Equilibrium Insurance Pricing, Market Value of Liabilities and Optimal Capitalization*

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June 5, 2003

Abstract

In this paper we review an insurance pricing model for a simple one-period economy with shares in productive firms, insurance firms, real assets and insurance contracts as developed in Turner [11] and extended in Taylor [13]. We consider the valuation of an insurance firm's liabilities in the model and the optimal capitalization of the insurance firm. The relationship of the results to standard financial theory are identified. The model does not allow us to say if the aggregate insurance market capital structure is optimal. Taylor's results will still hold for any given capital structure and the model can be calibrated to an aggregate insurance market capital structure. Implications of the model for the allocation of capital to lines of business is considered. Frictional costs such as transaction costs, taxes, bankruptcy costs or agency costs can be introduced into the model in order to derive an optimal capital structure.

Keywords: insurer liablities, fair value, equilibrium, capital structure

*Research supported by Australian Research Council Discovery Grant DP0345036 and financial support from the UNSW Actuarial Foundation of The Institute of Actuaries of Australia. This paper is to be presented to the 13th International AFIR Colloquium, Maastricht, Netherlands, September 2003.

1 Introduction

Taylor [13] extends an equilibrium model for insurance pricing, originally developed by Turner [11], to allow for insurer capitalization. We review the results and derive equilibrium insurance pricing formulae and make explicit the derivation of the price of risk and the discount factor. We consider the implications of the model for the market valuation of an insurer's liabilities and the optimal capitalization of insurers.

The model aims to include insurance firms in the equilibrium determination of prices and to also allow for heterogeneity and dependence in insured risks.

When considered more closely, the model does not allow the determination of a unique optimal capital structure for insurers. Frictional costs can be introduced into the model in order for it to be rich enough to include an optimal capital structure and to assess the resulting implications for fair pricing.

We consider fair insurance prices in the model allowing for limited liability, to reflect the option feature in the resulting insurance contract payoffs and the value of the equity in the insurer.

We discuss extensions of the model to address research questions of interest to both practitioners and academics.

2 The Model

The model is a single period model with individuals, productive firms and insurance firms. Individuals hold real assets, consume, invest in productive firms, invest in insurance firms, and purchase insurance over losses on their real assets. Productive firms only issue shares and are 100% equity financed. Insurance firms issue insurance policies on real assets, issue shares and purchase shares in productive firms.

2.1 Assumptions

The model makes the usual assumptions of perfect competition, no transaction costs or taxes, no restrictions on short sales, homogeneous expectations across all agents. All households are assumed to be risk averse, one period utility maximizers in terms of means and variances of end-of-period wealth.

Individual h is assumed to maximize a utility function

$$U_h\left(\mu_h, v_h, C_h\right)$$

where μ_h is expected terminal wealth of individual h, v_h is variance of terminal wealth of individual h, and C_h is current consumption of individual h. Individual h has current wealth W_h and has to decide how much to consume now, C_h, how many shares to purchase in productive firm i, denoted by v_{hi}, how many units of real asset k to own, given by a_{hk}, how many shares to purchase in insurer f, given by s_{hf}, and how many units of insurance on real asset k to be purchased from insurer f, given by n_{hfk}.

The budget constraint is then

$$W_h = \sum_i v_{hi} V_i + \sum_k a_{hk} A_k + \sum_f s_{hf} S_f + \sum_{fk} n_{hfk} P_{fk} + C_h$$

where V_i is the price of one share in productive firm i, A_k is the price of a unit of real asset k, S_f is the price of one share in insurer f, P_{fk} is the premium per unit of insurance for asset k charged by insurer f. The decision variables for household h are v_{hi}, a_{hk}, s_{hf}, n_{hfk}, and C_h.

Remark 1 *In the model, individuals can pool real assets and share risks costlessly. Under the mean-variance preference and other perfect markets assumptions, it is expected that classical financial theory results such as mutual fund theorems will hold. Details can be found in Jarrow [5]. Individuals will share risks optimally through diversification, and CAPM pricing will hold in the model. Borch [1] appears to be the first to apply financial economic models to insurance markets. Kihlstrom and Pauly [6] consider such an insurance equilibrium and conclude that "it is easy to show that the concept of an insurance equilibrium is equivalent to that of a contingent claims competitive equilibrium." The optimal holding of shares and insurance will take the form of a share in a mutual fund holding all the real assets and shares in the economy and a mutual insurer providing all the insurance contracts. There will be no additional insight gained from having more than one productive firm or more than one insurance firm in the model.*

Remark 2 *Productive firms in the model are 100% equity financed since they only issue shares and not debt. Insurance firms, in contrast, are financed by debt (policyholder funds) and equity. Insurance firms purchase only shares in productive firms and neither insurance firms nor productive firms own real assets.*

Remark 3 *The notation used in the Taylor [13] paper, which follows that in the original Turner [11] paper, is often difficult to follow. For instance, v_h is used for the variance of wealth and v_{hi} for shares in productive firms. We also see R_i and R_k used in a confusing manner - one is the end-of-period value of a particular share and the other the end-of-period value of a particular real asset assuming no insurable losses occur.*

Remark 4 *The model does not explicitly allow for inflation. The unit used is effectively money, not the consumption good, but the model results are consistent with the assumption of a fixed rate of inflation.*

Implicit in the model is an initial endowment to individual h of consumption goods $\widehat{C}_h$, shares in productive firm i denoted by $\widehat{v}_{hi}$, units of real asset k given by $\widehat{a}_{hk}$, shares in insurer f given by $\widehat{s}_{hf}$, and units of insurance on real asset k purchased from insurer f given by $\widehat{n}_{hfk}$ and of course with

$$W_h = \sum_i \widehat{v}_{hi} V_i + \sum_k \widehat{a}_{hk} A_k + \sum_f \widehat{s}_{hf} S_f + \sum_{fk} \widehat{n}_{hfk} P_{fk} + \widehat{C}_h$$

The terminal wealth of household h will be given by

$$T_h = \sum_i v_{hi} R_i + \sum_k a_{hk} \left(R_k - X_{hk}\right) + \sum_f s_{hf} Y_f + \sum_{fk} n_{hfk} X_{hk}$$

where R_i is the end-of-period value of share i, R_k is the value of real asset k, Y_f is the end-of-period value of a share in insurer f (assumed to ignore limited liability), X_{hk} are the losses for individual h on real asset k.

The model assumes initially that the loss to asset k, denoted by X_{hk}, is dependent on hazards of individual h. However, Taylor [13] assumes that X_{hk} and X_{gl} are stochastically independent if $k \neq l$. The implication of the above terminal wealth equation implies that individual h owns a_{hk} units of real asset k with end-of-period value of the real asset, after insurable losses, of $R_k - X_{hk}$. The end-of-period value of real asset k has a payoff that depends on who owns the asset. This assumption introduces heterogeneity across individuals in the model according to their loss experience for each real asset and, since the end of period payoff is assumed to depend on individual h, implies that each real asset will have a different value according to who owns the asset.

The same comment applies to the insurance contracts. Each individual will have a different risk and hence a different equilibrium premium. Heterogeneity of risks and the implications for asset prices and fair insurance prices is an important issue for further research. This is not addressed in this paper.

Kihlstrom and Pauly [6] develop a theory of insurance without information costs in a state contingent model. Such a model will be more appropriate to consider issues of heterogeneity of risks than is the mean-variance model considered here. Later we will assume that the payoff on real asset k is $R_k - X_k$ regardless of who owns the asset. Since the main aim of this paper is to consider the implications of the model for equilibrium capitalization, this assumption will not be critical to the conclusions.

2.2 Terminal wealth

The expected terminal wealth will be

$$\mu_h = E\left[T_h\right] = \sum_i v_{hi} \overline{R}_i + \sum_k a_{hk} \left(\overline{R}_k - \overline{X}_{hk}\right) + \sum_f s_{hf} \overline{Y}_f + \sum_{fk} n_{hfk} \overline{X}_{hk}$$

where an overbar indicates expected value.

The variance of terminal wealth is (see Taylor [13] Appendix A for derivations in the case where limited liability is assumed for the insurance firms)

$$\begin{aligned} v_h &= \sum_i \sum_j v_{hi} v_{hj} cov\left(R_i, R_j\right) + \sum_k \sum_j a_{hk} a_{hj} cov\left(\left(R_k - X_{hk}\right), \left(R_j - X_{hj}\right)\right) \\ &\quad + \sum_f \sum_g s_{hf} s_{hg} cov\left(Y_f, Y_g\right) + \sum_{fk} \sum_{gi} n_{hfk} n_{hgi} cov\left(X_{hk}, X_{gi}\right) \\ &\quad +2 \sum_i \sum_k v_{hi} a_{hk} cov\left(R_i, \left(R_k - X_{hk}\right)\right) + 2 \sum_i \sum_f v_{hi} s_{hf} cov\left(R_i, Y_f\right) \\ &\quad +2 \sum_i \sum_{fk} v_{hi} n_{hfk} cov\left(R_i, X_{hk}\right) + 2 \sum_k \sum_f a_{hk} s_{hf} cov\left(\left(R_k - X_{hk}\right), Y_f\right) \\ &\quad + \sum_k \sum_{fk} a_{hk} n_{hfk} cov\left(\left(R_k - X_{hk}\right), X_{hk}\right) + 2 \sum_f \sum_{fk} s_{hf} n_{hfk} cov\left(Y_f, X_{hk}\right) \end{aligned}$$

Note that economy wide end-of-period wealth is given by

$$\sum_h T_h = \sum_{hi} v_{hi} R_i + \sum_{hk} a_{hk}\left(R_k - X_{hk}\right) + \sum_{hf} s_{hf} Y_f + \sum_{hfk} n_{hfk} X_{hk}$$

Consider the balance sheet of the insurers. We will consider end-of-period values, but the same logic applies for the initial endowment (start of period values). For insurer f the assets will have end-of-period value,

$$\sum_i v_{fi} R_i$$

where v_{fi} are the number of shares of productive firm i owned by insurer f.

The liabilities will be the policy claims with end-of-period payoff

$$\sum_{hk} n_{hfk} X_{hk}$$

and the net end-of-period equity will be

$$\sum_h s_{hf} Y_f$$

Because of the accounting identity for the insurance firm's balance sheet we have

$$\sum_h s_{hf} Y_f + \sum_{hk} n_{hfk} X_{hk} = \sum_i v_{fi} R_i$$

If we now sum over all insurers we will have

$$\sum_{fh} s_{hf} Y_f + \sum_{hfk} n_{hfk} X_{hk} = \sum_{fi} v_{fi} R_i$$

We can therefore write the economy wide end-of-period wealth as

$$\sum_h T_h = \sum_{hi} v_{hi} R_i + \sum_{hk} a_{hk}\left(R_k - X_{hk}\right) + \sum_{fi} v_{fi} R_i$$

and we see that, in this model, insurance companies are just a means for individuals to hold shares in productive firms. Insurance contracts are in net zero supply in the economy. They are contingent claims similar to forwards, or, in the case of limited liability insurers, options in financial market models.

Remark 5 *The model, with zero transaction costs and frictions, does not provide a rationale for the existence of insurers. It is possible for individuals, in the model, to costlessly write insurance contracts between each other to achieve the same results as the insurance firms.*

Remark 6 *As noted earlier, the results of financial theory as developed under similar assumptions, tell us that all insurer's will have exactly the same balance sheet capital structure, and even assuming costly transactions, there need only be one insurer to pool risks.*

3 Utility Maximization

To begin with we will derive CAPM results for the model. We will then consider particular payoffs including limited liability of insurers.

The individual's utility optimization problem is to

$$\max_{v_{hi}, a_{hk}, s_{hf}, n_{hfk}, C_h} U_h\left(\mu_h, v_h, C_h\right)$$

subject to the budget constraint

$$\begin{aligned} W_h &= \sum_i \widehat{v}_{hi} V_i + \sum_k \widehat{a}_{hk} A_k + \sum_f \widehat{s}_{hf} S_f + \sum_{fk} \widehat{n}_{hfk} P_{fk} + \widehat{C}_h \\ &= \sum_i v_{hi} V_i + \sum_k a_{hk} A_k + \sum_f s_{hf} S_f + \sum_{fk} n_{hfk} P_{fk} + C_h \end{aligned}$$

Introducing a Lagrange multiplier for the constraint, the objective becomes

$$\max_{v_{hi}, a_{hk}, s_{hf}, n_{hfk}, C_h} \left[\begin{array}{c} U_h\left(\mu_h, v_h, C_h\right) \\ -\Psi_h \left(\begin{array}{c} \sum_i v_{hi} V_i + \sum_k a_{hk} A_k + \sum_f s_{hf} S_f \\ + \sum_{fk} n_{hfk} P_{fk} + C_h - W_h \end{array} \right) \end{array} \right]$$

Using the following notation,

$$U'_\mu = \frac{\partial U_h\left(\mu_h, v_h, C_h\right)}{\partial \mu_h}, U'_v = \frac{\partial U_h\left(\mu_h, v_h, C_h\right)}{\partial v_h}, U'_C = \frac{\partial U_h\left(\mu_h, v_h, C_h\right)}{\partial C_h}$$

the first order conditions for a maximum are:

$$\begin{aligned} \frac{\partial}{\partial v_{hi}} &= 0 \text{ (shares in productive firms)} \\ & U'_\mu \frac{\partial \mu_h}{\partial v_{hi}} + U'_v \frac{\partial v_h}{\partial v_{hi}} - \Psi_h V_i = 0 \qquad \text{for } i = 1, \ldots, I \end{aligned}$$

$$\frac{\partial}{\partial a_{hk}} = 0 \text{ (real assets)}$$

$$U'_\mu \frac{\partial \mu_h}{\partial a_{hk}} + U'_v \frac{\partial v_h}{\partial a_{hk}} - \Psi_h A_k = 0 \qquad \text{for } k = 1, \ldots, K$$

$$\frac{\partial}{\partial s_{hf}} = 0 \text{ (shares in insurers)}$$

$$U'_\mu \frac{\partial \mu_h}{\partial s_{hf}} + U'_v \frac{\partial v_h}{\partial s_{hf}} - \Psi_h S_f = 0 \qquad \text{for } f = 1, \ldots, F$$

$$\frac{\partial}{\partial n_{hfk}} = 0 \text{ (insurance policies on real asset } k \text{ with insurer } f\text{)}$$

$$U'_\mu \frac{\partial \mu_h}{\partial n_{hfk}} + U'_v \frac{\partial v_h}{\partial n_{hfk}} - \Psi_h P_{fk} = 0 \qquad \text{for } \begin{bmatrix} k = 1, \ldots, K; \\ f = 1, \ldots, F \end{bmatrix}$$

$$\frac{\partial}{\partial C_h} = 0 \text{ (optimal consumption)}$$

$$U'_\mu \frac{\partial \mu_h}{\partial C_h} + U'_v \frac{\partial v_h}{\partial C_h} + U'_C - \Psi_h = 0$$

$$\text{or } U'_C - \Psi_h = 0$$

$$\frac{\partial}{\partial \Psi_h} = 0 \text{ (budget constraint)}$$

$$\begin{array}{c} \sum_i v_{hi} V_i + \sum_k a_{hk} A_k + \sum_f s_{hf} S_f \\ + \sum_{fk} n_{hfk} P_{fk} + C_h - W_h \end{array} = 0$$

So we have $I + K + F + KF + 2$ equations to solve.

4 Market Valuation

Consider **shares in productive firms.** We have

$$\frac{\partial \mu_h}{\partial v_{hi}} = \overline{R}_i$$

and

$$\begin{aligned} \frac{\partial v_h}{\partial v_{hi}} &= 2 \sum_j v_{hj} cov\left(R_i, R_j\right) + 2 \sum_k a_{hk} cov\left(R_i, \left(R_k - X_{hk}\right)\right) \\ &+ 2 \sum_f s_{hf} cov\left(R_i, Y_f\right) + 2 \sum_{fk} n_{hfk} cov\left(R_i, X_{hk}\right) \end{aligned}$$

so the first order conditions become

$$\left[\begin{array}{c} U'_\mu \overline{R}_i + 2U'_v \left[\begin{array}{c} \sum_j v_{hj} cov\left(R_i, R_j\right) \\ +\sum_k a_{hk} cov\left(R_i, \left(R_k - X_{hk}\right)\right) \\ +\sum_f s_{hf} cov\left(R_i, Y_f\right) \\ +\sum_{fk} n_{hfk} cov\left(R_i, X_{hk}\right) \end{array}\right] \\ -\Psi_h V_i = 0 \end{array}\right] \qquad \text{for } i = 1, \ldots, I$$

Simplifying we obtain

$$\left[U'_\mu \overline{R}_i + 2U'_v cov\left(R_i, T_h\right) - \Psi_h V_i = 0\right] \qquad \text{for } i = 1, \ldots, I$$

We can do the same for the other first order conditions to derive the valuation results for each of the real assets, insurer equity and insurance policies:

Real assets

$$\left[U'_\mu \overline{R}_k + 2U'_v cov\left(R_k - X_{hk}, T_h\right) - \Psi_h A_k = 0\right] \qquad \text{for } k = 1, \ldots, K$$

and, as noted earlier, we will assume that $X_{hk} = X_k$ for all h to get

$$\left[U'_\mu \overline{R}_k + 2U'_v cov\left(R_k - X_k, T_h\right) - \Psi_h A_k = 0\right] \qquad \text{for } k = 1, \ldots, K$$

Thus regardless of who owns the real asset, the payoff is assumed to be the same and equal to $R_k - X_k$.

Shares in insurers

$$\left[U'_\mu \overline{Y_f} + 2U'_v cov\left(Y_f, T_h\right) - \Psi_h S_f = 0\right] \qquad for f = 1, \ldots, F$$

Insurance policies

$$\left[U'_\mu \overline{X}_{hk} + 2U'_v cov\left(X_{hk}, T_h\right) - \Psi_h P_{fk} = 0\right] \qquad \text{for } \left[\begin{array}{c} k = 1, \ldots, K; \\ f = 1, \ldots, F \end{array}\right]$$

Assume that there exists a risk free asset. For this asset, call it asset $i = 1$, we will have $\overline{R}_1 = 1 + r_f$, where r_f is the risk free rate of return, $V_1 = 1$ and $cov\left(R_1, R_i\right) = 0$ for $i = 1, \ldots, I$, $cov\left(R_1, R_k - X_{hk}\right) = 0$ for $k = 1, \ldots, K$, $cov\left(R_1, Y_f\right) = 0$ for $f = 1, \ldots, F$, and $cov\left(R_1, X_{hk}\right) = 0$ for $k = 1, \ldots, K$ from the definition of the risk free asset. Note that risk is covariance with an individual's optimal wealth in this model. The term risk free means that the asset value has no covariance with other assets in the economy. T_h is the end-of-period wealth of individual h and from the definition of risk free asset we have $cov\left(R_1, T_h\right) = 0$ for all h. If we denote the economy wide end-of-period wealth by $M = \sum_h T_h$ then we also have $cov\left(R_1, M\right) = 0$.

We can then solve for the Lagrange multiplier using

$$U'_\mu \overline{R}_1 + 2U'_v cov\left(R_1, T_h\right) - \Psi_h V_1 = 0$$

to get

$$\Psi_h = U'_\mu \left(1 + r_f\right)$$

Finally substituting for Ψ_h we obtain

$$\left[U'_\mu \overline{R}_i + 2U'_v cov\left(R_i, T_h\right) - U'_\mu \left(1 + r_f\right) V_i = 0\right] \qquad \text{for } i = 2, \ldots, I$$

Rearranging, we obtain each individual's pricing formula for asset i, at the optimal allocation in the economy, as

$$V_i = \frac{1}{\left(1 + r_f\right)} \left[\overline{R}_i + 2\frac{U'_v}{U'_\mu} cov\left(R_i, T_h\right)\right] \qquad \text{for } i = 2, \ldots, I$$

or

$$V_i = \frac{1}{\left(1 + r_f\right)} \left[\overline{R}_i + 2\lambda_h cov\left(R_i, T_h\right)\right] \qquad \text{for } i = 2, \ldots, I$$

where $\lambda_h = \frac{U'_v}{U'_\mu}$ is the marginal trade-off between variance (risk) and return for individual h. Similar results will apply for the real assets, shares in insurance firms and insurance policies.

4.1 Market Clearing

Competitive equilibrium requires that aggregate demand equal aggregate supply. If the asset markets are in equilibrium then so will be the consumption market (this is the Walras Law). So we need only consider asset market equilibrium. We can show this from the budget constraint, where $*$ indicates the equilibrium demand,

$$\sum_i \widehat{v}_{hi} V_i + \sum_k \widehat{a}_{hk} A_k + \sum_f \widehat{s}_{hf} S_f + \sum_{fk} \widehat{n}_{hfk} P_{fk} + \widehat{C}_h =$$
$$\sum_i v^*_{hi} V_i + \sum_k a^*_{hk} A_k + \sum_f s^*_{hf} S_f + \sum_{fk} n^*_{hfk} P_{fk} + C_h$$

so

$$\begin{aligned} & \widehat{C}_h - C_h \\ = & \left[\begin{array}{c} \sum_i \left(v^*_{hi} - \widehat{v}_{hi}\right) V_i + \sum_k \left(a^*_{hk} - \widehat{a}_{hk}\right) A_k \\ + \sum_f \left(s^*_{hf} - \widehat{s}_{hf}\right) S_f + \sum_{fk} \left(n^*_{hfk} - \widehat{n}_{hfk}\right) P_{fk} \end{array} \right] \end{aligned}$$

Summing over all individuals

$$
\begin{aligned}
& \sum_h \widehat{C}_h - \sum_h C_h \\
= & \left[\begin{array}{c} \sum_{hi} \left(v^*_{hi} - \widehat{v}_{hi}\right) V_i + \sum_{hk} \left(a^*_{hk} - \widehat{a}_{hk}\right) A_k \\ + \sum_{hf} \left(s^*_{hf} - \widehat{s}_{hf}\right) S_f + \sum_{hfk} \left(n^*_{hfk} - \widehat{n}_{hfk}\right) P_{fk} \end{array} \right] \\
= & \left[\begin{array}{c} \sum_i V_i \sum_h \left(v^*_{hi} - \widehat{v}_{hi}\right) + \sum_k A_k \sum_h \left(a^*_{hk} - \widehat{a}_{hk}\right) \\ + \sum_f S_f \sum_h \left(s^*_{hf} - \widehat{s}_{hf}\right) + \sum_{fk} P_{fk} \sum_h \left(n^*_{hfk} - \widehat{n}_{hfk}\right) \end{array} \right] \\
= & \sum_i V_i \sum_h \left(v^*_{hi} - \widehat{v}_{hi}\right) + \sum_k A_k \sum_h \left(a^*_{hk} - \widehat{a}_{hk}\right) + \sum_i V_i \sum_f \left(v^*_{fi} - \widehat{v}_{fi}\right) \\
= & \sum_i V_i \left[\sum_h \left(v^*_{hi} - \widehat{v}_{hi}\right) + \sum_f \left(v^*_{fi} - \widehat{v}_{fi}\right) \right] + \sum_k A_k \sum_h \left(a^*_{hk} - \widehat{a}_{hk}\right)
\end{aligned}
$$

When asset markets are in equilibrium, aggregate supply (the fixed endowment) for each of the productive firms shares, $\sum_{hi} \widehat{v}_{hi} + \sum_{fi} \widehat{v}_{fi}$, is equal to aggregate demand, $\sum_{hi} v^*_{hi} + \sum_{fi} v^*_{fi}$, and the aggregate supply of real assets, $\sum_h \widehat{a}_{hk}$, will equal aggregate demand, $\sum_h a^*_{hk}$.

This means that consumption good markets are also in equilibrium with aggregate supply (the fixed endowment of consumption goods) equal to aggregate demand, $\sum_h \widehat{C}_h = \sum_h C_h$.

We stress again that insurance contracts are in net zero supply in this model economy, as is the risk free asset.

4.2 Aggregation and Equilibrium Prices

To derive a pricing result in equilibrium, consider the first order conditions for shares in productive firms, assuming a risk free asset, rewritten in the form

$$
-\frac{1}{2\lambda_h} \left[\overline{R}_i - (1 + r_f) V_i\right] = cov\left(R_i, T_h\right) \qquad \text{for } i = 2, \ldots, I
$$

Now sum this over all individuals in the economy to get

$$
-\frac{1}{2} \sum_h \frac{1}{\lambda_h} \left[\overline{R}_i - (1 + r_f) V_i\right] = cov\left(R_i, \sum_h T_h\right) \qquad \text{for } i = 2, \ldots, I
$$

or

$$
\frac{1}{\lambda} \left[\overline{R}_i - (1 + r_f) V_i\right] = cov\left(R_i, M\right) \qquad \text{for } i = 2, \ldots, I
$$

where $\lambda = -2 \left[\sum_h \frac{1}{\lambda_h}\right]^{-1}$ and M is the end-of-period economy wide wealth.

We can do the same for real assets, shares in insurance firms and insurance policies to get:

Real assets

$$\frac{1}{\lambda}\left[\left(\overline{R}_k-\overline{X}_k\right)-(1+r_f)\,A_k\right]=cov\,(R_k-X_k,M)\qquad \text{for } k=1,\ldots,K$$

Shares in insurers

$$\frac{1}{\lambda}\left[\overline{Y}_f-(1+r_f)\,S_f\right]=cov\,(Y_f,M)\qquad \text{for } f=1,\ldots,F$$

Insurance policies

In the case where the payoff is assumed to be specific to individual h, so we have from the first order conditions

$$-\frac{1}{2\lambda_h}\left[\overline{X}_{hk}-(1+r_f)\,P_{fk}\right]=cov\,(X_{hk},T_h)\qquad \text{for } \begin{bmatrix} k=1,\ldots,K; \\ f=1,\ldots,F \end{bmatrix}$$

and summing over all individuals in the economy we get

$$\frac{1}{\lambda}\left[-\frac{\lambda}{2}\sum_h\frac{\overline{X}_{hk}}{\lambda_h}-[(1+r_f)\,P_{fk}]\right]=\sum_h cov\,(X_{hk},T_h)\qquad \text{for } \begin{bmatrix} k=1,\ldots,K; \\ f=1,\ldots,F \end{bmatrix}$$

or

$$\frac{1}{\lambda}\left[\overline{X}_k-[(1+r_f)\,P_{fk}]\right]=\sum_h cov\,(X_{hk},T_h)\qquad \text{for } \begin{bmatrix} k=1,\ldots,K; \\ f=1,\ldots,F \end{bmatrix}$$

where $X_k=-\frac{\lambda}{2}\sum_h\frac{X_{hk}}{\lambda_h}$. Additional assumptions are required for X_{hk} in order to simplify the right hand side into a form similar to that for the other assets in the economy.

If we assume that risks X_{hk} are independent of household h we can obtain the result

$$\frac{1}{\lambda}\left[\overline{X}_k-[(1+r_f)\,P_{fk}]\right]=cov\,(X_k,M)\qquad \text{for } \begin{bmatrix} k=1,\ldots,K; \\ f=1,\ldots,F \end{bmatrix}$$

Now to get our desired pricing equation in terms of market variables, we weight each of the previous equations by the total endowment (total supply) for each share, asset and insurance policy.

Thus, for each share in productive firm i we weight the equation for share in productive firm i by $\sum_h\widehat{v}_{hi}=\sum_h v_{hi}^*=v_i^*$ where v_{hi}^* is the optimal holding in equilibrium for individual h of shares in productive asset i. Similarly for each real asset we weight by $\sum_h\widehat{a}_{hk}=\sum_h a_{hk}^*=a_k^*$, for each share in insurer f we weight by $\sum_h\widehat{s}_{hf}=\sum_h s_{hf}^*=s_f^*$, and for each insurance policy $\sum_h\widehat{n}_{hfk}=\sum_h n_{hfk}^*=n_{fk}^*$.

We then sum to get

$$\frac{1}{\lambda}\begin{bmatrix} \sum_i v_i^*\left[\overline{R}_i-(1+r_f)\,V_i\right] \\ +\sum_k a_k^*\left[\left(\overline{R}_k-\overline{X}_k\right)-(1+r_f)\,A_k\right] \\ +\sum_f s_f^*\left[\overline{Y}_f-(1+r_f)\,S_f\right] \\ +\sum_{fk} n_{fk}^*\left[\overline{X}_k-[(1+r_f)\,P_{fk}]\right] \end{bmatrix}=\begin{bmatrix} \sum_i v_i^* cov\,(R_i,M) \\ +\sum_k a_k^* cov\,(R_k-X_k,M) \\ +\sum_f s_f^* cov\,(Y_f,M) \\ +\sum_{kf} n_{fk}^* cov\,(X_k,M) \end{bmatrix}$$

The risk free asset is assumed to be in zero net supply, so that for every amount lent at the risk free rate, there is an equal and opposite amount borrowed.

Now we know that the initial economy wide wealth less consumption, denoted by M_0, has value

$$M_0 = \sum_i \widehat{v}_i V_i + \sum_k \widehat{a}_k A_k + \sum_f \widehat{s}_f S_f + \sum_{fk} \widehat{n}_{fk} P_{fk}$$

and the end-of-period expected value of the economy wide wealth will be

$$\overline{M} = \sum_i v_i^* \overline{R}_i + \sum_k a_k^* \left(\overline{R}_k - \overline{X}_k\right) + \sum_f s_f^* \overline{Y}_f + \sum_{fk} n_{fk}^* \overline{X}_k$$

so that the above equation can be simplified to

$$\frac{1}{\lambda}\left[\overline{M} - (1 + r_f) M_0\right] = cov\,(M, M) = \sigma_M^2$$

where σ_M^2 is the variance of the end-of-period market wealth portfolio.

This allows us to solve for λ in terms of economy wide variables to get the market price of risk

$$\lambda = \frac{\left[\overline{M} - (1 + r_f) M_0\right]}{\sigma_M^2}$$

In equilibrium, each individual will alter their holdings of shares in productive firms, real assets, shares in insurance firms and insurance policies until their λ_h is the same as the market trade-off between risk (variance) and return (expected return) based on the economy wide wealth portfolio.

4.3 Equilibrium Market Values

The equilibrium pricing results can now be written as:

Shares in productive firms

$$\frac{\sigma_M^2}{\left[\overline{M} - (1 + r_f) M_0\right]}\left[\overline{R}_i - (1 + r_f) V_i\right] = cov\,(R_i, M) \qquad \text{for } i = 2, \ldots, I$$

or rearranging

$$V_i = \frac{1}{(1 + r_f)}\left[\overline{R}_i - \frac{\left[\overline{M} - (1 + r_f) M_0\right]}{\sigma_M^2} cov\,(R_i, M)\right] \qquad \text{for } i = 2, \ldots, I$$

In Gründl and Schmeiser [4], this is written as

$$V_i = \frac{1}{(1 + r_f)}\left[\overline{R}_i - \lambda^* \frac{cov\,(R_i, M)}{\sigma_M}\right]$$

with the market price of risk defined as

$$\lambda^* = \frac{\left[\overline{M} - (1 + r_f) M_0\right]}{\sigma_M}$$

4.3.1 CAPM

The equilibrium pricing results can be expressed in expected rate of return form, as is commonly used in the CAPM.

Define the rate of return on share in productive firm i as r_i with expected value

$$\overline{r}_i = \frac{\overline{R}_i}{V_i} - 1$$

Also define the expected rate of return on the economy wide wealth portfolio as

$$\overline{r}_M = \frac{\overline{M}}{M_0} - 1$$

Note that

$$cov\left(R_i, M\right) = V_i M_0 cov\left(r_i, \overline{r}_M\right)$$

and

$$\sigma_M^2 = M_0^2 \sigma_{\overline{r}_M}^2$$

which gives

$$\overline{r}_i - r_f = \frac{\left[\overline{r}_M - r_f\right]}{\sigma_{\overline{r}_M}^2} cov\left(r_i, \overline{r}_M\right) \qquad \text{for } i = 2, \ldots, I$$

or

$$\overline{r}_i - r_f = \beta_i \left[\overline{r}_M - r_f\right] \qquad \text{for } i = 2, \ldots, I$$

where

$$\beta_i = \frac{cov\left(r_i, \overline{r}_M\right)}{\sigma_{\overline{r}_M}^2}$$

We also have:

Real assets

$$A_k = \frac{1}{(1 + r_f)} \left[\left(\overline{R}_k - \overline{X}_k\right) - \lambda cov\left(R_k - X_k, M\right)\right] \qquad \text{for } k = 1, \ldots, K$$

Shares in insurers

$$S_f = \frac{1}{(1 + r_f)} \left[\overline{Y}_f - \lambda cov\left(Y_f, M\right)\right] \qquad \text{for } f = 1, \ldots, F$$

4.3.2 Insurance CAPM

The equivalent results for the market price of an insurance policy in an equilibrium model is related to the insurance CAPM. Taylor [12] sets out the different fair pricing models for insurance policies. The main issue in the fair pricing is to determine the profit loading for risk in the premium

The equilibrium premium of **Insurance policies** in the model is given by

$$P_{fk} = \frac{1}{(1 + r_f)} \left[\overline{X}_k - \lambda cov\left(X_k, M\right)\right] \qquad \text{for } \begin{bmatrix} k = 1, \ldots, K; \\ f = 1, \ldots, F \end{bmatrix}$$

and the profit loading for risk, expressed as a proportion of the (actuarial) net premium $\frac{1}{(1+r_f)}\overline{X}_k$ is

$$\frac{\lambda cov\left(X_k, M\right)}{\overline{X}_k}$$

5 An Alternative Derivation of the Results

Consider the first order condition derived earlier. For shares in productive firm i this was found to be, where we have removed the dependence of the loss on real asset k on who owns the asset,

$$\left[\begin{array}{c} U'_\mu \overline{R}_i + 2U'_v \left[\begin{array}{c} \sum_j v_{hj} cov\left(R_i, R_j\right) \\ +\sum_k a_{hk} cov\left(R_i, \left(R_k - X_k\right)\right) \\ +\sum_f s_{hf} cov\left(R_i, Y_f\right) \\ +\sum_{fk} n_{hfk} cov\left(R_i, X_k\right) \end{array}\right] \\ -U'_\mu \left(1+r_f\right) V_i = 0 \end{array}\right] \quad \text{for } i = 2, \ldots, I$$

We can rewrite this as

$$\begin{aligned} & -\frac{1}{2\lambda_h}\left[\overline{R}_i - \left(1+r_f\right) V_i\right] \\ = & \left[cov\left(R_i, R_2\right), \ldots, cov\left(R_i, X_k\right)\right] \left[\begin{array}{c} v_{h2} \\ \vdots \\ n_{hfk} \end{array}\right] \quad \text{for } i = 2, \ldots, I \end{aligned}$$

using vectors for the covariances and the optimal holdings.

The first order conditions for real assets, shares in insurance companies and insurance policies, can be written in vector and matrix format and the full set of equations becomes

$$\begin{aligned} & -\frac{1}{2\lambda_h}\left[\left[\begin{array}{c} \overline{R}_2 \\ \vdots \\ \overline{X}_k \end{array}\right] - \left(1+r_f\right)\left[\begin{array}{c} V_2 \\ \vdots \\ P_{fk} \end{array}\right]\right] \\ = & \left[\begin{array}{c} cov\left(R_i, R_2\right), \ldots, cov\left(R_i, X_k\right) \\ \vdots \\ cov\left(X_1, R_2\right), \ldots, cov\left(R_i, X_k\right) \end{array}\right]\left[\begin{array}{c} v_{h2} \\ \vdots \\ n_{hfk} \end{array}\right] \end{aligned}$$

This can be written in vector-matrix notation as

$$-\frac{1}{2\lambda_h}\left[\mathbf{E} - \left(1+r_f\right)\mathbf{P}\right] = \mathbf{\Omega}\mathbf{a}_h$$

where $\mathbf{E}$ is the vector of expected payoffs, $\mathbf{P}$ is the vector of values, $\mathbf{\Omega}$ is the variance-covariance matrix of the payoffs and $\mathbf{a}_h$ is the vector of holdings for individual h.

The optimal demand for each share, real asset and insurance policy is given by the elements in the vector $\mathbf{a}_h$ where

$$\mathbf{a}_h = \boldsymbol{\Omega}^{-1}\left[-\frac{1}{2\lambda_h}\left[\mathbf{E} - (1 + r_f)\,\mathbf{P}\right]\right]$$

Remark 7 *The mean-variance objective for each individual can easily be written in this vector and matrix notation from which the first order conditions can be derived directly.*

If we aggregate across all of the individuals we obtain

$$\frac{1}{\lambda}\left[\mathbf{E} - (1 + r_f)\,\mathbf{P}\right] = \boldsymbol{\Omega}\mathbf{m}$$

where $\mathbf{m}$ denotes the vector of the market portfolio holdings. The vector of prices is then given by

$$\mathbf{P} = \frac{1}{(1 + r_f)}\left[\mathbf{E} - \lambda\boldsymbol{\Omega}\mathbf{m}\right]$$

This holds for the portfolio of total economy wealth with value

$$\mathbf{m}^T\mathbf{P} = \frac{1}{(1 + r_f)}\left[\mathbf{m}^T\mathbf{E} - \lambda\mathbf{m}^T\boldsymbol{\Omega}\mathbf{m}\right]$$

from which we have

$$\lambda = \frac{\mathbf{m}^T\left[\mathbf{E} - (1 + r_f)\,\mathbf{P}\right]}{\mathbf{m}^T\boldsymbol{\Omega}\mathbf{m}}$$

Now each individual portfolio has market value

$$\mathbf{a}_h^T\mathbf{P} = \frac{1}{(1 + r_f)}\mathbf{a}_h^T\left[\mathbf{E} - \lambda\boldsymbol{\Omega}\mathbf{m}\right]$$

Notice that

$$-2\lambda_h\mathbf{a}_h = \boldsymbol{\Omega}^{-1}\left[\left[\mathbf{E} - (1 + r_f)\,\mathbf{P}\right]\right]$$

so that at the market equilibrium $\lambda_h\mathbf{a}_h$ does not depend on individual h.

The marginal trade-off between risk (variance) and return for individual h, λ_h, and the allocation for each of the shares, assets and insurance policies in their optimal wealth allocation in equilibrium, $\mathbf{a}_h$, for all individuals only depend on expected payoffs, covariances of payoffs, equilibrium prices and the risk free rate. This implies that all individuals must have an optimal wealth allocation that does not depend on individual characteristics. It also means that the marginal trade-off between risk (variance) and return for all individuals at the equilibrium will be the same. If the optimal wealth allocation for each individual in equilibrium does not depend on individual characteristics, then this can only be the case if they all hold portfolio weights that are the same as for the economy wide wealth portfolio. A formal proof of these results is readily derived from the results in Jarrow [5] Chapter 15.

Each individual holds the same mix of insurance policies and shares with each insurer. Each insurer will have the same capital structure, and this will be determined by the initial endowment in the economy of insurance policies and shares in insurers. It was shown earlier that insurers in this model, in aggregate, are a means for individuals to own shares in productive firms. The returns from the shares in productive firms held by insurers are distributed to the individuals holding shares in insurers and to policyholders in accordance with their claims experience. Insurance policies function as a risk pooling device.

6 Market Valuation of Insurer Liability

We have only made the assumption of mean-variance preferences and have made no assumption about the distribution of returns. So the above results hold for any distributions provided the moments exist. We will now consider the impact of limited liability of the insurers on the equilibrium premium and on insurer capitalization. If the insurance firms have limited liability, then pricing results need to allow for the option feature of the insurance policy payoff that results. Shares in insurers will also have option payoff features.

For insurance firm f, net equity will equal assets minus liabilities by definition. Assuming unlimited liability, in equilibrium, the payoffs at the end of the period will be such that

$$s_f^* Y_f = \sum_i v_{fi}^* R_i - \sum_{hk} n_{hfk}^* X_{hk}$$

so that net equity could be negative.

In Taylor [13], $X_f = \sum_{hk} n_{hfk} X_{hk}$ denotes total claims on insurer f, the payout ratio for insurer f is denoted by Q_f and the payout for an insurance policy will be $Q_f X_{hk}$ allowing for limited liability. The beginning of period equity (net assets) per share of insurer f is denoted by K_f and closing net assets per share for insurer f are defined as

$$E_f = Y_f + K_f R_f$$

where

$$R_f = \frac{\sum_i v_{fi} R_i}{\sum_i v_{fi} V_i}$$

is 1+ the rate of return on investments held by insurer f and Y_f is taken as the end of period cash flow per share assuming $K_f = 0$.

Funds held by insurer f to pay claims of X_f at the end of the period amount to

$$X_f + s_f E_f$$

and

$$Q_f = 1 + \min\left(0, \frac{s_f E_f}{X_f}\right)$$

Taylor [13] assumes the price of insurer f depends on K_f so that $S_f = S_f(K_f)$.

Insurer f has assets with start of period market value

$$\sum_i v_{fi} V_i$$

The shares in the insurance firms have start of period market value

$$s_f S_f(K_f)$$

and the start of period market value of the insurance liabilities are

$$P_f = \sum_{hk} n_{hfk} P_{fk}$$

The payoffs at the end of the period for each of these claims are

$$\sum_i v_{fi} R_i$$

$$s_f \max(E_f, 0)$$

and

$$\sum_{hk} n_{hfk} Q_f X_{hk}$$

By the definition of net equity

$$s_f \max(Y_f + K_f R_f, 0) + \sum_{hk} n_{hfk} Q_f X_{hk} = \sum_i v_{fi} R_i$$

The end of period liabilities (equity and insurance policies) of the insurer can be written

$$\begin{aligned} & s_f \max(E_f, 0) + \sum_{hk} n_{hfk} X_{hk} \left[1 + \min\left(0, \frac{s_f E_f}{X_f}\right)\right] \\ = \;& s_f \max(E_f, 0) + X_f \left[1 + \min\left(0, \frac{s_f E_f}{X_f}\right)\right] \\ = \;& X_f + s_f \max(E_f, 0) + s_f \min(0, E_f) \\ = \;& X_f + s_f E_f \end{aligned}$$

Based on equation (17) in Taylor [13] this equals

$$(P_f + s_f K_f) R_f = \sum_i v_{fi} R_i$$

since $(P_f + s_f K_f)$ are the beginning of period assets of insurer f.

The beginning of period balance sheet values for insurer f are

$$P_f + s_f S_f(K_f) = \sum_i v_{fi} V_i = \frac{\sum_i v_{fi} R_i}{R_f}$$

so we must then have

$$\left(P_f + s_f K_f\right) R_f = \left(P_f + s_f S_f\left(K_f\right)\right) R_f$$

or

$$S_f\left(K_f\right) = K_f$$

which has implications for much of the detail in Appendix A of Taylor [13].

However we can observe that the introduction of K_f does not allow the determination of an optimal capital structure for an insurer in the model. Fair prices of both shares in insurers and for insurance policies will reflect the option features that limited liability cause in the payoffs, but these are simply functions of whatever capital structure is assumed as the initial endowment in the model. The model will not generate cross sectional differences in insurer capitalization.

Consider the effect of limited liability on insurance fair pricing. Assuming that the insurer has limited liability, then if $\sum_i v_{fi} R_i \geq \sum_{hk} n_{hfk} X_{hk}$ policyholders are paid in full, otherwise they receive $\sum_i v_{fi} R_i$.

With limited liability we then have the payoff on a share in insurer f given by

$$\begin{aligned} Y_f^L &= \max\left[\frac{\sum_i v_{fi} R_i - X_f}{s_f}, 0\right] \\ &= \max\left[E_f, 0\right] \end{aligned}$$

If an insurer becomes insolvent then it is assumed that policyholders have their claims met pro-rata, so that the payoff X_{hk}^L allowing for limited liability will become

$$n_{hfk} X_{hk}^L = \frac{n_{hfk} X_{hk}}{X_f} \sum_i v_{fi} R_i$$

Given the model assumptions we know that

$$Y_f^L = \max\left[\frac{\sum_i v_{fi} R_i - X_f}{s_f}, 0\right]$$

We then have

$$\begin{aligned} X_{hk}^L &= \min\left(X_{hk}, \frac{X_{hk}}{X_f} \sum_i v_{fi} R_i\right) \\ &= X_{hk} + \min\left(\frac{X_{hk}}{X_f} \sum_i v_{fi} R_i - X_{hk}, 0\right) \\ &= X_{hk}\left(1 + \min\left(\frac{\sum_i v_{fi} R_i - X_f}{X_f}, 0\right)\right) \\ &= X_{hk}\left(1 + \min\left(\frac{s_f E_f}{X_f}, 0\right)\right) \\ &= X_{hk} Q_f \end{aligned}$$

as in Taylor [13].

The policyholder with insurer f has a policy that entitles them to payment of claims provided the assets of the company at the end of the period, $\sum_i v_{fi} R_i$, exceed the total claims payable by the company, X_f. Otherwise they are entitled to a share of the assets.

Summing over all insureds,

$$X_k^L = X_k Q_f$$

The insurance liability fair pricing equation will still apply since only the form of the payoffs have changed giving

$$P_{fk}^L = \frac{1}{(1+r_f)} \left[\overline{X}_k^L - \lambda cov \left(X_k^L, M \right) \right] = \qquad for \left[\begin{array}{c} k = 1, \ldots, K; \\ f = 1, \ldots, F \end{array} \right]$$

so that

$$P_{fk}^L = \frac{1}{(1+r_f)} \left[E \left[X_k Q_f \right] - \lambda cov \left(X_k Q_f, M \right) \right]$$

Rubinstein [10] derives valuation formulae for uncertain payoffs and shows how option based pricing results can be derived in a similar framework.

7 Valuation of the Limited Liability Exchange Option

The policyholder in insurer f has a policy that will pay the random claim amount X_{hk} assuming unlimited liability, and has sold an option to the share owners of insurer f to exchange the claim amount, if the policyholder is a claimant at the end of the year, for a share $\frac{X_{hk}}{X_f}$ of the assets of the company. Clearly X_{hk}, $\frac{X_{hk}}{X_f}$ and $\sum_i v_{fi} R_i$ are random variables and the valuation of this option is not straightforward, even assuming joint log-normality of asset returns and insurance losses. We will assume that each individual policyholder has a different claims distribution and that the fair market premium reflects that individual claims distribution. Thus P_{hk} will denote the fair premium assuming unlimited liability. We will denote the fair price of insurance allowing for limited liability by P_{hk}^L.

We will avoid issues of incomplete markets by assuming that contracts with payoffs depending on X_{hk} can be held by other individuals in the economy, even though in practice insurable interest would prevent this.

Note that we could just as easily treat the claim amount X_{hk} as a separate line of business rather than a separate individual risk.

We will value the exchange option using an approximation for the distribution of the return on the assets and the insurance claims on an insurer. The payment to the policyholder, allowing for limited liability will be

$$\begin{aligned} X_{hk}^{L} &= \min\left(X_{hk}, \frac{X_{hk}}{X_f}\sum_i v_{fi}R_i\right) \\ &= X_{hk} - \max\left(X_{hk} - \frac{X_{hk}}{X_f}\sum_i v_{fi}R_i, 0\right) \end{aligned}$$

so that the policyholder has an insurance contract equivalent that for an insurance company with unlimited liability along with an exchange option sold to the equity holders that allows them to exchange the insurance claim amount for a share in the value of the assets if the total claims exceed the value of the insurer assets.

The following assumptions are made in order to derive a valuation formulae for the policy. Individual asset returns and individual claim amounts are multivariate lognormal random variables. So that

$$\ln\left(\frac{R_i}{V_i}\right) \sim Normal\left(\mu_{R_i}, \sigma_{R_i}^2\right)$$

and

$$\ln\left(X_{hk}\right) \sim Normal\left(\mu_{X_{hk}}, \sigma_{X_{hk}}^2\right)$$

with correlation coefficients between log asset returns and log individual claim amounts given by $\rho_{R_i,X_{hf}}$, between log asset returns given by ρ_{R_i,R_j}, and between log claim amounts given by $\rho_{X_{hk},X_{jl}}$.

We assume that the return on the insurer assets and the total claims of the insurance company are also lognormal even though this is inconsistent with the assumption that individual asset returns and individual claim amounts are lognormal. This is expected to be a reasonable approximation to make. Note that 1 plus the return on the asset portfolio is

$$R_f = \frac{\sum_i v_{fi}R_i}{\sum_i v_{fi}V_i} = \sum_i \frac{v_{fi}V_i}{\sum_i v_{fi}V_i}\left(\frac{R_i}{V_i}\right) = \sum_i w_i\left(\frac{R_i}{V_i}\right)$$

which is a weighted sum of what we have assumed to be lognormal random variables, where the weights are

$$w_i = \frac{v_{fi}V_i}{\sum_i v_{fi}V_i}$$

We will assume the following approximations hold

$$\ln R_f \approx Normal\left(\mu_{R_f}, \sigma_{R_f}^2\right)$$

and

$$\ln X_f \approx Normal\left(\mu_{X_f}, \sigma_{X_f}^2\right)$$

where $\mu_{R_f}, \mu_{X_f}, \sigma^2_{R_f}$, and $\sigma^2_{X_f}$ are determined as functions of $\mu_{R_i}, \sigma_{R_i}, \sigma_{R_j}, \rho_{R_i,R_j}, \mu_{X_{hk}}, \sigma_{X_{hk}}, \sigma_{X_{jl}}$, and $\rho_{X_{hk},X_{jl}}$ to give the best fit for the lognormal approximation.

Given the assumptions that we have made, $ln\left[\frac{X_{hk}}{X_f}\sum_i v_{fi}R_i\right]$ has a normal distribution.

We denote the end-of-period assets value of insurer f as

$$A_f = \sum_i v_{fi}R_i = \left(\sum_i v_{fi}V_i\right)R_f$$

and the ratio of end-of-period assets to liabilities, ignoring limited liability, as

$$\frac{A_f}{X_f}$$

with initial value at the beginning of the period

$$\frac{A}{L}$$

where

$$A = \left(\sum_i v_{fi}V_i\right)$$

and

$$L = \sum_{hk} n_{hfk}P_{hk}$$

We can use the results of Margrabe [8] to derive a value of the exchange option. The fair price of the insurance contract becomes

$$\begin{aligned} P^L_{hk} &= P_{hk} - \left\{P_{hk}N(d_1) - P_{hk}\frac{A}{L}N(d_2)\right\} \\ &= P_{hk}\left[1 - \left\{N(d_1) - \frac{A}{L}N(d_2)\right\}\right] \end{aligned}$$

where

$$d_1 = \frac{\ln\frac{L}{A} + \frac{1}{2}\widehat{\sigma}^2}{\widehat{\sigma}}$$

$$d_2 = d_1 - \widehat{\sigma}$$

and $\widehat{\sigma}^2$ is the variance of $ln\left(\frac{X_{hk}}{\frac{X_{hk}}{X_f}\sum_i v_{fi}R_i}\right)$ which is the variance of $\ln\left(\frac{X_f}{A_f}\right)$. Hence

$$\widehat{\sigma}^2 = \sigma^2_{X_f} + A^2\sigma^2_{R_f} - 2Acov(X_f, R_f)$$

If we assume that the initial capital structure is to be a fixed ratio $\frac{A}{L}$ then we can determine the corresponding fair premium P^L_{hk}. Note that P_{hk} is the value of the insurance contract assuming unlimited liability and this is priced using the claims distribution ignoring the insolvency risk. This value allows for the risk in the claims distribution and the equilibrium price under the CAPM assumptions, where issues of incomplete markets are ignored, is given by

$$P_{hk} = \frac{1}{(1+r_f)}\left[\overline{X}_{hk} - \lambda cov\left(X_{hk}, M\right)\right] = \qquad for \left[\begin{array}{c} h = 1, \ldots, H; \\ k = 1, \ldots, K \end{array}\right]$$

In Myers and Read [9] these are referred to as the present value of losses and ignore the possibility of default by the insurance company. An actuarial valuation normally ignores the possibility of default and so these values correspond to a market based actuarial valuation. The allowance for default is given by the factor $1 - \left\{N(d_1) - \frac{A}{L}N(d_2)\right\}$ which multiplies the present value of losses to derive the value of the losses allowing for insurer default.

8 Implications for Capital Allocation

An important issue in the risk management of an insurer is the allocation of capital to lines of business. Myers and Read [9] present a marginal approach to allocating capital to lines of business. Cummins [2] reviews the approaches to capital allocation that have been proposed. Based on the model developed in this paper, it is possible to examine the implications for allocation of capital to lines of business, or in this case, individual policies.

The equity holders in the limited liability insurance company have effectively sold the policyholders an option to exchange the assets for the claim obligations under the policies if these are less than the total claims. They effectively hold the assets on behalf of the policyholders. The equity in the insurer, in the case of limited liability, has payoff

$$E_f^L = \max\left[A(R_f) - X_f, 0\right] = A(R_f) - X_f + \max\left[X_f - A(R_f), 0\right]$$

which using the Margrabe [8] exchange option formula and assumptions made earlier has fair or market value

$$A - L + L\left\{N(d_1) - \frac{A}{L}N(d_2)\right\}$$

Note that the value of the liabilities L is the value ignoring the possibility of default by the insurer. Myers and Read [9] refer to $A - L$ as surplus.

Consider the fair value of the policy liabilities derived earlier. This was the fair value assuming claims are paid in full less the value of the exchange option with payoff

$$\max\left(X_{hk} - \frac{X_{hk}}{X_f}\sum_i v_{fi}R_i, 0\right) = X_{hk}\max\left(1 - \frac{A(R_f)}{X_f}, 0\right)$$

The total payoff of the exchange options is

$$\sum_{hk} X_{hk}\max\left(1 - \frac{A(R_f)}{X_f}\right) = \max\left(X_f - A(R_f), 0\right)$$

So we can write the total equity value as

$$A - L + \sum_{hk} P_{hk} \left\{ N(d_1) - \frac{A}{L} N(d_2) \right\}$$

Now note that

$$L = \sum_{hk} P_{hk}$$

so we have total equity value given by

$$A - \sum_{hk} P_{hk} \left\{ 1 - \left\{ N(d_1) - \frac{A}{L} N(d_2) \right\} \right\}$$

If we allocate assets on a proportionate basis to policies we would obtain an expression for total equity value of

$$\sum_{hk} P_{hk} \left[\frac{A}{\sum_{hk} P_{hk}} - \left\{ 1 - \left\{ N(d_1) - \frac{A}{L} N(d_2) \right\} \right\} \right]$$

So if we allocated the total equity of the insurer to the policies according to

$$P_{hk} \left[\frac{A}{L} - \left\{ 1 - \left\{ N(d_1) - \frac{A}{L} N(d_2) \right\} \right\} \right]$$

then these would "add up" and will reflect the insolvency risk of the insurer at a company level.

Thus allocation to lines of business would be determined by valuing the losses for each line of business, adjusting for the risk of the liabilities but ignoring the possibility of default by the insurer. Thus P_{hk} is the market value of the losses for individual risk h on real asset k ignoring the default risk. This is the value of the liabilities that an actuarial valuation normally determines. An allowance is made for market risk in the liability value but not default risk of the insurer. Each liability value is multiplied by the same factor, $\left[\frac{A}{L} - \left\{ 1 - \left\{ N(d_1) - \frac{A}{L} N(d_2) \right\} \right\} \right]$, to determine the allocation of capital to the individual risk. This could just as easily be based on line of business rather than an individual risk. Thus capital is allocated in proportion to the market value of the losses by line of business, where the value is determined ignoring the insurer default risk.

9 Optimal Insurer Capitalization and Frictional Costs

As noted earlier, the model considered here does not allow an (unique) optimal determination of insurer capital structure. The capital structure of each insurer will be the same in equilibrium and this will be based on the initial endowments. The model can be calibrated to any given assumed capital structure and this could be based on aggregate industry data as in Taylor [13]. This is the classical

Miller-Modigliani result from finance theory for the optimal capital structure of firms as covered in standard finance texts. See for example Eichberger and Harper [3].

This can be demonstrated by noting that the beginning of period value of the liabilities and equity is $(P_f + s_f S_f)$ so that

$$\frac{A_f}{X_f} = \frac{(P_f + s_f S_f)\, R_f}{X_f}$$

With limited liability, P_f is the value of

$$X_f \text{ if } \frac{A_f}{X_f} > 1 \text{ and } A_f \text{ otherwise}$$

which in turn depends on P_f. The only way to determine a value of the liabilities is to fix the ratio $\frac{A}{L}$ which fixes the insurer capitalization. It is then possible to uniquely solve for the fair price of liabilities and the share price of the insurer. For each value of the ratio there is a unique fair price of the liabilities. Hence there is no optimal capital structure given by the model. Prices of liabilities and insurer share prices adjust to reflect the probability that the full claims will be paid based on the given initial capital structure of the company.

In order to include an optimal capital structure the model needs to be made richer. An approach to doing this would be to consider the impact of frictional costs. These costs include taxation, transaction costs, bankruptcy costs and agency costs. In a model with frictional costs, the value of insurer equity will depend on the variance of the firms end-of-period payoffs. The end-of-period payoffs for shares in the insurers will be the payoff ignoring these costs, less some function of variance to reflect the deadweight losses from frictional costs. This can be written as $Y_f - F\left(\sigma_f^2\right)$. MacMinn and Garven [7] consider the impact of frictional costs on insurance demand and pricing, but not on optimal insurer capital structure.

The model can also be made richer by allowing for asymmetric information, incomplete markets and should also be extended to a multi-period model. A continuous time model provides a modelling framework to effectively handle multi-periods. Issues concerning capital and regulation of insurers can be considered. The imposition of capital requirements is often justified by a situation where policyholders are unable to assess the financial condition of the insurance company and also by the fact that individuals can not diversify the risk of an insurance company becoming insolvent in their insurance purchases. These issues are related to information, transaction costs and incomplete markets.

The impact on fair insurance pricing and insurer capital structure under these alternative assumptions is clearly an important research issue of major significance to the financial and actuarial management of insurers, the fair treatment of policyholders and to insurance solvency regulation.

10 Conclusion

This paper has reviewed results from Taylor [13] and Turner [11]. We have derived the fair pricing results and reviewed the implications of the model for the optimal capital structure of an insurer. The model includes insurers, firms and households and the model assumptions are that individuals are mean-variance utility maximizers with perfect information and that markets have no frictions such as transactions costs, taxes, agency costs or bankruptcy costs. Standard results from finance theory for financial markets are shown to hold, implying that the model does not allow for a unique optimal capital structure for insurers. Limited liability has been considered and shown to alter the payoff to policyholders and share owners in insurers but not the form of the valuation formulae for the fair price of insurance. Related research has considered optimal demand for insurance and product design but very little research has considered the financial and risk management issues for insurers in this model framework.

Current research is aimed at extending the model to consider more general utility functions, frictional costs, incomplete markets as well as to a continuous time setting. In all of these cases the inclusion of frictional costs has the potential to provide new insights into fair insurance pricing and optimal capital structure for insurers. The model is also being extended to include both supply and demand in order to consider equilibrium pricing and capital. Standard financial and insurance models, including CAPM, assume a fixed supply given by the endowment in the economy.

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