Martingale Methods in Dynamic Portfolio Allocation with Distortion Operators*

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Abstract

Standard optimal portfolio choice models assume that investors maximise the expected utility of their future outcomes. However, behaviour which is inconsistent with the expected utility theory has often been observed.

In a discrete time setting, we provide a formal treatment of risk measures based on distortion functions that are consistent with Yaari's dual (non-expected utility) theory of choice (1987), and set out a general layout for portfolio optimisation in this non-expected utility framework using the risk neutral computational approach.

As an application, we consider two particular risk measures. The first one is based on the PH-transform and treats the upside and downside of the risk differently. The second one, introduced by Wang (2000) uses a distortion operator based on the cumulative normal distribution function.

1 Introduction

This paper considers the dynamic optimal consumption and portfolio selection problem, in a discrete-time setting. In the literature, the optimal portfolio problem is very widely considered, however, it is often treated in a continuous-time framework. Indeed, Merton (1971 [35]) proposed an explicit solution to this problem when the underlying security prices follow a geometric Brownian motion. The same problem was considered in general equilibrium asset pricing models, where many investors act so as to maximise their expected utility over consumption (see Lucas (1978 [32], Breeden (1979 [4]) and Cox et *al.* (1985 [12])). To solve the optimisation problem, they used stochastic dynamic programming that yields the Bellman-Dreyfus PDE which is in the general case non-linear and difficult to solve.

Harrison and Kreps (1979 [22]) introduced the martingale method to price contingent claims. This approach was applied by Karatzas *et al.* (1986 [27], 1987 [28]), Cox & Huang (1989 [10], 1991 [11]) and Pliska (1986 [38]) to provide a closed form solution for the optimal portfolio when the underlying security prices follow a general diffusion process. For a detailed treatment, see

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textbooks Merton [34], Duffie [13] and Karatzas and Shreve [29]. The basic idea is to use the completeness and the arbitrage-free property of the market to separate the computation of optimal consumption rules and that of a corresponding trading strategy. The optimal consumption is obtained by solving the first-order conditions, essentially stating that the agent's marginal utility process at the optimum is proportional to an Arrow-Debreu state price density process. A replicating financing strategy can then be constructed, just like in any option pricing technology. He and Pearson (1991 [24]) have extended the Cox & Huang approach to settings with incomplete markets.

In a discrete-time setting, Campbell and Viceira (1999 [8]) consider the problem when an infinitely-lived investor has utility over consumption and find an approximate solution to the discrete-time problem using an Euler approximation. Chamberlain and Wilson (2000 [9]) analyse the optimal consumption rules for an infinitely-lived investor who receives a random income at each period and saves for the future at a stochastic rate. Both approaches belong to the class of stochastic dynamic programming methods. Pliska in his textbook [39] uses the martingale approach to derive an exact solution of the portfolio problem. His approach is elegant and analogous to the continuous-time one, where the probability space is finite and the martingales are discrete.

The majority of portfolio choice models assume that preferences are represented by a von Neuman-Morgenstern utility function and individuals choose among risky alternatives so as to maximise the expectation of the utility of possible outcomes. Although the expected utility model has long been the standard for choice under uncertainty, questions have been raised concerning its validity, and behavior patterns which are systematic, yet inconsistent with expected utility theory have often been observed as in the Allais paradox (1953 [1]) and Kahneman & Tversky (1979 [26]). Machina (1982 [33]) considers expected utility without the "independence axiom". He shows that basic concepts and results of the expected utility framework do not depend on this axiom, but may be derived from the much weaker assumption of smoothness of preferences over alternative probability distributions. Fishburn (1988 [16]) surveys the reasons why the expected utility hypothesis fails. Camerer (1989 [7]) carries out empirical tests of several generalized models of utility theory. Yaari (1987 [50]) developed a dual theory of choice under risk where the roles of probabilities and payments are interchanged, so the wealth utility function is replaced by a probability distortion function. Some of the expected utility related paradoxes are resolved in the dual theory.

It can be argued that even though some other alternative models could handle the problems related to the expected utility model, this latter would still be appropriate for analyzing some types of choice under risk (see Neilson 1993 [36]). Other types of decision problems, however, would be modeled best using a non-expected utility framework. Therefore, expected utility theory should not be abandoned, but rather applied with more caution.

In a non-expected utility framework, Wang (1995 [46], 1996 [45]) proposes calculating insurance premiums by applying the proportional hazards transform to the decumulative distribution function, thereby introducing a new risk measure. This new measure turns out to be consistent with Yaari's dual theory of choice. Wang (2000 [48]) also uses a different class of distortion operators to recover the Black-Scholes formula. Sherris and van der Hoek (2001 [44]) introduce a new class of risk measures for asset allocation which is based on the distortion function approach to insurance risk.

Bufman and Leiderman (1990 [6]) use Israeli data between 1978 and 1986 to test an intertemporal consumption-investment model introduced by Epstein and Zin (1989 [15]) that uses Kreps-Porteus (1978 [31]) non-expected utility preferences. They find an evidence to reject the expected utility model and accept the non-expected utility one. Their results differ from those of Epstein and Zin (1989b [14]) and Giovannini and Jorion (1989 [17]) who took data from the tranquil postwar US economy. This suggests that a non-expected utility model may perform better in a volatile economy. The results of the empirical tests of the same model using French data from 1960 to 1994 conducted by Koskievic (1999 [30]) support those of Bufman and Leiderman (1990

[6]).

Ang et *al.* (2000 [2]) consider the dynamic portfolio choice using the disappointment aversion of Gul (1991 [18]). They find that moderately disappointment-averse investors, with low curvature utility, hold reasonable portfolios, unlike the expected utility investors whose optimal holdings include unreasonably large equity positions.

This paper is organised as follows. In section 2, we outline the risk neutral computational approach to dynamic asset allocation, and briefly discuss its advantage over dynamic programming. In section 3, we present the concept of risk aversion for non-expected utility and illustrate the idea of using a distortion function to price risk. A formal treatment of risk measures based on this concept follows the analysis. The new class of risk measures for portfolio selection based on the proportional hazards transform, proposed by Sherris and van der Hoek (2001 [44]) is then reviewed and extended to the multinomial case. This is the first step in setting out a general scheme for dynamic asset allocation when the risk measure is based on a distortion function. Some other properties, useful for the optimisation, are developed along the way. We then propose to solve the optimal portfolio problem using the risk neutral computational approach when the investor behaviour is modeled by this new class of risk measures. Some numerical examples for asset allocation are provided. We finally provide a comparison of the asset allocation when using the class of distortion operators proposed by Wang (2000 [48]). The conclusion highlights some further developments.

2 Risk-Neutral Computational Approach (RNCA)

The optimal portfolio proportions of an investor with a long horizon compared to those of an investor with a short horizon, such as what is typically assumed in 'tactical asset allocation' models, are significantly different (see Brennan, Schwartz & Lagnado (1997 [5]). This is why a multiperiod model is more realistic and extensively used. In the model presented here, the investor maximises the expected sum of discounted utility of consumption through time by choosing investment and consumption rules dynamically during the period of investment. This dynamic asset allocation is based on the information available in the market up to the time of revision of the portfolio.

This paragraph explains the risk neutral computational approach (RNCA) and outlines the main difference with dynamic programming. We follow Harrison and Pliska (1981 [21]) and Taqqu and Willinger (1987 [43]). For a detailed treatment, see Pliska [39] and Bingham and Kiesel [3].

Briefly, we recall some basic definitions:

Definition 1

- *A trading strategy $H = (H_0, H_1, ..., H_N)$ is a vector of stochastic processes $H = \{H_n(t); t = 1, 2, ..., T\}$, for $n = 0, 1, ..., N$. $H_n(t)$ is the number of units of securities that an investor carries forward from time $t-1$ to time t ($H_n(0)$ is not defined).*

- *The value process $V = \{V_t; t = 0, 1, ..., T\}$ is a stochastic process defined by:*

$$V_t = \begin{cases} H_0(1).B_0 + \sum_{n=1}^{N} H_n(1).S_n(0), & t = 0 \\ H_0(t).B(t) + \sum_{n=1}^{N} H_n(t).S_n(t), & t \geq 1 \end{cases}$$

  V_t is the time-t value of the portfolio before any transactions are made at that same time.

- *A consumption process $C = \{C_t, t = 0, ..., T\}$ is a non-negative, adapted stochastic process with C_t representing the amount of funds consumed by the investor at time t.*

- *A consumption-investment plan (C, H) is said to be self-financing if:*

$$V_t = C_t + H_0(t+1).B(t) + \sum_{n=1}^{N} H_n(t+1).S_n(t), \qquad t = 0, ..., T-1$$

An investor with a von Neuman-Morgenstern utility function u chooses the consumption-investment plan that maximises the expected utility of consumption through time according to the following problem:

$$\begin{cases} \max\limits_{(C,H)} & \mathbb{E}\left[\sum_{t=0}^{T}\beta^t.u(C_t)\right] \\ subject\ to & v = \text{initial wealth} = V_0 \\ & V_t = C_t + H_0(t+1).B(t) + \sum_{n=1}^{N} H_n(t+1).S_n(t), \qquad t = 0,...,T-1 \\ & V_T = C_T \end{cases} \tag{1}$$

In problem (1), the investor seeks to maximise the objective function when facing the trade-off between the consumption and investment strategy at the beginning of each time period $[t, t+1]$, $t = 0...T-1$. Notice that the vector of trading strategies H, which is a control variable in the optimisation problem, appears in the constraints but not in the objective function. The standard way to solve this problem is to compute, working backward in time t in a recursive manner, a value function called the indirect utility, which represents the maximum expected utility of consumption through time T, starting with wealth V_t and consumption C_t given the history $\mathcal{F}_t$. This is the dynamic programming approach which carries out T optimisations to solve for the optimal consumption and investment rules at each period $[t, t+1]$, $t = 0...T-1$.

The idea of the risk-neutral computational approach is to replace the constraints in problem (1) by equivalent constraints that don't involve the investment strategy vector H. The only control variable being the vector of consumption C, the problem can be solved using the Lagrange method for a standard static optimisation problem. The assumption of a complete arbitrage-free market ensures the existence of a unique equivalent martingale probability Q, such that the initial wealth is equal to the expectation under Q of the discounted consumption through time. Hence, as shown in Pliska [39] the problem (1) is equivalent to:

$$\begin{cases} \max\limits_{C} & \mathbb{E}\left[\sum_{t=0}^{T}\beta^t.u(C_t)\right] \\ subject\ to & \mathbb{E}_Q\left[\sum_{t=0}^{T}\frac{C_t}{B_t}\right] = v \\ & C \text{ is an adapted process} \end{cases}$$

where the first expectation is taken under the original probability P and the second expectation is taken under the equivalent martingale probability Q. Let $L = \dfrac{Q}{P}$ denote the state price density, and define the conditional discounted state price density

$$N_t(\omega) = \frac{\mathbb{E}[L|\mathcal{F}_t](\omega)}{B_t} \forall \omega \in \Omega$$

The problem (1) is equivalent to:

$$\begin{cases} \max\limits_{C} & \mathbb{E}\left[\sum_{t=0}^{T}\beta^t.u(C_t)\right] \\ subject\ to & \mathbb{E}\left[\sum_{t=0}^{T}C_t N_t\right] = v \\ & C \text{ is an adapted process} \end{cases}$$

Using Lagrange multipliers to solve the problem:

$$\max_{C}\mathbb{E}\left[\sum_{t=0}^{T}\beta^t.u(C_t) - \lambda\sum_{t=0}^{T}C_t N_t\right]$$

The first order conditions are :

$$C_t(\omega) = I\left(\frac{\lambda N_t(\omega)}{\beta^t}\right) \qquad \forall \omega \in \Omega, \qquad \forall t = 0, 1, ...T \tag{2}$$

Where $I = [u']^{-1}$ is the inverse of the marginal utility, and λ is a Lagrange multiplier determined by substituting the optimal consumption policy into the appropriate static budget constraint:

$$\mathbb{E}\left[\sum_{t=0}^{T} I\left(\frac{\lambda N_t}{\beta^t}\right) N_t\right] = v$$

Once the optimal consumption rules are obtained, the self-financing condition is used to compute the trading strategies generating C_t:

$$H_0(t).B(t) + \sum_{n=1}^{N} H_n(t).S_n(t) = C_t + H_0(t+1).B(t) + \sum_{n=1}^{N} H_n(t+1).S_n(t) \tag{3}$$

To sum up, in the first step, the (RNCA) yields the optimal consumption at each period in a straightforward optimisation. In the second step, the optimal trading strategies that finance this consumption are computed step by step, working backward in time, by solving a system of linear equations at each time step.

For a discussion of implementation in a binomial and trinomial lattice, with the HARA class of utility, see Hamada (2001 [19]). The next paragraphs exploit this idea and solve the optimal portfolio problem in a non-expected utility framework.

3 Asset allocation with a risk measure based on a distortion function

3.1 Risk aversion in utility theory and its dual

Decision makers with a von Neumann-Morgenstern utility function are said to be risk averse if they prefer to have the expected value of a gamble rather than facing the gamble itself, i.e. $U(W) > EU(W + X)$ for a level of wealth W and all gambles with $\mathbb{E}(X) = 0$ and positive variance.

It can be proved (see Ingersoll [25]) that decision makers are risk averse if and only if their von Neumann-Morgenstern utility function of wealth is strictly *concave* at the relevant wealth level. Moreover, the intensity of risk aversion is measured by the degree of concavity of the utility function. This is determined by Arrow-Pratt's absolute risk aversion index[1]. The larger the index, the more risk-averse the agent.

To induce a risk-averse individual to undertake a fair gamble, a compensatory risk premium $\Pi_c(X)$ has to be offered. Or dually, to avoid a present gamble, a risk averse individual would be willing to pay an insurance risk premium $\Pi_i(X)$. These risk premiums are depicted as follows:

$$\begin{aligned} \mathbb{E}[U(W + \Pi_c(X) + X)] &= U(W) \\ \mathbb{E}[U(W + X)] &= U(W - \Pi_i(X)) \end{aligned}$$

The amount $W - \Pi_i(X)$ is the amount which, when received with certainty, is considered by the investor as good as $W + X$. It is called the certainty equivalent of the gamble $W + X$

[1] Arrow-Pratt's absolute risk aversion is defined as $A(W) = -\frac{u''(W)}{u'(W)}$

In the expected utility theory, suppose that an individual must choose among lotteries with at most n outcomes $x_1, x_2, ..., x_n$, with respective probabilities $p_1, p_2, ..., p_n$, then there exists a utility function U such that this individual's choice criterium is to maximise

$$\mathbb{E}[U(X)] = \sum_{i=1}^{n} p_i U(x_i) = \int_{\Omega} u(X(\omega))dP(\omega)$$

Note that this objective function is linear in probabilities and distorts the payoffs.

In the dual theory of choice introduced by Yaari [50], the certainty equivalent to X is defined as[2]:

$$\Pi(X) = \int g(S_X(t))dt$$

where g is a "dual utility" or a distortion function (continuous and non-decreasing) $g : [0, 1] \rightarrow [0, 1]$ with $g(0) = 0$ and $g(1) = 1$, applied to the probability decumulative distribution:

$$S_X(t) = \Pr[X > t]$$

If X is a positive random variable representing a loss amount then $\Pi(X)$ is the certainty equivalent of the risk X. In the dual theory, given a choice among risky prospects, the agent would prefer risks having the greatest certainty equivalent.

It can be proved (see Yaari [50]) that the investor is risk averse if and only if g is convex. An intuitive interpretation of this property follows in the case when g is differentiable:

$$\Pi(X) = \int g(S_X(t))dt = \int tg'(S_X(t))dF_X(t)$$

Recall that:

$$\mathbb{E}[X] = \int tdF_X(t)$$

Comparing $\Pi(X)$ to $\mathbb{E}[X]$, $\Pi(X)$ can be thought of as a corrected mean of X where the payment t receives a weight $g'(S_X(t)) \geq 0$. Note that these weights sum up to 1, *i.e.* $\int g'(S_X(t))dF_X(t) = 1$.[3]

If g is convex, then,

$$t_1 > t_2 \Rightarrow S_X(t_1) < S_X(t_2) \Rightarrow g'(S_X(t_1)) < g'(S_X(t_2))$$

Therefore, the weight assigned to a high outcome is less than the weight assigned to a low outcome. Hence, by distorting the probabilities with a convex function, the agents behave pessimistically, in the sense that they assign high probability to bad outcomes and low probability to good outcomes.

The comparison of risk aversion in this framework is naturally based on the convexity of the function g representing the agent's preference function. The more convex the function g, the more risk averse the agent.[4]

To sum up, while risk aversion in utility theory is measured by the utility function, in the dual theory, it is measured by the probability distortion function. The choice of the distortion function g determines the properties of the certainty equivalent.

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[2] This general form of $\Pi(X)$ is valid for continuous and discrete time cases, where the integral sign will be a summation sign in a discrete case, and the appropriate formula is developed in a later paragraph.

[3] $\int g'(S_X(t))dF_X(t) = \int \frac{d}{dt}[-g(S_X(t))]dt = g(1) - g(0) = 1$

[4] The dual Arrow-Pratt risk aversion would be in this case $\frac{g''(p)}{g'(p)}$ for $0 < p < 1$, as defined in Yarri (1986 [49]). In the sense of Ross (1981 [42]), agents are strongly more risk averse, if they require a larger compensation for any mean preserving spread in their prospects, even if the initial situation is not one of perfect certainty. Risk aversion measurement in the sense of Yaari (1986 [49]) and Ross (1981 [42]) are discussed in Röel (1985 [41])

In the literature, Wang (1996 [46]) proposes a general class of distortion operators to use in pricing insurance premiums. When the distortion function is a power function, i.e., $g(x) = x^r$, the mapping

$$S_X(t) \rightarrow g(S_X(t))$$

is called the PH-transform. Applications and implementation of the PH-transform in insurance is discussed in Wang (1998 [47]). Although the PH-transform enjoys desirable properties in insurance pricing, it cannot be applied to assets and liabilities simultaneously. Wang (2000 [48]) proposes another class of distortion operators

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha]$$

where $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the standard normal cumulative distribution function and shows how the mean of the distorted decumulative distribution can be used as another alternative to the risk-neutral valuation in asset pricing. Sherris and van der Hoek (2001 [44]) (hereafter VHS (2001) introduced another framework for pricing asset and liabilities, based on distortion of the probability distribution. They use two different distortion operators, g and h to allow a different pricing of the upside and downside of the risk. The specification of g and h is not given, thereby allowing for a general pricing framework. In the following, we shall consider the certainty equivalent in discrete-time, then overview the risk measure introduced by VHS (2001), develop new properties which are useful for optimisation, and use these results to solve the optimal portfolio problem.

3.2 Discrete version of the certainty equivalent in the dual theory

Let g be a continuous, non-decreasing function, $g : [0,1] \rightarrow [0,1]$ with $g(0) = 0$ and $g(1) = 1$, and X be a positive random variable representing a risk. The risk X is measured by its certainty equivalent defined as:

$$\Pi(X) = \int_0^\infty g(S_X(t))dt$$

In the discrete-time case, X takes n possible values $(X(\omega_1), X(\omega_2), ..., X(\omega_n))$ with probabilities $(p_1, p_2, ..., p_n)$, where $n \in \mathbb{N}^*$. In probabilistic notation, let $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ be the probability space where ω_i , $i \in \{1, .., n\}$ are the states of the world, then $P[\omega_i] = p_i \ \forall \ i \in \{1, .., n\}$.

This is typically the case of a tree model, where the number of states grows as time evolves. The following theorem gives an expression of $\Pi(X)$ in the discrete case.

Theorem 1 (Certainty equivalent) *If X is a random variable taking n possible values $(X(\omega_1)$, $X(\omega_2), ..., X(\omega_n))$ with probabilities $(p_1, p_2, ..., p_n)$, such that $X(\omega_i) \neq X(\omega_j)$ if $i \neq j$, then,*

$$\Pi(X) = \widehat{\mathbb{E}}_g[X]$$

where $\widehat{\mathbb{E}}_g[X]$ is a weighted average of possible values of X, such that the weight $P^g(\omega_i)$ assigned to $X(\omega_i)$ is given by:

$$P^g(\omega_i) = g\left(P[X \geq X(\omega_i)]\right) - g\left(P[X > X(\omega_i)]\right) \quad \forall \omega_i \in \Omega$$

$\widehat{\mathbb{E}}_g[X]$ can be thought of as an expectation where the probability assigned to a possible value of X depends also on all the other possible values.

Proof. To simplify notation, let $x_i = X(\omega_i)$, $i \in \{1,..,n\}$. Without loss of generality, the possible values of X can be ordered such that: $x_1 < x_2 < ... < x_n$ and we then have:

$$\Pr[X > t] = \begin{cases} 1 & if\ x_1 > t \\ \sum_{k=2}^{n} p_k & if\ x_2 > t \geq x_1 \\ \sum_{k=3}^{n} p_k & if\ x_3 > t \geq x_2 \\ .... & ..... \\ p_n & if\ x_n > t \geq x_{n-1} \\ 0 & if\ t \geq x_n \end{cases}$$

Therefore,

$$\begin{aligned} \Pi(X) &= \int_0^{x_1} g(1)dt + \int_{x_1}^{x_2} g(\sum_{k=2}^{n} p_k)dt + \int_{x_2}^{x_3} g(\sum_{k=3}^{n} p_k)dt + ... + \int_{x_{n-1}}^{x_n} g(p_n)dt \\ &= x_1 + \sum_{i=1}^{n-1} g_i^p . [x_{i+1} - x_i] \\ &= \sum_{i=1}^{n} \left[g_{i-1}^p - g_i^p \right] x_i \end{aligned}$$

where

$$\begin{aligned} g_i^p &= g\left(\sum_{k=i+1}^{n} p_k \right) \\ &= g\left(P[X > x_i] \right) \end{aligned}$$

Since g is an increasing function from $[0,1]$ to $[0,1]$, and $\forall i$ $\sum_{k=i}^{n} p_k > \sum_{k=i+1}^{n} p_k$ then:

$$\forall i \in \{1..n\},\ P_i^g \equiv g\left(\sum_{k=i}^{n} p_k \right) - g\left(\sum_{k=i+1}^{n} p_k \right) \in [0,1]$$

Moreover, $\sum_{i=1}^{n} P_i^g = g\left(\sum_{k=1}^{n} p_k\right) - g\left(0\right) = g(1) = 1$. Therefore $\{P_1^g, P_2^g, ..., P_n^g\}$ define a probability measure on the probability space $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$. ∎

This theorem states that in the discrete-time case, the certainty equivalent of X is nothing but an expectation under another probability measure. This has been shown when all the possible values of X are distinct. The question that arises immediately is: what happens in the case when some possible values of X coincide? The following example provides an insight into this question.

Example 1 *Let X be a random variable taking the values $(x_1, x_2, ..., x_9)$ with probabilities $(p_1, p_2, ..., p_9)$, such that:*

$$x_1 < x_2 = x_3 < x_4 < x_5 = x_6 = x_7 < x_8 = x_9$$

We have:

$$P[X > t] = \begin{cases} 1 & if\ x_1 > t \\ \sum_{k=2}^{9} p_k & if\ x_2 > t \geq x_1 \\ \sum_{k=4}^{9} p_k & if\ x_4 > t \geq x_2 \\ \sum_{k=5}^{9} p_k & if\ x_5 > t \geq x_4 \\ \sum_{k=8}^{9} p_k & if\ x_8 > t \geq x_5 \\ 0 & if\ t \geq x_8 \end{cases}$$

So:

$$
\begin{aligned}
\Pi(X) &= \int_0^{x_1} g(1)dt + \int_{x_1}^{x_2} g(\sum_{k=2}^{9} p_k)dt + \int_{x_2}^{x_4} g(\sum_{k=4}^{9} p_k)dt + \int_{x_4}^{x_5} g(\sum_{k=5}^{9} p_k)dt + \int_{x_5}^{x_8} g(\sum_{k=8}^{9} p_k)dt \\
&= \left(g(1) - g(\sum_{k=2}^{9} p_k)\right) x_1 + \left(g(\sum_{k=2}^{9} p_k) - g(\sum_{k=4}^{9} p_k)\right) x_2 + \left(g(\sum_{k=4}^{9} p_k) - g(\sum_{k=5}^{9} p_k)\right) x_4 + \\
&\quad \left(g(\sum_{k=5}^{9} p_k) - g(\sum_{k=8}^{9} p_k)\right) x_5 + g(\sum_{k=8}^{9} p_k) x_8
\end{aligned}
$$

Define the new random variable Y taking the values $(y_1, y_2, y_3, y_4, y_5) = (x_1, x_2, x_4, x_5, x_8)$ with probabilities $(p'_1, p'_2, p'_3, p'_4, p'_5) = (p_1, p_2 + p_3, p_4, p_5 + p_6 + p_7, p_8 + p_9)$. The new elements y_i satisfy $y_1 < y_2 < y_3 < y_4 < y_5$, so, applying the theorem above we have:

$$\Pi(Y) = \sum_{i=1}^{n} \left[g_{i-1}^{p'} - g_i^{p'} \right] y_i$$

where

$$g_i^{p'} = g\left(\sum_{k=i+1}^{n} p'_k \right)$$

By expanding the expression $\Pi(Y)$, we have:

$$\Pi(X) = \Pi(Y)$$

Hence, the problem can be reduced to the case where the possible values are all distinct.

The example illustrates the following idea. In the general case when some values of X coincide, order them in an increasing order, then from each set of equal values keep only one value and assign the probability of the set to this value. Thus, a new variable Y is defined in such a way that all the elements of Y are strictly increasing with adjusted probability weights such that the identity $\Pi(X) = \Pi(Y)$ is satisfied.

Another approach consists of keeping the redundant values and dividing the probability weights by the number of these values. This is the idea of the next corollary:

Corollary 1 *If X is a random variable taking n possible values $(X(\omega_1), X(\omega_2), ..., X(\omega_n))$ with probabilities $(p_1, p_2, ..., p_n)$, then,*

$$\Pi(X) = \widetilde{\mathbb{E}}_g[X]$$

where the expectation is taken under the distorted probability measure $\widetilde{P}^g$ defined by:

$$\widetilde{P}^g(\omega_i) = \frac{g\left(P[X \geq X(\omega_i)]\right) - g\left(P[X > X(\omega_i)]\right)}{\#\{X(\omega_j) : X(\omega_j) = X(\omega_i)\}} \quad \forall \omega_i \in \Omega$$

The notation $\#\{X(\omega_j) : X(\omega_j) = X(\omega_i)\}$ stands for the number of values $X(\omega_j)$ equal to $X(\omega_i)$

Proof. The idea of the proof is given in the previous example. In formal terms, let

$$\Psi_s = \{i \in \{1, ..., n\} : X(\omega_i) = X(\omega_s)\}, \quad s \in \{1, ..., n\}$$

and

$$P^g(\omega_i) = g\left(P[X \geq X(\omega_i)]\right) - g\left(P[X > X(\omega_i)]\right) \quad \forall \omega_i \in \Omega$$

We have: $\forall i \in \Psi_s$, $P^g(\omega_i) = P^g(\omega_s)$ and so

$$\sum_{i \in \Psi_s} P^g(\omega_i) X(\omega_i) = |\Psi_s| \, P^g(\omega_s) X(\omega_s)$$

where

$$|\Psi_s| = \#\{X(\omega_j) : X(\omega_j) = X(\omega_s)\}$$

Define a new random variable Y and a subset of indices $S \subseteq \{1, ..., n\}$, such that:

$$\begin{aligned} Y(\omega_i) &= X(\omega_i) \ \forall i \in S \\ Y(\omega_i) &\neq Y(\omega_j) \text{ for } i \neq j, \forall (i,j) \in S^2 \\ \mathrm{Im}(Y) &= \mathrm{Im}(X) \end{aligned}$$

where $\mathrm{Im}(X)$ is the set of all possible values taken by X.

By applying the theorem to Y whose values are all distinct, we have:

$$\begin{aligned} \Pi(Y) &= \sum_{s \in S} P^g(\omega_s) Y(\omega_s) \\ &= \sum_{s \in S} \frac{|\Psi_s| \, P^g(\omega_s) X(\omega_s)}{|\Psi_s|} \\ &= \sum_{s \in S} \frac{1}{|\Psi_s|} \sum_{i \in \Psi_s} P^g(\omega_i) X(\omega_i) \\ &= \sum_{i=1}^{n} \frac{P^g(\omega_i) X(\omega_i)}{\#\{X(\omega_j) : X(\omega_j) = X(\omega_i)\}} \end{aligned}$$

Since

$$\Pr[X > t] = \Pr[Y > t] \quad \forall t \geq 0$$

then,

$$\Pi(Y) = \Pi(X)$$

■

With these ingredients, we propose to study some classes of risk measures based on distortion functions and consider the application to asset allocation.

3.3 Van der Hoek and Sherris class of risk measures

In their paper, van der Hoek and Sherris define the certainty equivalent of a random variable X by:

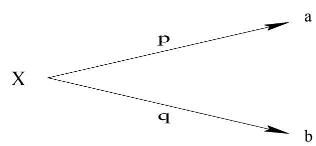
$$\begin{aligned} H(X) \equiv H_{\alpha,g,h}(X) &= \alpha + H_h((X-\alpha)^+) - H_g((\alpha - X)^+) \\ &= \alpha + \int_0^\infty h\left\{\Pr[(X-\alpha)^+ > t]\right\} dt - \int_0^\infty g\left\{\Pr[(\alpha - X)^+ > t]\right\} dt \end{aligned} \tag{4}$$

Where α is a real constant, and h is a convex and increasing function on $[0,1]$ with $h(0) = 0$ and $h(1) = 1$, and g is a concave and increasing function on $[0,1]$ with $g(0) = 0$ and $g(1) = 1$. The convexity and concavity of h and g ensures the concavity of $H_{\alpha,g,h}(X)$ which is an appealing property in portfolio optimisation.

Definition 2 *The functions g and h are said to be conjugate if and only if: $h(x) = 1 - g(1-x)$ $\forall x \in [0,1]$*

In what follows, we provide an expression of $H(X)$ in the discrete-time case, but let us consider some simple cases first.

If X is a binomial random variable taking two possible values a with probability p, and b with probability q (with $p+q=1$), then $H(X)$ takes two forms depending on whether $a > b$ or $a < b$.



Binomial random variable

- If $a > b$, then:

$$H(X) = \alpha + (b-\alpha)^+ - (\alpha - a)^+ + h(p)[(a-\alpha)^+ - (b-\alpha)^+] - g(1-p)[(\alpha-b)^+ - (\alpha-a)^+]$$

When g and h are conjugate, then the above expression simplifies to:

$$H(X) \quad = h(p)a + [1-h(p)]b$$

$H(X)$ does not depend on the parameter α anymore.

- If $a < b$, then

$$H(X) = \alpha + (a-\alpha)^+ - (\alpha - b)^+ + h(1-p)[(b-\alpha)^+ - (a-\alpha)^+] - g(p)[(\alpha-a)^+ - (\alpha-b)^+]$$

When h and g are conjugate, then:

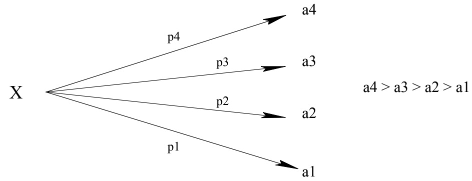
$$H(X) \quad = h(q)b + [1-h(q)]a$$

Thus, in the case when g and h are conjugate, if $a \neq b$, then,

$$H(X) = h\left(P[X = \max(a,b)]\right).\max(a,b) + g\left(P[X = \min(a,b)]\right).\min(a,b)$$

3.3.1 Quadrinomial case

Before generalising the properties of H for the multinomial case, this paragraph treats the quadrinomial case in order to give an intuition for the more general case. If X is a Bernoulli random variable X having four possible values (x_1, x_2, x_3, x_4) with probabilities (p_1, p_2, p_3, p_4), such that $\sum_{i=1}^4 p_i = 1$



Quadrinomial random variable

$$
\begin{aligned}
H(X) \;=\; & \alpha+(x_1-\alpha)^+-(\alpha-x_4)^+ + h(p_2+p_3+p_4)[(x_2-\alpha)^+-(x_1-\alpha)^+]+ \\
& h(p_3+p_4)\ [(x_3-\alpha)^+-(x_2-\alpha)^+] + h(p_4)[(x_4-\alpha)^+-(x_3-\alpha)^+]- \\
& g(p_1+p_2+p_3)[(\alpha-x_3)^+-(\alpha-x_4)^+] - g(p_1+p_2)[(\alpha-x_2)^+-(\alpha-x_3)^+] \\
& -g(p_1)[(\alpha-x_1)^+-(\alpha-x_2)^+]
\end{aligned}
$$

When g and h are conjugate, then $H(X)$ has a simpler form that does not depend on α :

$$H(X)=x_1+h(p_4)[x_4-x_3]+h(p_3+p_4)[x_3-x_2]+h(p_2+p_3+p_4)[x_2-x_1]$$

If we denote $X(\omega_i)=x_i$, the values of the random variable X in the states $\{\omega_i,\ i\in\{1,2,3,4\}\}$, then:

$$H(X)=p^h(\omega_1)X(\omega_1)+p^h(\omega_2)X(\omega_2)+p^h(\omega_3)X(\omega_3)+p^h(\omega_4)X(\omega_4)$$

where:

$$
\begin{aligned}
p^h(\omega_1) &= 1-h(p_2+p_3+p_4) \\
p^h(\omega_2) &= h(p_2+p_3+p_4)-h(p_3+p_4) \\
p^h(\omega_3) &= h(p_3+p_4)-h(p_4) \\
p^h(\omega_4) &= h(p_4).
\end{aligned}
$$

Since $\forall x\in[0,1]$, $h(x)\in[0,1]$ and h is increasing on $[0,1]$, then the weights $p^h(\omega_i)\in[0,1]$ for all $i\in\{1,2,3,4\}$. Moreover, $\sum_{i=1}^4 p^h(\omega_i)=1$, so $\{p^h(\omega_i)\}_i$ define a probability measure on the probability space $\Omega=\{\omega_1,\omega_2,\omega_3,\omega_4\}$. We can then write $H(X)$ as an expectation under the distorted probabilities $\{p^h(\omega_i)\}_i$:

$$H(X)=\mathbb{E}_h[X]$$

3.3.2 Multinomial case

Proposition 1 (Order assumption) *If X is a multinomial discrete random variable taking the values $(x_1,x_2,...,x_n)$ such that $x_1<x_2<...<x_n$, with probabilities $(p_1,p_2,...,p_n)$, then,*

$$
\begin{aligned}
H(X) &= \alpha+(x_1-\alpha)^+-(\alpha-x_n)^+ + \sum_{i=1}^{n-1}\left\{\begin{array}{c} h_i^p.\left[(x_{i+1}-\alpha)^+-(x_i-\alpha)^+\right] \\ +g_i^p.\left[(\alpha-x_{i+1})^+-(\alpha-x_i)^+\right]\end{array}\right\} \\
&= \alpha+\sum_{i=1}^{n}\left[h_{i-1}^p-h_i^p\right](x_i-\alpha)^+-\left[g_i^p-g_{i-1}^p\right](\alpha-x_i)^+ \qquad (5)
\end{aligned}
$$

where

$$h_i^p=h\left(1-\sum_{k=1}^{i}p_k\right)\ \text{and}\ g_i^p=g\left(\sum_{k=1}^{i}p_k\right)$$

Proof. The idea of the proof is the same as in the proof of Theorem (Certainty equivalent). A detailed proof is provided in the appendix. ■

In its general form, $H(X)$ is a piecewise linear function, so it is not differentiable. However, when $x_1,x_2,...,x_n$ and α can be ordered, then $H(X)$ has a simple differentiable form given by the following corollary:

Corollary 2 (Position of α) *If X is a multinomial discrete random variable taking the values $(x_1, x_2, ..., x_n)$ such that $x_1 < x_2 < ... < x_n$, with probabilities $(p_1, p_2, ..., p_n)$, then,*

- *If $\alpha \leq x_1$, then:*

$$H(X) = x_1 + \sum_{i=1}^{n-1} h_i^p \left[x_{i+1} - x_i\right] = \sum_{i=1}^{n} \left[h_{i-1}^p - h_i^p\right] x_i$$

- *If $\alpha \geq x_n$, then:*

$$H(X) = x_n + \sum_{i=1}^{n-1} g_i^p \left[x_i - x_{i+1}\right] = \sum_{i=1}^{n} \left[g_i^p - g_{i-1}^p\right] x_i$$

- *If $\alpha \in [x_r, x_{r+1})$ where $r \in \{1, ..., n-1\}$, then:*

$$H(X) = \left[1 - h_r^p - g_r^p\right] \alpha + \sum_{i=1}^{r} \left[g_i^p - g_{i-1}^p\right] x_i + \sum_{i=r+1}^{n} \left[h_{i-1}^p - h_i^p\right] x_i$$

where

$$h_i^p = h\left(1 - \sum_{k=1}^{i} p_k\right) \text{ and } g_i^p = g\left(\sum_{k=1}^{i} p_k\right)$$

This corollary illustrates the idea of pricing the upside and the downside of the risk differently. In effect, for outcomes x_i's below the level α, the probability distribution is distorted by the function g, and for outcomes x_i's above the level α, the probability distribution is distorted by the function h. This is a flexible way to price risk around some benchmark return α. The choice of the distortion functions g and h reflects the risk behaviour of the investor. Indeed, h is convex[5], so

$$h\left(1 - \sum_{k=1}^{i-1} p_k\right) - h\left(1 - \sum_{k=1}^{i} p_k\right) < h\left(1 - \sum_{k=1}^{i-2} p_k\right) - h\left(1 - \sum_{k=1}^{i-1} p_k\right)$$

or

$$h_i^p < h_{i-1}^p$$

So the probability assigned to the outcome x_i is less than the probability assigned to x_{i-1}. In other terms, the investor assigns lower probabilities to higher outcomes. The more risk averse the investor, the more convex the distortion function h. The same argument applies to the concavity of g.

Furthermore, the choice of h with respect to g reflects how the investor considers the risk with respect to the benchmark α. For some choice of g and h, pricing risk does not depend on α. This is the idea of the following proposition:

[5] If h is a convex function, then $2h(p) \leq h(p-1) + h(p+1)$ wherever h is defined.

Proposition 2 (Conjugate) *Let X be a random variable taking n possible values $(X(\omega_1), X(\omega_2), ..., X(\omega_n))$ with probabilities $(p_1, p_2, ..., p_n)$, such that $X(\omega_i) \neq X(\omega_j)$ if $i \neq j$. In the case when h and g are conjugate, we have*

$$H(X) = \mathbb{E}_h[X] \tag{6}$$

where the expectation is taken under the probability measure P^g given by:

$$P^g(\omega_i) = h\left(P[X \geq X(\omega_i)]\right) - h\left(P[X > X(\omega_i)]\right) \quad \forall \omega_i \in \Omega \tag{7}$$

Proof. In the case when h and g are conjugate, i.e., $h(1-x) = 1 - g(x)$, in the appendix we show that:

$$H(X) = H_{0,0,h}(X) = \int_0^{\infty} h(P[X > t])dt$$

Therefore, the proposition follows from Theorem (Certainty equivalent). ■

Remark 1 *How do we compute $H(X)$ if the order $X(\omega_1) < X(\omega_2) < ... < X(\omega_n)$ changes?*

Put $X = [X(\omega_1), X(\omega_2), ..., X(\omega_i), ..., X(\omega_j), ..., X(\omega_n)]$. X is the vector of the realizations of X in an increasing order. Put $P = [p_1, p_2, ..., p_i, ..., p_j, ..., p_n]$ where $P(i) = p_i = P[X = X(\omega_i)]$. Define $P^g = [P^g(1), P^g(2), ..., P^g(n)]$ where $P^g(i) = g\left(\sum_{k=1}^{i} P(k)\right) - g\left(\sum_{k=1}^{i-1} P(k)\right)$. We have $H(X) = \langle P^g, X\rangle$ the inner product of the vectors P^g and X.

If a permutation σ applies to X, then:

$X := \sigma(X)$

$P := \sigma(P)$

$P^g := [P^g(1), P^g(2), ..., P^g(n)]$, *where* $P^g(i) = g\left(\sum_{k=1}^{i} P(k)\right) - g\left(\sum_{k=1}^{i-1} P(k)\right)$

$H(X) := \langle P^g, X\rangle$

It is clear that the order of possible values $(x_1, x_2, ..., x_n)$ of the risk X plays a central role in the evaluation of $H(X)$. This is due to the fact that $H(X)$ is based on a transformation of the decumulative distribution function $P[X > x]$. In an optimisation problem, this is not an appealing feature, since the possible values of X are in general the control variables, i.e. $H(X)$ is optimised by choosing $x_1, x_2, ..., x_n$. Sorting these values at each step of the optimisation is costly in time. In the following proposition, we do not get around this problem, however, the order is implicitly taken care of in the coefficients of x_i's.

Proposition 3 (No order assumption) *For a multinomial discrete random variable X taking the values $(x_1, x_2, ..., x_n)$, with probabilities $(p_1, p_2, ..., p_n)$, $x_i \neq x_j$ if $i \neq j$ then,*

$$H(X) = Const + \sum_{i=1}^{n} \left(g_i^{p,X} I_{\{x_i \leq \alpha\}} + h_i^{p,X} I_{\{x_i > \alpha\}}\right) x_i$$

Where

$$\begin{aligned} g_i^{p,X} &= g\left(\sum_{\{x_k \leq x_i\}} p_k\right) - g\left(\sum_{\{x_k < x_i\}} p_k\right) \\ h_i^{p,X} &= \left[h\left(1 - \sum_{\{x_k < x_i\}} p_k\right) - h\left(1 - \sum_{\{x_k \leq x_i\}} p_k\right)\right] \end{aligned}$$

$$Const = \left[1 - h\left(1 - \sum_{\{x_k \le \alpha\}} p_k\right) - g\left(\sum_{\{x_k \le \alpha\}} p_k\right)\right]\alpha$$

and the notation

$$\sum_{\{x_k \le \alpha\}} p_k = \sum_{k=1}^{n} p_k I_{\{x_k \le \alpha\}}$$

where $I_{\{x_k \le \alpha\}}$ *is the indicator function on the set* $\{x_k \le \alpha\}$.

3.3.3 Optimal portfolio choice in a multiperiod model

In the dual theory of choice under uncertainty, investors evaluate a risk X by calculating its certainty equivalent $H(X)$ as defined in equation (4), using the distortion functions g and h representing their preference order. Given a choice among many risks, an investor would then pick the one having the greatest certainty equivalent.

In a multiperiod asset allocation, investors are faced with a series of decisions where at the beginning of each investment period, they have to choose the optimal amount of consumption, and the rest is invested in the market. The optimal consumption level C_t at the beginning of the period $[t, t+1]$ is a risky prospect. Formally, C_t, $t = 0, ..., T$ is a non-negative and bounded random variable defined on some probability space. By virtue of Theorem 2 in Yaari ([49]), the scheme $(C_0, C_1, ..., C_T)$ is preferred to $(C'_0, C'_1, ..., C'_T)$ if and only if there exists an increasing continuous function $U : R_+^n \to R_+$ such that:

$$U(H(C_0), H(C_1), ..., H(C_T)) \ge U(H(C'_0), H(C'_1), ..., H(C'_T))$$

In words, the investor chooses the consumption stream that maximises an increasing function of the certainty equivalents of consumption at each period. In what follows, we choose

$$\begin{array}{rcl} U & : & R_+^n \to R_+ \\ (x_1, x_2, ..., x_n) & \mapsto & \sum \beta^i x_i \end{array}$$

where $0 < \beta \le 1$ is the time preference factor. The consumption-investment problem (1) becomes:

$$\left\{ \begin{array}{ll} \underset{C_0, C_1, ..., C_T}{Max} & \sum_{t=0}^{T} \beta^t H(C_t) \\ s.t & v = \sum_{t=0}^{T} \mathbb{E}_Q \left[\dfrac{C_t}{(1+r)^t}\right] \\ & C_t \ge 0 \quad \forall t \in [0, T] \end{array} \right. \tag{8}$$

Where v is the initial wealth and r is a constant interest rate.

Problem (8) is set up in the discrete-time case. However, it can be solved numerically either with a finite number or a continuum range of states at each time period.

Let us consider the case when at each time t, C_t takes $(2t + 1)$ possible values (recombining trinomial tree), then C_t is a vector of $2t + 1$ control variables, i.e.

$$C_t = [C_{t,-t}, ..., C_{t,0}, ..., C_{t,t}] \in \mathbb{R}_+^{2t+1}$$

For each vector C_t, there exists a corresponding vector

$$p_t = [p_{t,-t}, ..., p_{t,0}, ..., p_{t,t}] \in [0, 1]^{2t+1}$$

where $p_{t,i} = P[C_t = C_{t,i}]$.

From Proposition (No order assumption), one way to write the term $H(C_t)$ in the objective function is:

$$H(C_t) = \sum_{i=-t}^{t} F_{C_{t,i}}(p_t).C_{t,i}$$

Where the function

$$F_{C_{t,i}} : [0,1]^{2t+1} \longrightarrow \mathbb{R}_+$$

$$p_t = [p_{t,-t}, ..., p_{t,0}, ..., p_{t,t}] \longmapsto \left\{ \begin{array}{c} \left[g\left(\sum_{\{C_{t,k} \leq C_{t,i}\}} p_k\right) - g\left(\sum_{\{C_{t,k} < Ct,i\}} p_k\right)\right] I_{\{C_{t,i} \leq \alpha\}} \\ + \left[h\left(1 - \sum_{\{C_{t,k} < C_{t,i}\}} p_k\right) - h\left(1 - \sum_{\{C_{t,k} \leq C_{t,i}\}} p_k\right)\right] I_{\{C_{t,i} > \alpha\}} \end{array} \right\} \tag{9}$$

Remark 2 *Another formulation is also possible using Proposition (Order assumption), where the control variables are ordered explicitly.*

The general expression (9) can be drastically simplified in the case when g and h are conjugate functions and the consumption at each time is distributed with the same probability over the states, i.e.

$$g(x) = 1 - h(1-x) \quad \forall x \in [0,1]$$

and

$$P[C_t = C_{t,i}] = q_t = \frac{1}{2t+1} \quad \forall t \in [1,T], \forall i \in [-t,t]$$

Then, using Proposition (conjugate), we have

$$F_{C_{t,i}}(p_t) = g\left(\frac{1}{2t+1}\#\{C_{t,k} \leq C_{t,i}\}\right) - g\left(\frac{1}{2t+1}\#\{C_{t,k} < C_{t,i}\}\right)$$

where $\#\{C_{t,k} \leq C_{t,i}\}$ denotes the number of variables $C_{t,k}$, $k \in \{-t, ..., t\}$, such that $C_{t,k} \leq C_{t,i}$.

In this case, the problem (8) can be solved using one of the simplicial algorithms used in rank regression problems. Osborne (2001 [37]) is an excellent reference for solving such types of problems.

3.3.4 Optimal portfolio choice in a one-period binomial model

One risky security and one riskless asset In the case of a one-period model with 2 assets, there are 5 unknown variables: C_0, the consumption at time 0, C_1^u and C_1^d, the consumption at time 1 for the up and down states and H_0 and H_1 the investment positions in the safe and the risky asset respectively. To solve this problem, we start from the budget constraints that involve both consumption and investment strategies and express all the variables in terms of C_1^u and C_1^d. We then show how this is equivalent to using the risk-neutral computational approach directly, which from the start determines the constraints in terms of C_1^u and C_1^d.

The consumption-investment problem is:

$$\begin{array}{ll} \max\limits_{(C,H)} & H(C_0) + \beta.H(C_1) \\ \textit{subject to:} & C_0, C_1 \geq 0 \\ \textit{and} & \textit{Budget constraints,} \end{array} \tag{10}$$

where the budget constraints are:

$$W_0 - C_0 = H_0.B_0 + H_1.S_0 \tag{11}$$
$$W_1 = C_1 = H_0.B_1 + H_1.S_1 \tag{12}$$

We can rewrite all the variables in terms of C_1^u and C_1^d, which become the control variables in problem (10).

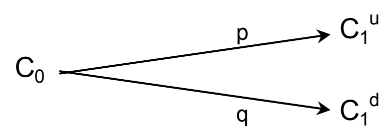


Figure 1: Consumption nodes in one period binomial model

Equation (12) yields two equations:

$$\begin{aligned} C_1^u =& \quad H_0.B_1 + H_1.S_1^u \\ C_1^d =& \quad H_0.B_1 + H_1.S_1^d \end{aligned}$$

Solving then for H_0 and H_1 :

$$\begin{aligned} H_0 =& \quad -\tfrac{S_1^d}{B_1(S_1^u - S_1^d)} C_1^u + \tfrac{S_1^u}{B_1(S_1^u - S_1^d)} C_1^d \\ H_1 =& \quad \tfrac{1}{S_1^u - S_1^d} C_1^u - \tfrac{1}{S_1^u - S_1^d} C_1^d \end{aligned} \tag{13}$$

Using equations (11) and (13), C_0 can be expressed in terms of C_1^u and C_1^d :

$$C_0 = W_0 - b_1 C_1^u - b_2 C_1^d \tag{14}$$

where:

$$b_1 = \frac{B_1 S_0 - S_1^d}{B_1(S_1^u - S_1^d)} \quad \text{and} \quad b_2 = \frac{S_1^u - B_1 S_0}{B_1(S_1^u - S_1^d)}$$

Interpretation of b_1 and b_2 :

For the choice of $S_1^d < B_1 S_0 < S_1^u$, it is easy to check that $Q\{\omega_1\} = \dfrac{B_1 S_0 - S_1^d}{(S_1^u - S_1^d)}$ and $Q\{\omega_2\} = \dfrac{S_1^u - B_1 S_0}{(S_1^u - S_1^d)}$ define a martingale measure for the discounted price $\dfrac{S_1}{B_1}$. Therefore.

$$b_1 C_1^u + b_2 C_1^d = \mathbb{E}_Q\left[\frac{C_1}{B_1}\right]$$

And equation (14) can be written as:

$$W_0 = C_0 + \mathbb{E}_Q\left[\frac{C_1}{B_1}\right]$$

The absence of arbitrage condition is equivalent to $Q\{\omega_i\} \in (0,1)$ for $i \in \{1,2\}$. Provided that $B_1 = (1+r) \geq 1$, we also have $b_i \equiv \dfrac{Q\{\omega_i\}}{B_1} \in (0,1)$.

The problem (10) is then equivalent to:

$$\begin{array}{ll} \max\limits_{(C,H)} & H(C_0)+\beta.H(C_1) \\ \textit{subject to:} & C_0, C_1 \geq 0 \\ & W_0 = C_0 + \frac{1}{1+r}\mathbb{E}_Q\left[C_1\right] \end{array} \tag{15}$$

This would be the starting point if the optimal portfolio problem (10) was formulated within the risk-neutral computational approach.

The constraints of Problem (10) are:

1. $C_1^u \geq 0$ and $C_1^d \geq 0$
2. $W_0 - C_0 \geq 0 \;\Rightarrow\; b_1 C_1^u + b_2 C_1^d \geq 0$ (using (14))
3. $C_0 \geq 0 \;\Rightarrow\; b_1 C_1^u + b_2 C_1^d \leq W_0$ (using (14))

Solving the problem - Feasible region In this paragraph, we consider the special case when the distortion functions g and h are conjugate, and we solve the problem analytically. The general case of non-conjugate distortion functions is left to the next paragraph dealing with the case of 2 risky assets and 1 riskless security.

Let us write all the variables in terms of C_1^u and C_1^d :

- $H(C_0) = C_0 = W_0 - b_1 C_1^u - b_2 C_1^d$
- $H(C_1) = \begin{cases} h(p)C_1^u + [1-h(p)]C_1^d & \text{if } C_1^u \geq C_1^d \\ h(q)C_1^d + [1-h(q)]C_1^u & \text{if } C_1^u \leq C_1^d \end{cases}$

Therefore, the objective function in the problem (15) is equal to:

$$H(C_0)+\beta H(C_1) = \begin{cases} W_0 + \{\beta h(p) - b_1\}C_1^u + \{\beta(1-h(p)) - b_2\}C_1^d & \text{if } C_1^u \geq C_1^d \\ W_0 + \{\beta(1-h(q)) - b_1\}C_1^u + \{\beta h(q) - b_2\}C_1^d & \text{if } C_1^u \leq C_1^d \end{cases}$$

We consider then the two maximization problems:

$$P_1 = \begin{cases} \max\limits_{C_1^u, C_1^d} & \{\beta h(p) - b_1\}C_1^u + \{\beta(1-h(p)) - b_2\}C_1^d \\ \textit{subject to:} & C_1^u \geq 0, C_1^d \geq 0 \\ & C_1^u \geq C_1^d \\ & b_1 C_1^u + b_2 C_1^d \leq W_0 \end{cases} \tag{16}$$

and

$$P_2 = \begin{cases} \max\limits_{C_1^u, C_1^d} & \{\beta(1-h(q)) - b_1\}C_1^u + \{\beta h(q) - b_2\}C_1^d \\ \textit{subject to:} & C_1^u \geq 0, C_1^d \geq 0 \\ & C_1^u \leq C_1^d \\ & b_1 C_1^u + b_2 C_1^d \leq W_0 \end{cases} \tag{17}$$

Maximizing (10) is equivalent to solving P_1 and P_2 and choosing the solution that corresponds to the higher objective value function. The two maximization problems are linear programs that can be solved numerically using any optimisation software package. However, since there are only two choice variables, an analytical solution can be given explicitly.

Remark 3 *If the probability distribution is such that $P\{\omega_1\} = P\{\omega_2\} = \frac{1}{2}$, then P_1 and P_2 have the same solution and therefore, it suffices to solve either the problem P_1 or P_2.*

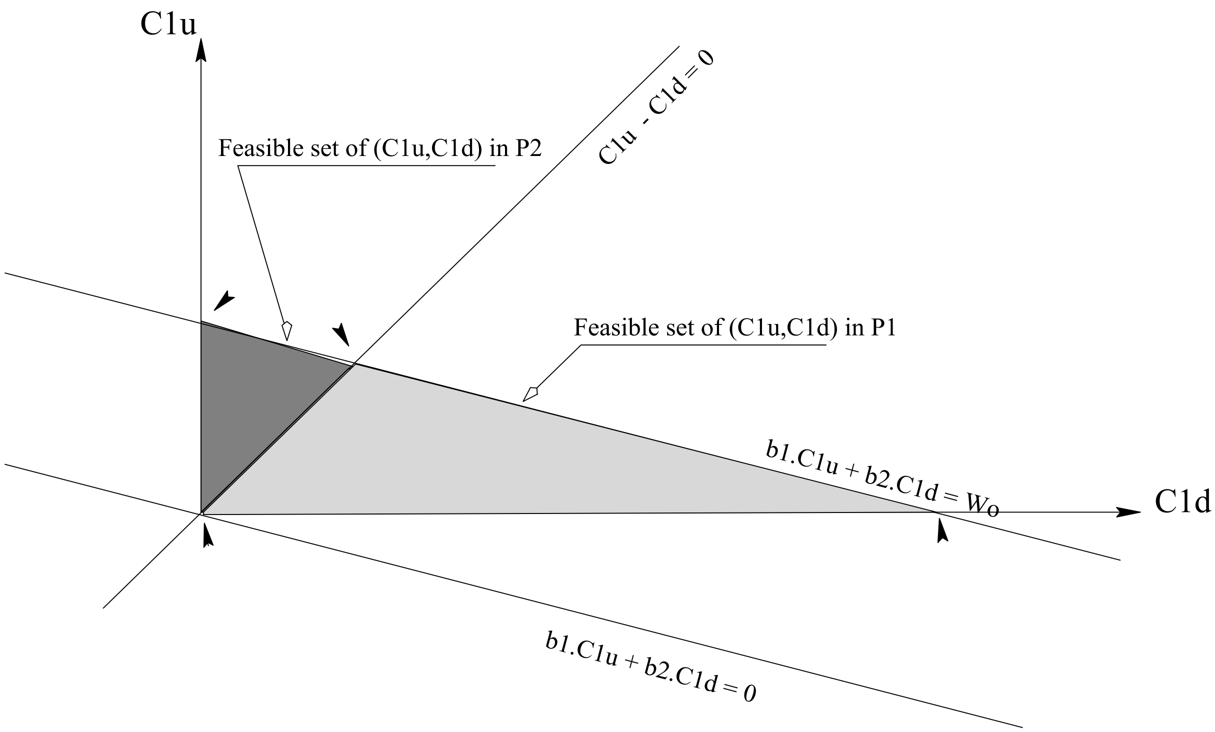


Figure 2: Feasible regions for consumption in one-period model

Given the constraints, feasible sets of the solution to P_1 and P_2 are illustrated in figure (2).

The objective function in the problem (16) is linear in C_1^u and C_1^d. The equation of the line representing the level curve of the objective function at a possible value z_1 is given by:

$$C_1^d = \frac{z_1}{\beta(1-h(p))-b_2} - \frac{\beta h(p)-b_1}{\beta(1-h(p))-b_2} C_1^u \tag{18}$$

There are four possible solutions of the problem corresponding to the summits of the feasible region:

$$(0,0), \quad (0, \frac{W_0}{b_2}), \quad (\frac{W_0}{b_1+b_2}, \frac{W_0}{b_1+b_2}) \text{ and } (\frac{W_0}{b_1}, 0). \tag{19}$$

Relation between the solution and the risk-averse index (ρ) The solution takes one of the four possible values given in (19). For the choice of the function $h(p) = 1-(1-p)^\rho$, where ρ is a risk aversion parameter, the intercept and the slope of (18) depend on ρ. The level curve equation (18) can be written as:

$$C_1^d = \frac{z_1}{A(\rho)} - \frac{B(\rho)}{A(\rho)} C_1^u \tag{20}$$

where $A(\rho) = \beta(1-h(p)) - b_2$ and $B(\rho) = \beta h(p) - b_1$

$$A(\rho) \geq 0 \;\Leftrightarrow\; \rho \leq \frac{\ln(b_2/\beta)}{\ln(1-p)} \quad \text{and} \quad B(\rho) \geq 0 \;\Leftrightarrow\; \rho \geq \frac{\ln(1-b_1/\beta)}{\ln(1-p)}$$

Put $\rho_1 = \dfrac{\ln(1-b_1/\beta)}{\ln(1-p)}$ and $\rho_2 = \dfrac{\ln(b_2/\beta)}{\ln(1-p)}$, we have: $\rho_1 < \rho_2$ if and only if $\beta > b_1 + b_2 = \frac{1}{1+r}$.

Case 1 *$\beta > \frac{1}{1+r}$, then $\rho_1 < \rho_2$, the following table summarizes the signs of $A(\rho)$ and $B(\rho)$, given the position of ρ in relation to ρ_1 and ρ_2 :*

ρ	*0*	ρ_1		ρ_2	∞
$A(\rho)$	+	+	+	*0*	-
$B(\rho)$	-	*0*	+	+	+
$\frac{-B(\rho)}{A(\rho)}$	+	0	-	∞	+

- *On $(0, \rho_1)$ the slope is positive and $A(\rho)$ is positive. Maximizing z_1 is equivalent to increasing the intercept in (20) (shifting the curve up). The solution is then $(W_0(1+r), W_0(1+r))$*[6].
- *On (ρ_1, ρ_2) the slope is negative and $A(\rho)$ is positive. Maximizing z_1 is equivalent to increasing the intercept in (20) (shifting the curve up). The solution is then*

$$\begin{cases} (\frac{W_0}{b_1}, 0) & \textit{if } \rho \in (\rho_1, \rho_3) \\ (W_0(1+r), W_0(1+r)) & \textit{if } \rho \in (\rho_3, \rho_2) \end{cases}$$

  where $\rho_3 = \dfrac{\ln(1-b_1(1+r))}{\ln(1-p)}$

- *On (ρ_2, ∞) the slope is positive and $A(\rho)$ is negative. Maximizing z_1 is equivalent to reducing the intercept in (20) (shifting the curve down). The solution is then $(\frac{W_0}{b_1}, 0)$.*

Case 2 *$\beta < \frac{1}{1+r}$, then $\rho_1 > \rho_2$, the following table summarizes the signs of $A(\rho)$ and $B(\rho)$, given the position of ρ in relation to ρ_1 and ρ_2 :*

ρ	*0*	ρ_2		ρ_1	∞
$A(\rho)$	+	*0*	-	-	-
$B(\rho)$	-	-	-	*0*	+
$\frac{-B(\rho)}{A(\rho)}$	+	∞	-	0	+

- *On $(0, \rho_2)$ the slope is positive and $A(\rho)$ is positive. Maximizing z_1 is equivalent to increasing the intercept in (20) (shifting the curve up). The solution is then $(0, 0)$*[7].
- *On (ρ_2, ρ_1) the slope is negative and $A(\rho)$ is negative. Maximizing z_1 is equivalent to reducing the intercept in (20) (shifting the curve down). The solution is then $(0, 0)$.*
- *On (ρ_1, ∞) the slope is positive and $A(\rho)$ is negative. Maximizing z_1 is equivalent to reducing the intercept in (20) (shifting the curve down). The solution is then $(\frac{W_0}{b_1}, 0)$.*

In sum, the possible values for optimal consumption (C_1^u, C_1^d) are $(0, 0)$, $(\frac{W_0}{b_1}, 0)$ and $(\frac{W_0}{b_1+b_2}, \frac{W_0}{b_1+b_2})$. The solution depends on the risky security distribution (values of (ρ_1, ρ_2)) and the sign of $(\beta - \frac{1}{1+r})$.

The same analysis applies to the problem P_2 (17). Put $\rho'_1 = \dfrac{\ln(b_1/\beta)}{\ln(p)}$ and $\rho'_2 = \dfrac{\ln(1-b_2/\beta)}{\ln(p)}$, we have: $\rho'_1 < \rho'_2$ if and only if $\beta < b_1 + b_2 = \frac{1}{1+r}$.

Case 1 *$\beta > \frac{1}{1+r}$, then $\rho'_1 > \rho'_2$,*

[6] Since the slope $-\frac{B(\rho)}{A(\rho)} \leq 1$, then the solution $(0, 0)$ is eliminated

[7] Since the slope $-\frac{B(\rho)}{A(\rho)} \geq 1$, then the solution $(W_0(1+r), W_0(1+r))$ is eliminated

- *On* $(0, \rho_2')$ *the solution is* $(0,0)$
- *On* (ρ_2', ρ_1') *the solution is*

$$\begin{cases} (W_0(1+r), W_0(1+r)) & \text{if } \rho \in (\rho_2', \rho_3') \\ (0, \frac{W_0}{b_2}) & \text{if } \rho \in (\rho_3', \rho_1') \end{cases}$$

where $\rho_3' = \dfrac{\ln(b_1(1+r))}{\ln(p)}$

- *On* (ρ_2, ∞) *the slope is positive and* $A(\rho)$ *is negative. Maximizing* z_1 *is equivalent to reducing the intercept in (20) (shifting the curve down). The solution is then* $(0, \frac{W_0}{b_2})$.

Case 2 $\beta < \frac{1}{1+r}$, *then* $\rho_1' < \rho_2'$,

- *On* $(0, \rho_1')$ *the solution is* $(W_0(1+r), W_0(1+r))$.
- *On* (ρ_1', ρ_2') *the solution is* $(0,0)$.
- *On* (ρ_2', ∞) *the solution is* $(0, \frac{W_0}{b_2})$.

Figure 3 plots the solution for the case where $\beta = 0.95$, $r = 10\%$ and $\rho = 0.8$.

Optimal investment strategies Once the optimal consumption rules are obtained, the optimal investment strategies follow from the budget equations (13).

- When $(C_1^u, C_1^d) = (0,0)$, then $(H_0, H_1) = (0,0)$
- When $(C_1^u, C_1^d) = (\frac{W_0}{b_1}, 0)$, then $(H_0, H_1) = (-W_0 \frac{S_1^d}{B_1 S_0 - S_1^d}, W_0 \frac{B_1}{B_1 S_0 - S_1^d})$.
- When $(C_1^u, C_1^d) = (\frac{W_0}{b_1+b_2}, \frac{W_0}{b_1+b_2})$, then $(H_0, H_1) = (W_0, 0)$. This situation is referred to as "plunging".
- When $(C_1^u, C_1^d) = (0, \frac{W_0}{b_2})$, then $(H_0, H_1) = (W_0 \frac{S_1^u}{S_1^u - B_1 S_0}, -W_0 \frac{B_1}{S_1^u - B_1 S_0})$.

Two correlated risky securities and one riskless asset

Discretising the process Consider two risky securities, which are log-normally distributed and one riskless asset. In order to obtain completeness of the market, we consider a trinomial lattice, where the bivariate log-normal process is described by:

$$dS_{t,1} = \mu_1 S_{t,1} dt + \sigma_1 S_{t,1} dW_{t,1}$$
$$dS_{t,2} = \mu_2 S_{t,2} dt + \sigma_2 S_{t,2} \left(\rho dW_{t,1} + \sqrt{1-\rho^2} dW_{t,2} \right)$$

where W_1 and W_2 are two independent Brownian motions, σ_i is the volatility of the instantaneous return on stock i, and ρ is the correlation coefficient between the instantaneous returns of the two stocks. Following He (1990, [23]), we approximate the increments of $(W_{t,1}\ , W_{t,2})$ by two random variables $(\varepsilon_1, \varepsilon_2)$ such that:

$$\Pr\left[\varepsilon_1 = \frac{\sqrt{3}}{\sqrt{2}}, \varepsilon_2 = \frac{1}{\sqrt{2}}\right] = \frac{1}{3}$$
$$\Pr\left[\varepsilon_1 = 0, \varepsilon_2 = \frac{-2}{\sqrt{2}}\right] = \frac{1}{3}$$
$$\Pr\left[\varepsilon_1 = \frac{-\sqrt{3}}{\sqrt{2}}, \varepsilon_2 = \frac{1}{\sqrt{2}}\right] = \frac{1}{3}$$

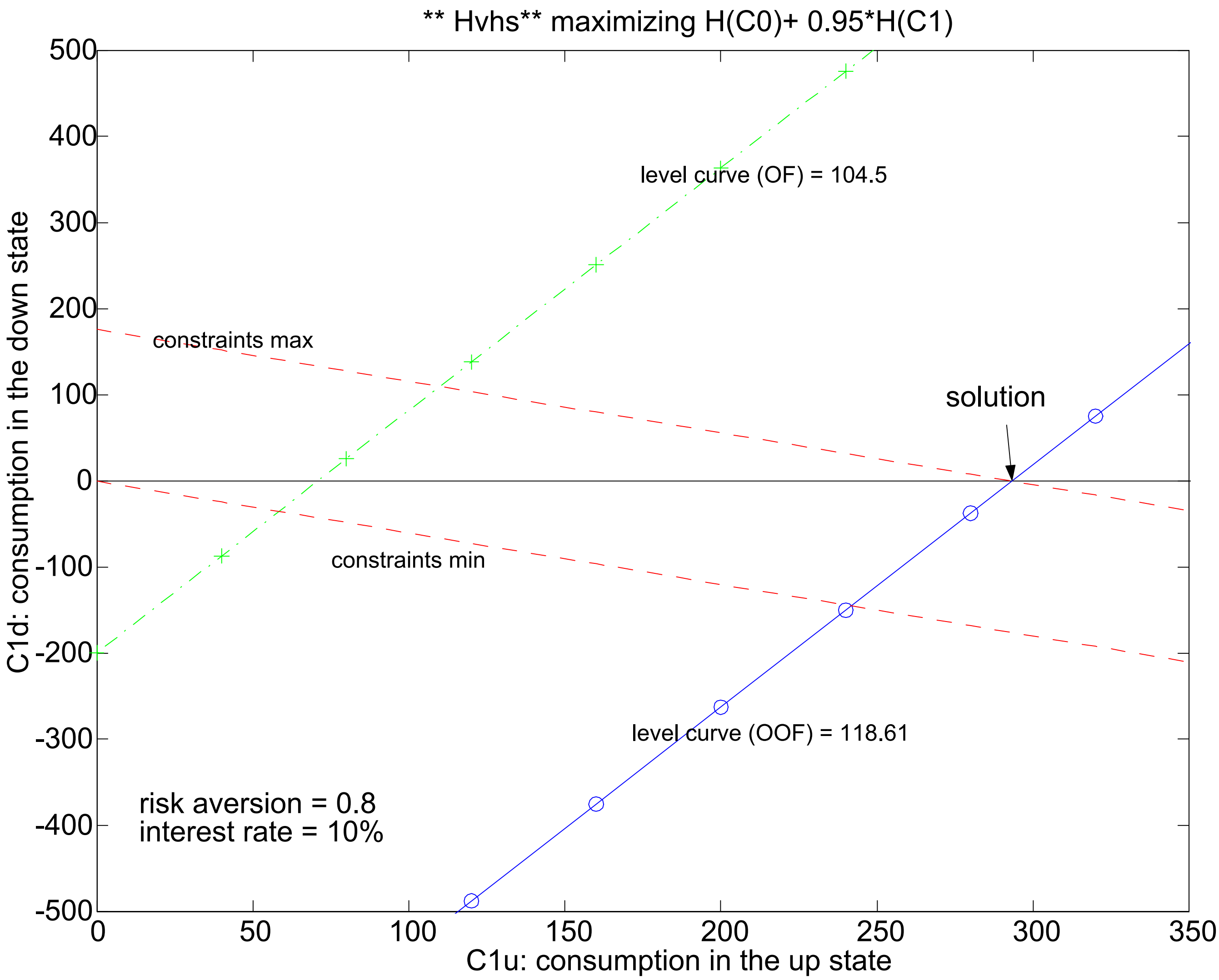


Figure 3: Solution to the portfolio problem in one period

So,

$$S_{k+1,1} = \begin{cases} S_{k,1}\left(1 + \frac{\mu_1}{n} + \sigma_1 \frac{\sqrt{3}}{\sqrt{2n}}\right) \\ S_{k,1}\left(1 + \frac{\mu_1}{n}\right) \\ S_{k,1}\left(1 + \frac{\mu_1}{n} - \sigma_1 \frac{\sqrt{3}}{\sqrt{2n}}\right) \end{cases}$$

$$S_{k+1,2} = \begin{cases} S_{k,2}\left(1 + \frac{\mu_2}{n} + \sigma_2 \rho \frac{\sqrt{3}}{\sqrt{2n}} + \sigma_2 \sqrt{1-\rho^2}\frac{1}{\sqrt{2n}}\right) \\ S_{k,2}\left(1 + \frac{\mu_2}{n} - \sigma_2 \sqrt{1-\rho^2}\frac{2}{\sqrt{2n}}\right) \\ S_{k,2}\left(1 + \frac{\mu_2}{n} - \sigma_2 \rho \frac{\sqrt{3}}{\sqrt{2n}} + \sigma_2 \sqrt{1-\rho^2}\frac{1}{\sqrt{2n}}\right) \end{cases}$$

where n is the number of time steps. Note that the choice of $(\varepsilon_1, \varepsilon_2)$ approximating the increments of $(W_{t,1}\ , W_{t,2})$ is not unique.

The trinomial lattice with two securities is a complete market model. It is arbitrage-free if and only if there exists a unique risk-neutral measure Q. That is:

$$\begin{aligned} E_Q[\Delta S_1^*] &= 0 \\ E_Q[\Delta S_2^*] &= 0 \end{aligned} \tag{21}$$

The model has 5 degrees of freedom $(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$. With the conditions (21), there remains 3 degrees of freedom. We can let $(\sigma_1, \sigma_2, \rho)$ be free and choose (μ_1, μ_2) so that the system (21) has a solution $(Q(\omega_1), Q(\omega_2), Q(\omega_3)) \in (0,1)^3$.

$$\mu_1 = n(B_1 - 1 - \sigma_1 \sqrt{\frac{3}{2n}}\,(Q(\omega_1) - Q(\omega_3))$$

$$\mu_2 = n(B_1 - 1 - \sigma_2 \rho \sqrt{\frac{3}{2n}}\,(Q(\omega_1) - Q(\omega_3)) + \sigma_2\sqrt{1-\rho^2}\sqrt{\frac{1}{2n}}\,(-Q(\omega_1) + 2Q(\omega_2) - Q(\omega_3))$$

For $Q(\omega_1) = Q(\omega_2) = Q(\omega_3) = \frac{1}{3}$, we obtain $\mu_1 = \mu_2 = (B_1 - 1) = r$.

One can choose suitable values for $(Q(\omega_1), Q(\omega_2), Q(\omega_3))$ so as to obtain sensible values for $(\mu_1, \mu_2, \sigma_1, \sigma_2)$. For example, $(\mu_1, \sigma_1) = (10\%, 20\%)$ for a share and $(\mu_2, \sigma_2) = (5\%, 8\%)$ for a Treasury bill.

When the lattice is recombining, we lose 2 degrees of freedom because of the conditions:

$$\begin{aligned} u_1 \cdot d_1 &= m_1^2 \\ u_2 \cdot d_2 &= m_2^2 \end{aligned}$$

However, the existence of the risk neutral measure Q still holds.

Solving the problem There is not a great deal of difference between the case of one risky and one riskless asset and the case of two risky and one riskless asset. In effect, the mathematical form of the optimisation problem is exactly the same, however, the number of states increases by one to ensure the completeness of the market model. This added dimension is translated by the addition of a new control variable C_1^m which represents the value of the consumption at time 1 in the middle state. The problem is:

$$\begin{aligned} &\max_{C_0, C_1} && H(C_0) + \beta . H(C_1) \\ &\textit{subject to:} && v = C_0 + \frac{1}{1+r}\mathbb{E}_Q[C_1] \\ & && \text{and } C_0, C_1 \geq 0 \end{aligned}$$

It can be shown that for the choice of the risk neutral probabilities $q_u \leq q_m \leq q_d$, the order $C_1^u \geq C_1^m \geq C_1^d$ is optimal. An application of Proposition (Order assumption) in the case of a trinomial random variable gives:

$$\begin{aligned} H(C_1) = \alpha &+ (C_1^u - \alpha)^+ \\ &+ \left[(C_1^m - \alpha)^+ - (C_1^u - \alpha)^+\right].h(1-p_u) \\ &+ \left[(C_1^d - \alpha)^+ - (C_1^m - \alpha)^+\right].h(1-p_u-p_m) \\ &- (\alpha - C_1^d)^+ \\ &- \left[(\alpha - C_1^m)^+ - (\alpha - C_1^d)^+\right].g(1-p_d) \\ &- \left[(\alpha - C_1^u)^+ - (\alpha - C_1^m)^+\right].g(1-p_d-p_m) \end{aligned}$$

This is a piecewise linear function with control variables C_1^u, C_1^m and C_1^d. Using the corollary (Position of α), we transform it to four linear functions depending on the value of α with respect to the control variables.

$$H(C_1) = \begin{cases} (h_0 - h_1)C_1^u + (h_1 - h_2)C_1^m + (h_2 - h_3)C_1^d & \text{if } \alpha < C_1^d \\ \alpha(1 - h_1 - g_1) + (g_1 - g_{0)}C_1^u + (h_1 - h_2)C_1^m + (h_2 - h_3)C_1^d & \text{if } C_1^d \leq \alpha < C_1^m \\ \alpha(1 - h_2 - g_2) + (g_1 - g_{0)}C_1^u + (g_2 - g_1)C_1^m + (h_2 - h_3)C_1^d & \text{if } C_1^m \leq \alpha < C_1^u \\ (g_1 - g_{0)}C_1^u + (g_2 - g_1)C_1^m + (g_3 - g_2)C_1^d & \text{if } C_1^u \leq \alpha \end{cases}$$

where

$$h_i = h\left(1 - \sum_{k=1}^{i} p_k\right) \text{ and } g_i = g\left(\sum_{k=1}^{i} p_k\right)$$

We then solve four linear programs (L_i) for $i = 1, .., 4$.

$$(L_i) = \begin{cases} \max\limits_{C=(C_1^u, C_1^m, C_1^d)} & v - \frac{1}{1+r}\mathbb{E}_Q[C_1] + \beta.H(C_1) \\ \textit{subject to:} & \mathbb{E}_Q[C_1] \leq v(1+r) \\ & 0 \leq C_1^i \leq \frac{v(1+r)}{q_i} \quad i = d, m, u \\ & C_1^d \leq C_1^m \leq C_1^u \\ & \alpha \in [C_1^i, C_1^{i+1}) \end{cases} \tag{22}$$

The optimal consumption is the one that corresponds to the maximum of the four optimal objectives resulting from solving the four linear programs defined above.

3.4 Wang's class of distortion operators

Now consider another class of distortion operators introduced by Wang (2000 [48]). Wang shows that applying this distortion operator to a stock price distribution, the risk neutral valuation of stock prices can be recovered in the normal and the lognormal cases. Further investigations, however, should be carried out to check whether this statement is true for any contingent claim, and also when there is no normality assumption on the underlying asset prices. Hamada & Sherris (2001 [20]) provide some insight into this question.

3.4.1 The operator

Let X be a random variable with a decumulative distribution function $S_X(x) = P[X > x]$. The expectation of X is alternatively given by:

$$\mathbb{E}[X] = \int_{-\infty}^{0} [S_X(x) - 1]dx + \int_{0}^{\infty} S_X(x)dx$$

Let $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ be the standard normal cumulative function and $\alpha \in \mathbb{R}$, the distortion operator is defined as:

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha]$$

for u in $[0,1]$. The risk-adjusted premium of X, as defined by Wang (2000) admits the following Choquet representation

$$H[X,\alpha] = \int_{-\infty}^{0} \{g_\alpha[S_X(x)] - 1\}dx + \int_{0}^{\infty} g_\alpha[S_X(x)]dx$$

When X is positive, we have:

$$H[X,\alpha] = \int_{0}^{\infty} g_\alpha[S_X(t)]dt$$

The risk-adjusted premium is evaluated as in Yaari's dual theory of choice under uncertainty. The tail distribution $S_X(t)$ is distorted by the function $g_\alpha(p) = \Phi[\Phi^{-1}(p) + \alpha]$. This operator shifts the p^{th} quantile of X by a positive or negative value α and reevaluates the normal cumulative probability of the shifted quantile.

If $\alpha > 0$, then $g_\alpha(p) > p$, if $\alpha < 0$, then $g_\alpha(p) < p$. Since g_α is continuous and $g_\alpha(p) \in [0,1]$, then:

$$\begin{aligned} g_\alpha \text{ is convex if } \alpha &< 0 \\ g_\alpha \text{ is concave if } \alpha &> 0 \end{aligned}$$

The investor behaves pessimistically by shifting the quantiles to the left, thereby assigning high probabilities to low outcomes, and behaves optimistically by shifting the quantiles to the right thereby assigning high probabilities to high outcomes. Typically, an insurer has a lower α than a reinsurer when pricing the same risk.

3.4.2 The portfolio problem

In asset allocation, at each time period, the consumption C_t is a positive random variable. The investor seeks to maximise the discounted sum of the certainty equivalents of consumption through time, as described by the problem:

$$\begin{cases} \max\limits_{C} & \sum_{t=0}^{T} \beta^t \cdot H[C_t,\alpha] \\ \textit{subject to} & \sum_{t=0}^{T} B_t^{-1} \cdot \mathbb{E}_Q\left[C_t\right] = v \\ & C \text{ is an adapted process} \end{cases} \tag{23}$$

where

$$H[C_t,\alpha] = \int_0^\infty g_\alpha\left(P[C_t > x]\right)dx$$

If the model consists of a finite number of states at each time period, then C_t takes n_t possible values $c_{t,1}, c_{t,2}, ..., c_{t,n_t}$ with respective probabilities $p_{t,1}, p_{t,2}, ..., p_{t,n_t}$, where n_t is the number of states at time t. Using Corollary (Certainty equivalent),

$$H[C_t,\alpha] = \sum_{i=1}^{n_t} \frac{\left[g_\alpha\left(P[C_t \geq c_{t,i}]\right) - g\left(P[C_t > c_{t,i}]\right)\right]}{\#\{c_{t,j} : c_{t,j} = c_{t,i}\}} c_{t,i}$$

The probability that $C_t \geq c_{t,i}$ is equal to the sum of the probability weights p_k, such that $c_{t,k} \geq c_{t,i}$, i.e.

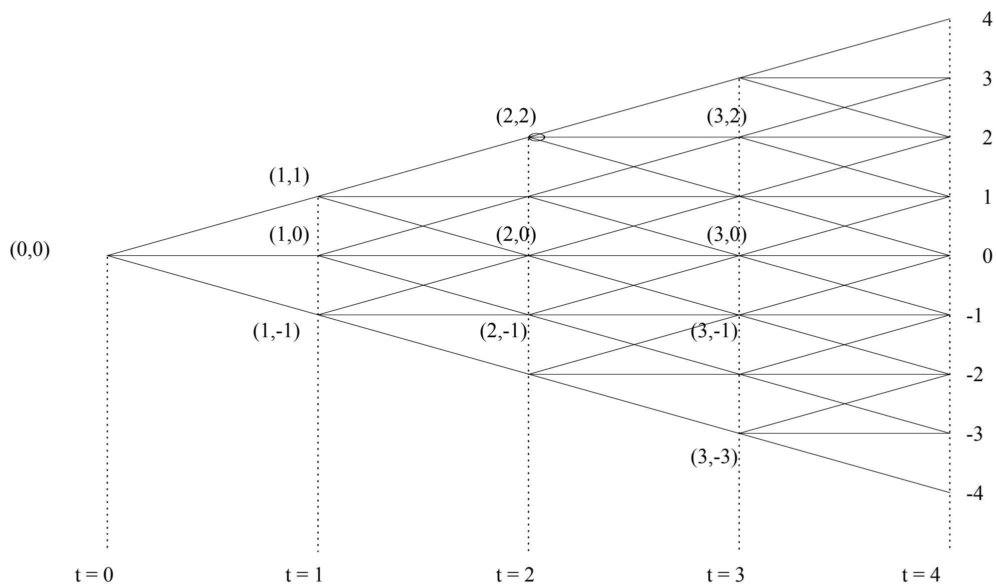
$$P[C_t \geq c_{t,i}] = \sum_{\{k: c_{t,k} \geq c_{t,i}\}} p_{t,k}$$

Hence,

$$H[C_t, \alpha] = \sum_{i=1}^{n_t} \frac{\left[g_\alpha\left(\sum_{\{k:c_{t,k}\geq c_{t,i}\}} p_{t,k}\right) - g\left(\sum_{\{k:c_{t,k}>c_{t,i}\}} p_{t,k}\right)\right]}{\#\{c_{t,j} : c_{t,j} = c_{t,i}\}} c_{t,i}$$

By defining the risk-neutral probability and using the expression above, the description of the problem is complete. This is not a linear program, however it can be solved using an optimisation package. The next paragraph shows how to solve it on a trinomial lattice and provides some results in two periods.

3.4.3 How to compute $H[C, \alpha]$ over a lattice?



Fix a time t, and consider the distribution of the consumption represented by the vertical nodes $(t, i)_{-t \leq i \leq t}$. At time t, the consumption C_t takes $2t+1$ possible values $c_{t,i}$ with probabilities $p_{t,i} = P[S = S_{t,i}]$. Hence,

$$H[C_t, \alpha] = \sum_{i=-t}^{t} \frac{\left[g_\alpha\left(\sum_{\{k:c_{t,k}\geq c_{t,i}\}} p_{t,k}\right) - g\left(\sum_{\{k:c_{t,k}>c_{t,i}\}} p_{t,k}\right)\right]}{\#\{c_{t,j} : c_{t,j} = c_{t,i}\}} c_{t,i}$$

The probabilities $p_{t,i}$, $t \in \{1, ..., T\}$, $i \in \{-t, ..., 0, ..., t\}$ can be specified as follows:

$$\begin{aligned} p_{t,i} &= P\left[S = S_{t,i}\right] \\ &= \sum_{\max(0,i) \leq k \leq \left[\frac{t+i}{2}\right]} \binom{t}{k} \binom{t-k}{k-i} \cdot p_u^k \cdot p_d^{k-i} \cdot (1 - p_u - p_d)^{t-2k+i} \end{aligned} \tag{24}$$

where p_u and p_d are respectively the probabilities of up and down jumps and $[x]$ is the integer part of x. The expression (24) is a generalisation of the probabilities in a binomial model. Likewise, the expression $\mathbb{E}_Q[C_t]$ in the constraints can be computed. In effect,

$$\mathbb{E}_Q[C_t] = \sum_{i=-t}^{t} Q_{t,i} \cdot c_{t,i}$$

where:

$$\begin{aligned} Q_{t,i} &= Q[S = S_{t,i}] \\ &= \sum_{\max(0,i) \leq k \leq \left[\frac{t+i}{2}\right]} \binom{t}{k}\binom{t-k}{k-i} \cdot q_u^k \cdot q_d^{k-i} \cdot (1 - q_u - q_d)^{t-2k+i} \end{aligned}$$

where q_u and q_d are respectively the risk-neutral probabilities of up and down jumps. For a detailed discussion of the implementation, see Hamada (2001 [19])

3.4.4 Numerical examples

This paragraph provides two numerical examples of portfolio allocation using the classes of distortion operators introduced earlier.

The first example considers Wang's distortion operator. Suppose that there are three dates $t = 0, 1, 2$ and five states of the world. This corresponds to a recombining trinomial lattice. We numerically solve the problem (23) for $T = 2$. For a loading parameter $\alpha = 0.5$, Figure (4) shows the consumption and investment strategies as well as the wealth process for a two period example. The discount factor $\beta = 0.9$, the risk-free interest rate $r = 10\%$ and initial wealth $v = \$10$.

The jump probabilities are $p_u = p_m = p_d = \frac{1}{3}$ and $q_u = \frac{2}{15}$, $q_m = \frac{1}{3}$ and $q_d = \frac{8}{15}$. For this choice of risk neutral probabilities, and a correlation coefficient $\rho = 0.75$, the means and volatilities of the two risky securities are respectively : $mean_1 = 13.46\%$, $volatility_1 = 07.07\%$, and $mean_2 = 15.20\%$, $volatility_2 = 10.61\%$.

The intermediate consumption C is null and the consumption is strictly positive at the highest state of the world. However, intermediate positions nr, $r1$ and $r2$ in the riskless asset, the first security and the second security respectively are nonzero.

To see the impact of the loading parameter α on the consumption stream, Figure (5) plots the optimal consumption for different values of α.

Around the value $\alpha = -0.8$, there is a switch in consumption from the lowest state, where it is nonzero and null elsewhere, to the highest state.

A closer look at the consumption process around $\alpha = -0.8$, is represented in Figure (6). This figure shows that in the transitory passage across the level $\alpha = 0.8$, the intermediate consumption becomes nonzero.

From the examples above, it is clear that the linearity of the dual utility in consumption results in a corner solution in the optimisation problem. This is not a desirable feature in portfolio selection, although, as shown in the example, with 3 assets, diversification is possible. On the other hand, within the expected utility framework, a risk averse investor is always diversifying provided that the expected return of the risky asset is positive.

The second example is a numerical solution to the problem (22) where the risk measure is the one introduced by Van der Hoek and Sherris. Figure (7) shows the optimal consumption and trading strategies for the parameters values indicated in the figure.

This numerical example shows that consumption at the end of the investment period is positive in all the states. This is due to the asymmetry resulting from pricing the downside of the risk

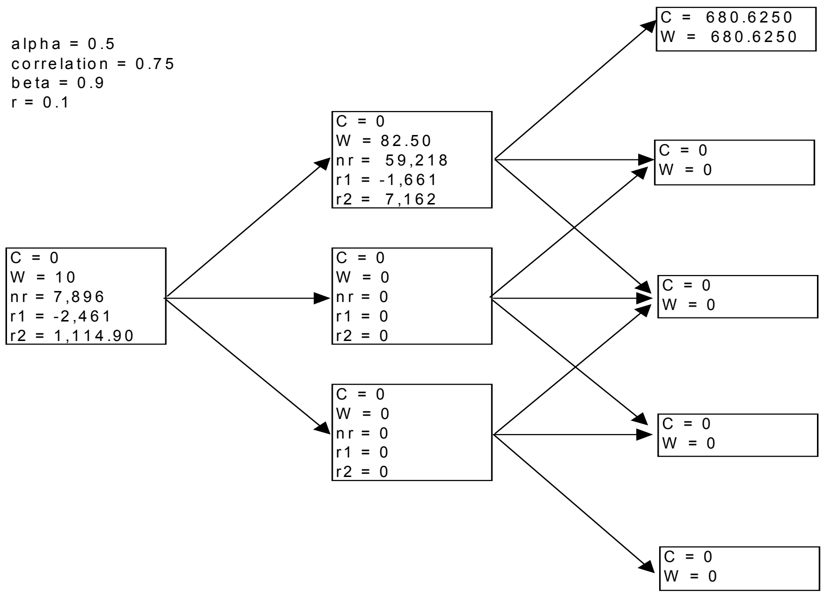


Figure 4: Consumption and investment strategies using Wang's distortion operator

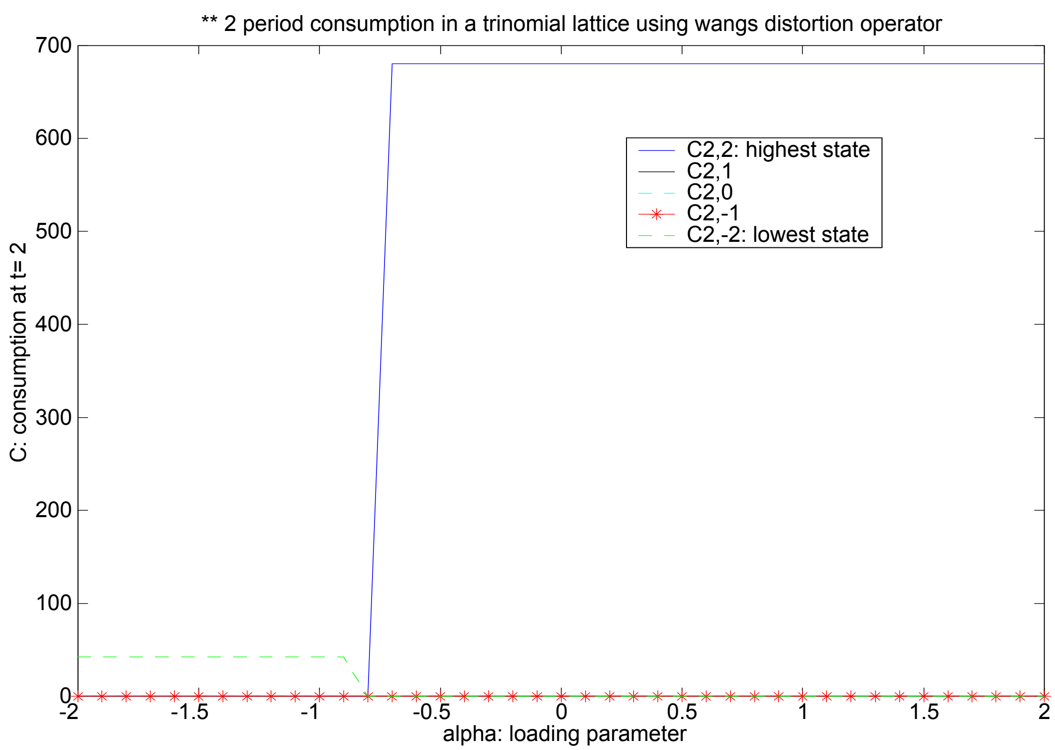


Figure 5: Consumption process as function of the loading parameter α

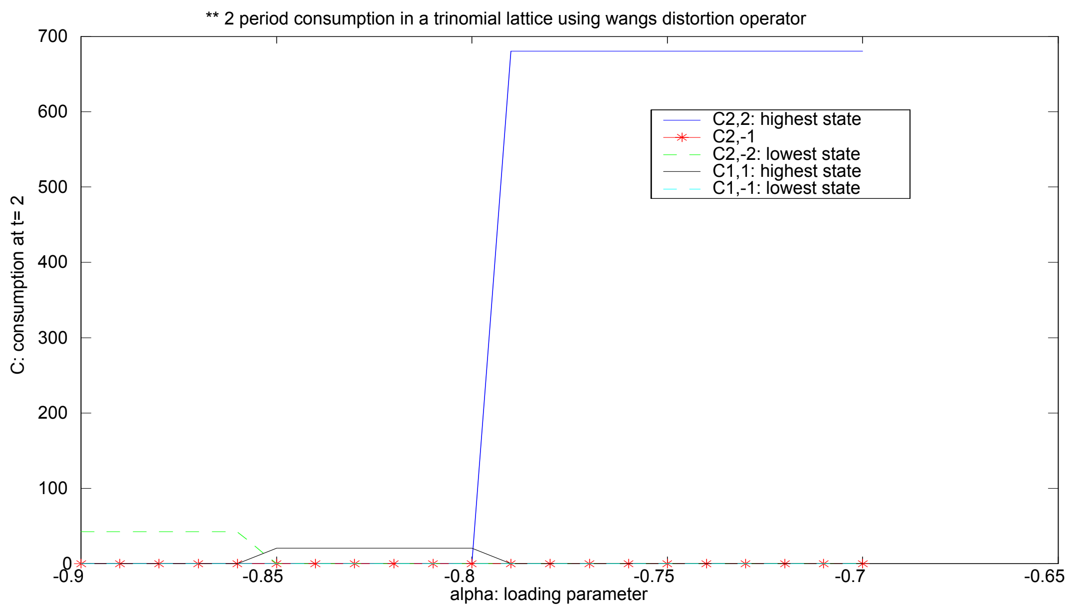


Figure 6: Consumption process as function of the loading parameter α (a zoom into -0.8)

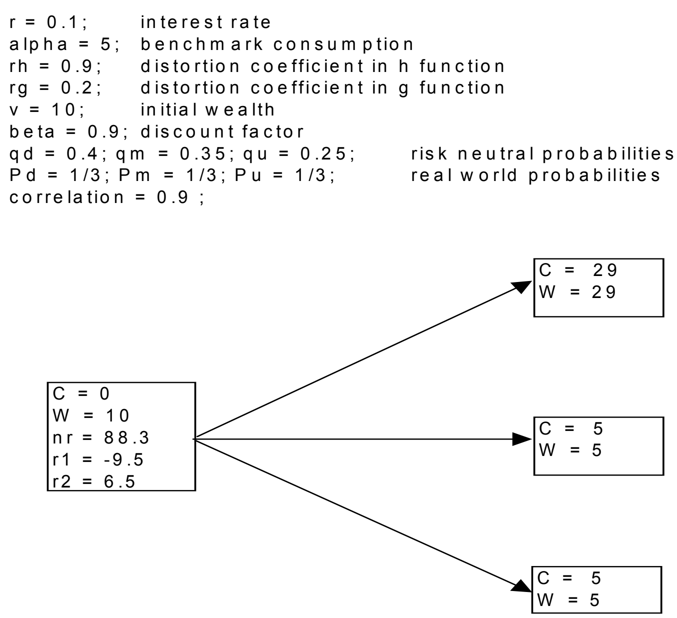


Figure 7: Optimal consumption and trading strategies using PH risk measure

using the distortion function $g(x) = x^{0.2}$ and the upside of the risk using $h(x) = 1 - (1 - x)^{0.9}$. It is worth noting that the consumption in the middle and the down state equals the benchmark consumption $\alpha = 5$. This is consequence of the linearity in Problem (22) where $C = \alpha$ is a corner solution.

4 Conclusion

In this paper we have provided a formal treatment of risk measures based on distortion functions in discrete-time setting. We have also shown that the risk neutral computational approach is well adapted to portfolio optimisation with such measures that don't lie within the expected utility framework.

The application to two different distortion operators shows that the portfolio consumption and investment rules are different from the expected utility results since the optimisation leads to corner solutions resulting from the linearity of the objective in the control variables. This is an undesirable feature and an important area that needs to be addressed before these non-expected utility risk measures can be confidently applied to asset allocation. This is an area for future research. One possibility is to consider combining expected and non-expected utility measures as in Quiggin (1982 [40]).

References

[1] M. Allais. Le comportement de l'homme rationnel devant le risque. *Economertica 21*, pages 503–546, 1953.

[2] A. Ang, G. Bekaert, and J. Liu. Why stocks may disappoint. 2000.

[3] N. Bingham and R. Kiesel. *Risk Neutral Valuation : Pricing and Hedging of Financial Derivatives.* London, New York : Springer, 1998.

[4] D. T. Breeden. An intertemporal capital asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics 7*, pages 265–296, 1979.

[5] M. J. Brennan, E. S. Schwartz, and R. Lagnado. Strategic asset allocation. *Journal of Economic Dynamics and Control, 21*, pages 79–88, June 1997.

[6] G. Bufman and L. Leiderman. Consumption and asset returns under non-expected utility, some new evidence. *Economics letters 34*, pages 231–235, 1990.

[7] C. F. Camerer. An experimental test of several generalized utility theories. *Journal of Risk and Uncertainty, 2, no. 1*, pages 61–104, April 1989.

[8] J. Y. Campbell and L. M. Viceira. Consumption and portfolio decisions when expected returns are time varying. *Quarterly Journal of Economics, vol. 114*, pages 433–95, May, 1999.

[9] G. Chamberlain and C. A. Wilson. Optimal intertemporal consumption under uncertainty. *Review of Economic Dynamics, 3*, pages 365–395, 2000.

[10] J. C. Cox and C.-F. Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *J. Econom. Theory 49*, pages 33–83, 1989.

[11] J. C. Cox and C.-F. Huang. A variational problem arising in financial economics. *Journal of Mathematical Economics, vol. 20, no. 5*, pages 465–87, 1991.

[12] J. C. Cox, J. E. Ingersoll, and S. A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica 53*, pages 363–384, 1985.

[13] D. Duffie. *Dynamic Asset Pricing Theory.* Princeton University Press, 1992.

[14] L. G. Epstein and S. E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: An empirical analysis. *Carnegie-Mellon University, Pittersburg, PA*, 1989.

[15] L. G. Epstein and S. E. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica, vol. 57, no. 4*, pages 937–69, July 1989.

[16] P. C. Fishburn. Nonlinear preference and utility theory. *Johns Hopkins Series in the Mathematical Sciences, no 5. Baltimore and London*, pages xiv, 259, 1988.

[17] A. Giovannini and P. Jorion. Time-series tests of a non-expected utility model of asset pricing. *Columbia University, New York*, 1989.

[18] F. Gul. A theory of disappointment aversion. *Econometrica*, 1991.

[19] M. Hamada. Implementation of dynamic portfolio optimisation using the risk neutral computational approach. *Working paper*, 2001.

[20] M. Hamada and M. Sherris. On the relationship between risk neutral valuation and pricing using distortion operators. *Working paper*, 2001.

[21] J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications 11*, pages 215–260, 1981.

[22] M. J. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory, vol 2*, pages 381–408, June 1979.

[23] H. He. Convergence from discrete to continuous-time contingent claims prices. *The Review of Financial Studies, vol. 3, no. 4*, pages 523–546, 1990.

[24] H. He and N. D. Pearson. Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case. *Journal of Economic Theory 54, no. 2*, pages 259–304, August 1991.

[25] J. E. Ingersoll. *Theory of Financial Decision Making*. Totowa, N.J. : Rowman and Littlefield, 1987.

[26] D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica 47*, pages 263–291, 1979.

[27] I. Karatzas, J. P. Lehoczky, S. P. Sethi, and S. E. Shreve. Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research 11*, pages 261–294, 1986.

[28] I. Karatzas, J. P. Lehoczky, and S. E. Shreve. Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. *SIAM Journal on Control and Optimization 25*, pages 1557–1586, 1987.

[29] I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance*. New York, London : Springer, 1998.

[30] J.-M. Koskievic. An intertemporal consumption-leisure model with non-expected utility. *Economics Letters 64*, pages 285–289, 1999.

[31] D. M. Kreps and E. L. Porteus. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica, vol. 46, no. 1*, pages 185–200, Jan. 1978.

[32] R. Lucas. Asset prices in an exchange economy. *Econometrica 46*, pages 1426–1446, 1978.

[33] M. J. Machina. "expected utility" analysis without the independence axiom. *Econometrica, vol. 50*, pages 277–323, March 1982.

[34] R. C. Merton. *Continous-Time Finance*. Oxford, U.K.: Basil Blackwell, rev. ed. edition, 1992.

[35] R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, December 1971.

[36] W. S. Neilson. An expected utility-user's guide to nonexpected utility experiments. *Eastern Economic Journal 19, no. 3*, pages 257–74, Summer 1993.

[37] M. Osborne. *Simplicial Algorithms for Minimizing Polyhedral Functions*. Cambridge University Press, 2001.

[38] S. R. Pliska. A stochastic calculus model of continuous trading: Optimal portfolios. *Mathematics of Operations Research 11*, pages 370–382, 1986.

[39] S. R. Pliska. *Introduction to Mathematical Finance: Discrete Time Models.* Blackwell Publishers, 1997.

[40] J. Quiggin. A theory of anticipated utility. *Journal of Economic Behavior and Organization*, 3:323–343, 1982.

[41] A. Roel. Risk aversion in quiggin and yaari's rank-order model of choice under uncertainty. *The Economic Journal*, 1985.

[42] S. A. Ross. Some stronger measures of risk aversion in the small and in the large with aplications. *Econmetrica*, 49:621–638, 1981.

[43] M. S. Taqqu and W. Willinger. The analysis of finite security markes using martingales. *Adv. Applied Probability 19*, pages 1–25, 1987.

[44] J. Van der Hoek and M. Sherris. A class of non-expected utility risk measures and implications for asset allocations. *Insurance: Mathematics and Economics*, 28:69–82, 2001.

[45] S. Wang. Ambiguity aversion and the economics of insurance. *University of Waterloo, research report 96-04*, 1996.

[46] S. Wang. Premium calculation by transforming the layer premium density. *Astin Bulletin 26*, pages 71–92, 1996.

[47] S. Wang. Implementation of proportional hazards transforms in ratemaking. *Proceedings of the Casualty Actuarial Society*, LXXXV, 1998. Available for download on http://www.casact.org/pubs/proceed/proceed98/index.htm.

[48] S. S. Wang. A class of distortion operators for pricing financial and insurance risks. *The Journal of Risk and Insurance, vol. 36, no. 1*, pages 15–36, 2000.

[49] M. E. Yaari. *Univariate and Multivariate Comparison of Risk Aversion: A New Approach.* Cambridge University Press, 1986.

[50] M. E. Yaari. The dual theory of choice under risk. *Econometrica, vol. 55*, pages 95–115, January 1987.

A Proof of Proposition (Order assumption)

Proof. For $x_1 < x_2 < ... < x_n$ we have:
$(x_1-\alpha)^+ \le (x_2-\alpha)^+ \le ... \le (x_n-\alpha)^+$ and $(\alpha-x_1)^+ \ge (\alpha-x_2)^+ \ge ... \ge (\alpha-x_n)^+$, so

$$\Pr\left[(X-\alpha)^+ > t\right] = \begin{cases} 1 & if\ (x_1-\alpha)^+ > t \\ \sum_{k=2}^n p_k & if\ (x_2-\alpha)^+ > t \ge (x_1-\alpha)^+ \\ \sum_{k=3}^n p_k & if\ (x_3-\alpha)^+ > t \ge (x_2-\alpha)^+ \\ .... & ..... \\ p_n & if\ (x_n-\alpha)^+ > t \ge (x_{n-1}-\alpha)^+ \\ 0 & if\ t \ge (x_n-\alpha)^+ \end{cases}$$

and

$$\Pr\left[(\alpha-X)^+ > t\right] = \begin{cases} 1 & if\ (\alpha-x_n)^+ > t \\ \sum_{k=1}^{n-1} p_k & if\ (\alpha-x_{n-1})^+ > t \ge (\alpha-x_n)^+ \\ \sum_{k=1}^{n-2} p_k & if\ (\alpha-x_{n-2})^+ > t \ge (\alpha-x_{n-1})^+ \\ .... & ..... \\ p_1 & if\ (\alpha-x_1)^+ > t \ge (\alpha-x_2)^+ \\ 0 & if\ t \ge (\alpha-x_1)^+ \end{cases}$$

Therefore,

$$H(X) = \alpha + \int_0^{(x_1-\alpha)^+} h(1)dt + \int_{(x_1-\alpha)^+}^{(x_2-\alpha)^+} h(\sum_{k=2}^n p_k)dt + \int_{(x_2-\alpha)^+}^{(x_3-\alpha)^+} h(\sum_{k=3}^n p_k)dt + ... + \int_{(x_{n-1}-\alpha)^+}^{(x_n-\alpha)^+} h(p_n)dt -$$
$$\int_0^{(x_n-\alpha)^+} g(1)dt - \int_{(\alpha-x_n)^+}^{(\alpha-x_{n-1})^+} g(\sum_{k=1}^{n-1} p_k)dt - \int_{(\alpha-x_{n-1})^+}^{(\alpha-x_{n-2})^+} g(\sum_{k=1}^{n-2} p_k)dt - ... - \int_{(\alpha-x_2)^+}^{(\alpha-x_1)^+} g(p_1)dt$$

By using

$$\int_{(x_i-\alpha)^+}^{(x_{i+1}-\alpha)^+} h(\sum_{k=i+1}^n p_k)dt = h\left(1-\sum_{k=1}^i p_k\right)\left[(x_{i+1}-\alpha)^+ - (x_i-\alpha)^+\right]$$

and

$$\int_{(\alpha-x_{i+1})^+}^{(\alpha-x_i)^+} g(\sum_{k=1}^i p_k)dt = -g\left(\sum_{k=1}^i p_k\right)\left[(\alpha-x_{i+1})^+ - (\alpha-x_i)^+\right]$$

we get the desired result. ■

B The case when g and h are conjugate

We propose to show that

$$H(X) = H_{0,0,h}(X) = \int_0^\infty h(P[X>t])dt$$

First, it is easy to check that, for $t \ge 0$,

$$\begin{aligned} S_{(X-\alpha)^+}(t) &= S_X(\alpha+t) \\ S_{(\alpha-X)^+}(t) &= 1 - S_X(\alpha-t) \end{aligned}$$

and since h and g are conjugate, we have:

$$h'(x) = g'(1-x)$$

$$\begin{aligned}
H(X) &= \alpha + \int_0^\infty h\left(S_{(X-\alpha)^+}(t)\right).dt - \int_0^\infty g\left(S_{(\alpha-X)^+}(t)\right).dt \\
&= \alpha + \int_0^\infty h\left(S_X(\alpha+t)\right).dt + \int_0^\infty g\left(1-S_X(\alpha-t)\right).dt \\
&= \alpha - \int_0^\infty th'\left(S_X(\alpha+t)\right).dS_X(\alpha+t) - \int_0^\infty tg'\left(1-S_X(\alpha-t)\right).dS_X(\alpha-t) \\
&= \alpha - \int_\alpha^\infty (u-\alpha)h'\left(S_X(u)\right).dS_X(u) - \int_\alpha^{-\infty} (\alpha-v)g'\left(1-S_X(v)\right).dS_X(v)
\end{aligned}$$

$$\begin{aligned}
&= \alpha + \alpha\int_\alpha^\infty h'\left(S_X(u)\right).dS_X(u) - \int_\alpha^\infty uh'\left(S_X(u)\right).dS_X(u) \\
&\quad -\alpha\int_\alpha^{-\infty} h'\left(S_X(v)\right).dS_X(v) + \int_\alpha^{-\infty} vh'\left(S_X(v)\right).dS_X(v) \\
&= \alpha + \alpha\int_{-\infty}^\infty h'\left(S_X(u)\right).dS_X(u) - \int_{-\infty}^\infty vh'\left(S_X(v)\right).dS_X(v) \\
&= \alpha - \alpha\underbrace{\int_{-\infty}^\infty h'\left(S_X(u)\right).dF_X(u)}_{=1} - \int_{-\infty}^\infty vh'\left(S_X(v)\right).dS_X(v) \\
&= \int_{-\infty}^\infty vh'\left(S_X(v)\right).dF_X(v) \\
&= \int_0^\infty h\left(S_X(v)\right).dv
\end{aligned}$$