framework for pricing options based on probability distortion functions, which, as it will be shown later, is consistent with the standard option pricing approach.

The concept of probability distortion functions is widely used in insurance risk pricing. The idea is to transform the real world probability distribution of the contingent claim to adjust for risk. The link between the probability distortion, change of measure and entropy is discussed in Reesor and McLeish (2001). Probability distortion is used in Yaari (1987) in the theory of choice under uncertainty. The certainty equivalent<sup>1</sup> of a risk is computed as the mean of the distorted cumulative distribution function of the underlying risk.

Wang (2000) proposes a class of probability distortion functions that aims to integrate financial and insurance pricing theories. The probability distortion function proposed is based on the standard cumulative Normal distribution. In his paper, Wang claims that the new distortion function connects four different approaches:

1. the traditional actuarial standard deviation principle,
2. Yaari's (1987) economic theory of choice under uncertainty,
3. CAPM, and
4. option-pricing theory.

<sup>1</sup>The certainty equivalent of a risk is the amount which when received with certainty, is regarded as good as taking the risk itself.

Let us recall some definitions of the probability distortion functions. Consider a random variable  $X$  with a decumulative distribution function  $S_X(x) = P[X > x]$ . Let  $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$  be the standard Normal cumulative distribution function and define

$$g_\alpha(p) = \Phi[\Phi^{-1}(p) + \alpha]$$

for  $p$  in  $[0, 1]$  and  $\Phi^{-1}$  denotes the inverse of  $\Phi$ . The distortion function  $g_\alpha$  shifts the  $p^{\text{th}}$  quantile of  $X$ , assuming that  $X$  is normally distributed, by a constant  $\alpha$  and re-evaluates the normal cumulative probability for the shifted quantile. Wang (2000) shows that  $g_\alpha(p)$  is concave for positive  $\alpha$  and convex for negative  $\alpha$ . In fact, it is easy to see that if  $\alpha > 0$ , then  $g_\alpha(p) > p$ , and if  $\alpha < 0$ , then  $g_\alpha(p) < p$ . Since  $g_\alpha$  is continuous and  $g_\alpha(p) \in [0, 1]$ , then it follows that

$$\begin{aligned} g_\alpha \text{ is convex if } \alpha &< 0 \\ g_\alpha \text{ is concave if } \alpha &> 0 \end{aligned}$$

Intuitively, with this distortion function, an individual behaves pessimistically by shifting the quantiles to the left, thereby assigning higher probabilities to low outcomes, and behaves optimistically by shifting the quantiles to the right thereby assigning higher probabilities to high outcomes.

Wang (2000) defines the risk-adjusted premium for a risk  $X$ , by the Choquet integral representation

$$H[X; \alpha] = \int_{-\infty}^0 \{g_\alpha[S_X(x)] - 1\} dx + \int_0^\infty g_\alpha[S_X(x)] dx$$

where  $X$  will be negative for insurance losses and will be positive for payoffs from limited-liability assets. This new risk pricing measure has many advantages and seems to perform well if normality of the underlying risk is assumed. However, it is not clear why it should work for non-normal case.

Hamada and Sherris (2003) applied Wang transform to price European call option written on a security with prices following a geometric Brownian motion and they derived the Black and Scholes option price formula. This consistency with financial theory is not obtained when the underlying is not log-normal. The case of constant elasticity of variance (CEV) process was considered to show this inconsistency. This is due to Wang's choice of the distortion function based on the cumulative normal distribution.

## 5.1 Probability distortion functions in the non-normal case

In the case where the underlying is not log-normal, a fair price can still be obtained by choosing a distortion function based on the cumulative distribution function of the underlying security. Indeed, in Hamada and Sherris (2003), the following proposition was shown.

**Proposition 2** *Let  $X$  be a random variable with cumulative distribution function  $F$  and density function symmetric about 0. Furthermore, let the contingent claim  $Y_T$  be a function of  $X$  such that  $Y_T = h(X)$  where  $h$  is a continuous, positive and increasing function. The fair price of  $Y_T$  at time 0 is given by:*

$$Y_0 = e^{-rT} I[Y_T; -\alpha_T]$$

where

$$I[Y_T; -\alpha_T] = \int_0^\infty F[F^{-1}(P[Y_T > s]) - \alpha_T] ds,$$

$r$  is a continuously compounded risk-free rate, and  $\alpha_T$  is a parameter calibrated to the market price.

The proposition above states that a fair price for the claim is given by its certainty equivalent, where the certainty equivalent is defined as the expected value of the distorted decumulative distribution function. This is consistent with insurance pricing theory, introduced by Yaari (1987). The question that arises is whether this provides an arbitrage-free price? If the underlying is log-normally distributed, then  $F = \Phi$  and it is proven in Hamada and Sherris (2003) that indeed one can obtain an arbitrage-free price using this probability distortion function, specifically, the Black-Scholes price for options.

Now, if the underlying is not Normal, then the answer to the above question is not clear in all cases. It can be argued that the pricing formula in the above proposition can be used since it is founded on non-expected utility theory.

The above formula seems difficult to compute, however, for symmetric distributions, where elliptical are a special case, we have a simpler representation, given in the following proposition.

**Proposition 3** *With the same set-up as in the previous proposition, we have*

$$I[Y_T; -\alpha_T] = \mathbb{E}[h(X - \alpha_T)]. \quad (35)$$

**Proof.** See Hamada and Sherris (2003) ■

## 5.2 Option pricing when the underlying is log-elliptically distributed

Let  $Z \sim E_1(0, 1, \psi)$ , a standard elliptical random variable, and define the process  $(X_t)_{t \geq 0}$  such that

$$X_t = \mu t + \sigma \sqrt{t} Z \sim E_1(\mu t, \sigma^2 t, \psi)$$

for each time  $t \geq 0$ . Let  $(\mathcal{F}_t)$  denote the natural filtration generated by the process  $(X_t)$  and define the process  $(S_t)_{t \geq 0}$  of the underlying prices as

$$S_t = S_0 e^{X_t},$$

where  $S_0$  is the security price at time 0 which is known and fixed.

Based on these assumptions, the market is complete as it is driven by a one dimensional process  $(X_t)$ . The underlying price is a strictly increasing function of  $(X)$ , so it is adapted to the filtration  $(\mathcal{F}_t)$ .

At each time  $t$ , we have

$$S_t = S_0 e^{X_t} \sim L E_1(\log S_0 + \mu t, \sigma^2 t, \psi).$$

From Proposition (2), the fair premium at time 0 of a European call option maturing at  $T$  written on a security with price process  $S$  is given by

$$C_0 = e^{-rT} I[(S_T - K)^+; -\alpha_T]$$

where

$$I[(S_T - K)^+; -\alpha_T] = \int_0^\infty F[F^{-1}(P[(S_T - K)^+ > s]) - \alpha_T] ds$$

Since the density of  $Z$  is symmetric about 0 and  $(S_T - K)^+ = h(Z)$  where

$$h(z) = \left( S_0 e^{\mu T + \sigma \sqrt{T} z} - K \right)^+,$$

then using Proposition (3), we have

$$I[(S_T - K)^+; -\alpha_T] = \mathbb{E}[h(Z - \alpha_T)].$$

One can explicitly evaluate the above expectation and as a result obtain a fair premium for the option. We summarized this result in the following theorem.

**Theorem 4** *The fair premium of a European call option, with exercise price  $K$  and maturity  $T$ , written on a security with prices having elliptical distribution with parameters defined above, is given by:*

$$\begin{aligned} I[(S_T - K)^+; -\alpha_T] &= e^{\mu T + \sigma\sqrt{T}\alpha_T} \cdot \psi(-\frac{1}{2}\sigma^2 T) F_{Z^*} \left( \frac{\mu T + \sigma\sqrt{T}\alpha_T - \log K/S_0}{\sigma\sqrt{T}} \right) \\ &\quad - K F_Z \left( \frac{\mu T + \sigma\sqrt{T}\alpha_T - \log K/S_0}{\sigma\sqrt{T}} \right) \end{aligned} \quad (36)$$

where  $Z$  is spherically-distributed random variable with characteristic generator  $\psi$ ,  $Z^*$  is a random variable with as density the Esscher transform given by

$$f_{Z^*}(x) = \frac{e^{\sigma\sqrt{T}x}}{\psi(-\frac{1}{2}\sigma^2 T)} f_Z(x),$$

and  $\alpha_T$  is a parameter calibrated to the market prices of the underlying security.

**Remark:** Even if the market is complete, the option price here is expressed in terms of a density of Esscher transform type.

**Proof.** From Theorem 7 of Dhaene and Valdez (2003), we have

$$\mathbb{E}[(Y - K)^+] = e^\mu \cdot \psi(-\frac{1}{2}\sigma^2) F_{Z^*} \left( \frac{\mu - \log K}{\sigma} \right) - K F_Z \left( \frac{\mu - \log K}{\sigma} \right)$$

for any  $Y \sim LE_1(\mu, \sigma^2, \psi)$  and  $Z \sim S_1(\psi)$  (spherical distribution) with density  $f_Z$  and cdf  $F_Z$ , where

$Z^*$  is a random variable with as density the Esscher transform given by

$$f_{Z^*}(x) = \frac{e^{\sigma x}}{\psi(-\frac{1}{2}\sigma^2)} f_Z(x).$$

Now since

$$I[(S_T - K)^+; -\alpha_T] = \mathbb{E}[h(Z - \alpha_T)],$$

we can evaluate the expression in (35) where  $h(Z - \alpha_T) = \left( e^{\mu T + \sigma\sqrt{T}\alpha_T + \sigma\sqrt{T}Z} - K \right)^+ = (Y - K)^+$  and  $Y \sim LE_1(\mu T + \sigma\sqrt{T}\alpha_T, \sigma^2 T, \psi)$ . The result in (36) now clearly follows. ■

This price is similar to the Black-Scholes option pricing formula. It is indeed straightforward to show that it collapses to the Black-Scholes price when the underlying is geometric Brownian motion.

## 6 Conclusion

The purpose of this paper was twofold. the first is deriving a new version of Stein's Lemma for a bivariate elliptical random variable and using it to re-derive the C.A.P.M. The second is using the probability distortion functions approach to derive a closed form solution of a call option price when the underlying is elliptically distributed. This generalizes the work of Hamada and Sherris (2003) where consistency of Black-Scholes option pricing and probability distortion functions is proven in the case of normality.

The setup of the first part of the paper is a discrete time, one period model. This can be extended in future research to a continuous-time CAPM with elliptical returns, and the link between CAPM and option pricing of the second part can be established.

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