Continuous Compounding, Volatility and Beta: Review of and Response to Fitzherbert

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Abstract

In a recent paper, Fitzherbert claims that if the expected return used in empirical studies were the average continuously compounded return then the CAPM relationship between expected return and β would be questionable. We demonstrate that an arithmetic average of returns should be used to estimate the expected return in the standard CAPM and show that using a geometric average is incorrect. We derive the CAPM based on continuous compounding returns and show that the compounding frequency does not alter the relationship between expected return and beta, when the correct model assumptions are used. The paper concludes with a discussion of the finance coverage in the education syllabus and examinations for actuaries.

1 Introduction

A recent paper by Fitzherbert (2001) [7] discusses the Capital Asset Pricing Model (CAPM), and claims that the relationship between beta and return demonstrated in an early empirical study of the CAPM "almost disappears when the definition of mean return is changed from arithmetic average of discrete returns to continuous compounding. In consequence, many of the empirical studies of β values and return may need to be re-interpreted". The implication is that the use of the arithmetic average is not correct in empirical tests of the standard CAPM. He discusses a number of the empirical studies of the CAPM, particularly the early studies. There is very little reference to more recent research in asset pricing.

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The paper is incorrect in its claim that arithmetic averaging of returns shouldn't be used for determining an estimate of the expected return for testing of the standard discrete time period CAPM. It is also confusing in its discussion of continuous compounding and discrete compounding and arithmetic and geometric average returns. In a later paper, Fitzherbert (2002) [8], the discussion of "mean rate of return" is clearer, but the same incorrect claim with respect to the determination of an estimate of the expected return is made. In Fitzherbert (2001) [7] the difference between arithmetic and geometric averages is not explained clearly. We demonstrate that the arithmetic average is the correct mean return to use for the expected return in the standard CAPM.

Many of the misconceptions in the Fitzherbert paper were not mentioned in the discussion of the paper at the Institute of Actuaries of Australia Horizons Meetings in both Sydney and Melbourne. A press release was issued by The Institute of Actuaries of Australia on 15 October 2001 headed "Challenging the conventional risk return theory". In that press release it states "Fitzherbert argues from the perspective of long-term investors, **time weighted** returns (or their equivalent) need to be calculated to assess investment theories".

We clarify when an arithmetic average should be used to estimate an expected return and discuss the estimation of returns for use in the standard discrete time CAPM. We discuss the use of continuous compounding returns in the CAPM. We also briefly comment on the empirical studies mentioned in the Fitzherbert paper.

To conclude we cover actuarial education in the area of asset pricing and finance theory more broadly. We discuss implications for the education syllabus and examinations of actuarial students and also for continuing education of practitioners.

2 Continuous Compounding and Discrete Compounding

Fitzherbert (2001) [7] confuses compounding frequency with methods of estimating returns from historical data - geometric or arithmetic averages. He refers to continuous compounding as if this is the same as geometric average returns. It is not. The discussion of mean returns is clearer in Fitzherbert (2002) [8] but the error with respect to use of the arithmetic average remains.

In Fitzherbert (2001) [7] Section 3.1 discusses the meaning of "mean return". Realised returns on an investment can be calculated over a time period allowing for the cash flows using the internal rate of return. The internal rate of return can assume any frequency of compounding of interest including continuous compounding. The internal rate of return is referred to as the money-weighted rate of return. This calculation is sometimes modified when cash flows alternate in sign since it is then possible for there to be more than one solution to the IRR net present value equation. This often happens in tax based financing transactions such as leveraged leases. Modified internal of return methods have been developed for these cases.

The rate of return can be expressed assuming any compounding frequency. An internal rate of return calculated and expressed as an annual (nominal) return assuming $m$thly compounding is easily converted to a frequency assuming $n$thly compounding using the relationship

$$\left(1+\frac{j^{(m)}}{m}\right)^m = \left(1+\frac{j^{(n)}}{n}\right)^n$$

Continuous compounding is a special case where the continuous compounding equivalent return, denoted by δ, is defined to be

$$e^{\delta} = \lim_{m\to\infty}\left(1+\frac{j^{(m)}}{m}\right)^m$$

When comparing alternative investments for performance measurement purposes an adjustment for the timing of cash flows is made. The return is called a time weighted rate of return. The time weighted rate of return determines the return between cash flows by valuing the portfolio on the date of cash flows. These returns are then compounded together to determine the time weighted return.

The compounding frequency of an interest rate has nothing to do with the method of averaging returns. Any frequency of compounding can be used to specify an interest rate and any of these rates can be averaged using either geometric or arithmetic averaging.

3 Geometric and Arithmetic Averages

3.1 Fitzherbert's Example

Fitzherbert (2001) [7] uses the following two portfolios to illustrate the problems with using an arithmetic mean rate of return:

Portfolio	**A**	**B**
Time	**Value $**	**Value $**
0	100	100
1	200	110
2	80	121

For Portfolio A, Fitzherbert calculates the per period returns as $\frac{200}{100}-1=100\%$ for the first period and $\frac{80}{200}-1=-60\%$ for the second period. He then calculates an average return of these 2 returns of 20% per period. For Portfolio B, Fitzherbert calculates the per period returns as $\frac{110}{100}-1=10\%$ for the first period and $\frac{121}{110}-1=10\%$ for the second period which gives an average return of 10% per period.

Note that we can express these returns as continuous compounding equivalents using $e^{\delta}=\ln(1+i)$ where i is the per period return. Thus for Portfolio

A, for the first period the continuous compounding equivalent is $\ln(2) = 0.6931$ or 69.31%, and for the second period it is $\ln(0.4) = -0.9163$ or -91.63%. The per period arithmetic average return has a continuous compounding equivalent of 0.1823 or 18.23%. The arithmetic average of the continuous compounding equivalent per period returns is -11.16% and the standard deviation is 64.76%.

Fitzherbert calculates the geometric mean for the two time periods for each investment as $\left[\left(\frac{200}{100}\right)\left(\frac{80}{200}\right)\right]^{\frac{1}{2}} - 1 = -10.5573\%$ for Portfolio A and $\left[\left(\frac{110}{100}\right)\left(\frac{121}{110}\right)\right]^{\frac{1}{2}} - 1 = 10\%$ for Portfolio B. These are discrete per period returns. They can also be calculated as continuous compounding equivalents using $e^{\delta} = \left[\left(\frac{200}{100}\right)\left(\frac{80}{200}\right)\right]^{\frac{1}{2}}$ or $\delta = -0.1116$ (or 11.16% per period) for Portfolio A and 0.0953 or 9.53% for Portfolio B.

Fitzherbert states that the final portfolio value can only be determined using the geometric mean for the two periods. The arithmetic mean for the two periods does not allow the determination of the exact final value of the portfolio.

3.2 The Error in Fitzherbert

Kritzman (1995) Chapter 5 [11] sets out, in a very clear way, the correct use of arithmetic and geometric averages in calculating returns. In this section of the paper we identify the error in Fitzherbert's investment return calculation example. It is important to remember that returns are random variables. Future returns can not be predicted with certainty. A sample of historical returns from a portfolio is one realisation of returns from a distribution of possible values in each period.

To illustrate when the arithmetic average should be used to determine expected return we can modify the example in Fitzherbert [7] as follows:

Portfolio	**A**	**B**
Time	**Value \$**	**Value \$**
0	100	100
1	200	110
2	40	121

For Portfolio B we will assume that the Portfolio is invested so that it is guaranteed to earn 10% per period.

For Portfolio A, we will assume that the values are one possible path of returns drawn from a probability distribution of returns for each time period. Assume that over any period the value of Portfolio A can either double with probability 0.5, with a return of 100%, or decrease by 80% with probability 0.5, a return of -80%. The expected return for a time period for Portfolio A will then be $\frac{100-80}{2} = 10\%$. There would then be 4 possible paths or realisations of values of Portfolio A over the 2 time periods given in the following diagram:

Time	0	1	2
			400
		200	
			40
Value	100		
			40
		20	
			4
Average portfolio value		110	121
Average return % p.a.		10%	10%

The values for Portfolio A in the modified example are a realisation along only one of the possible paths of values for the portfolio. All the paths are equally likely by construction. Now the arithmetic average of the possible returns in each period is 10%. The arithmetic average for the possible returns in period 2 is also 10%.

If we want to estimate the expected return of the probability distribution generating the returns in any period then we must use the arithmetic average across all possible equally likely paths. In practice, historical data provides only a single realisation for the portfolio value and we can not average across all possible values in a given time period. However, if we assume a stationary return probability distribution[1] then we can use the arithmetic average of the returns along a path of realised returns to estimate the expected return.

3.3 When to use Arithmetic and Geometric Averages

Portfolio A and B in the modified example both have the same expected return per period and the same expected portfolio value at the end of any time period. Portfolio A has random returns whereas Portfolio B does not.

To estimate the expected portfolio value at the end of any time period, assuming a stationary probability distribution generating the returns, it is necessary to use the arithmetic average of the per period realised returns and to estimate the expected portfolio value in t time periods as

$$\overline{P}_t = P_0 \left(1 + \overline{r}_a\right)^t$$

where P_0 is the initial portfolio value and $\overline{r}_a$ is the arithmetic average return.

If we want to accumulate funds along a particular path of historical returns then the value will be

$$\begin{aligned} P_t &= P_0 \left(1 + r_1\right)\left(1 + r_2\right) \ldots \left(1 + r_t\right) \\ &= P_0 \prod\nolimits_{i=1}^{t} \left(1 + r_i\right) \end{aligned}$$

[1] Readers who not familiar with the meaning of a stationary return distribution are referred to the material used for Subject 103 of the professional syllabus of the Institute of Actuaries

where r_i is the realised return in period i with period i being from time $i-1$ to i.

Now by definition the geometric average is

$$\overline{r}_g = \left[\prod_{i=1}^{t} (1+r_i)\right]^{\frac{1}{t}} - 1$$

so that

$$P_t = P_0 (1+\overline{r}_g)^t$$

where $\overline{r}_g$ is the geometric average of the discrete per period returns along a particular realisation of the portfolio value.

If we denote the random future value of the portfolio by $\widetilde{P}_t$ then taking expectations we have

$$\mathbb{E}\left[\widetilde{P}_t\right] = P_0 \prod_{i=1}^{t} (1+\mathbb{E}[r_i])$$

and assuming independent and identically distributed returns ensures that $\mathbb{E}[r_i]$ is the same for each time period and the expected values of cross products of returns are zero, so

$$\mathbb{E}\left[\widetilde{P}_t\right] = P_0 (1+\mathbb{E}[r])^t$$

and if we have enough observations then the arithmetic average along a single realisation of the returns will estimate $\mathbb{E}[r]$.

Note that

$$\mathbb{E}\left[P_0 (1+\overline{r}_g)^t\right] \neq P_0 (1+\mathbb{E}[\overline{r}_g])^t .$$

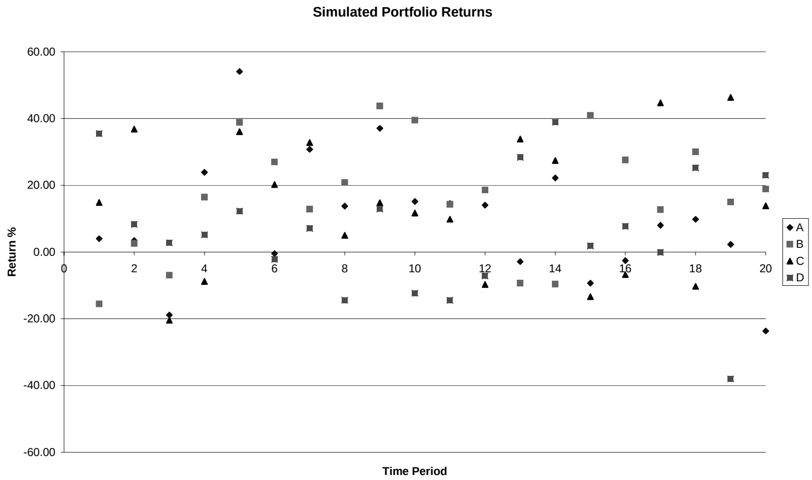
The CAPM is a model for the expected returns for a single time period. The appropriate average return to use will be the arithmetic average. Fitzherbert's claim that the use of arithmetic averages is the reason the empirical tests of the CAPM are questionable is not correct. There are many reasons that the CAPM does not explain actuarial security price data and the method of averaging the returns is not one of them.

As we show in this paper, there are forms of the CAPM that use continuous compounding returns as well as discrete per period returns. In all cases the correct estimate of the expected return for any period, assuming a stationary probability distribution for returns, is the arithmetic average of historical returns. This arithmetic average could be of continuous compounding returns if the continuous version of the CAPM is used as the model for expected returns, as we will show in this paper.

If returns are not stationary then it is necessary to estimate the expected return allowing for the assumed model of returns.

4 Normal distributions

The above example can easily be made less simplistic by assuming a probability distribution for returns. The CAPM is often derived assuming a normal distribution for returns. Assuming returns in each time period are generated from a Normal distribution with mean 10% and standard deviation of 20%, we can simulate equally likely paths of returns using a random number generator. This is easy to do in an Excel spreadsheet. We have done this for 4 simulated portfolios over 20 time periods and the returns are shown in the graph:



Along each path we have calculated the arithmetic (sample) average for each sample of 20 returns, as well as the standard deviation and the geometric average return. They are given in the following table:

Portfolio	A	B	C	D
Arithmetic Average	9.77	16.92	13.96	6.01
Standard Deviation	17.95	17.27	20.14	18.20
Geometric Average	8.32	15.59	12.13	4.34
AA-0.5SD^2	8.16	15.43	11.93	4.35

Because we have only 20 periods the arithmetic average is an estimate of the true mean of 10% and this has its own statistical distribution. In fact we know that the arithmetic average is normally distributed with mean 10% and a standard error of 6.324%. The sample standard deviation is also an estimate of the variance based on the sample.

The last row shows the arithmetic average minus half the standard deviation squared. Thus $0.0816 = 0.0977 - \frac{1}{2}(0.1795)^2$ for Portfolio A and similarly for the other portfolios. The interesting thing to note is that this is almost equal to the geometric average.

It appears that

$$\overline{r}_g \doteqdot \overline{r}_a - \frac{1}{2}s^2$$

for these simulated portfolios.

Fitzherbert (2001) [7] claims that knowing the arithmetic average return does not allow us to forecast the final portfolio value. This is certainly the case if we assume that returns are deterministic - an often used assumption in actuarial practice. However, if we recognise that the returns in each period are random and drawn from a probability distribution, and we also know the standard deviation, then we can also forecast the final portfolio value, at least to within sampling error. Fitzherbert (2001) [7] attempts to develop an approximate relationship between geometric and arithmetic average returns which is similar to that above.

If we assume that one plus the per period return $1+r$ has a lognormal distribution with $\mathbb{E}\left[\ln\left(1+r\right)\right]=\mu$ and $Variance\left[\ln\left(1+r\right)\right]=\sigma^2$ then $\ln\left(1+r\right)$ has a normal distribution with mean μ and variance σ^2. To estimate μ from a sample from this distribution we would use the arithmetic average of $\ln\left(1+r_i\right)$ where r_i is the realised return in time period i.

Consider the value of a portfolio at time t, earning a return of r_i in each time period, with $\ln(1+r_i)$ independent and identically normally distributed with mean μ and variance σ^2. We have

$$\widetilde{P}_t = P_0 \prod\nolimits_{i=1}^{t} \left(1+r_i\right)$$

so

$$\ln \widetilde{P}_t = \ln P_0 + \sum\nolimits_{i=1}^{t} \ln\left(1+r_i\right)$$

and therefore $\ln \widetilde{P}_t$ is normally distributed with mean $\ln P_0 + n\mu$ and variance $n\sigma^2$.

The continuous compounding per period return $\delta = \ln\left(1+r\right)$ is a random variable as is the future portfolio value at time t given by $P_t = P_0 e^{\delta t}$. However

$$\mathbb{E}\left[\widetilde{P}_t\right] = P_0 \mathbb{E}\left[e^{\delta t}\right]$$

and since δ is a normally distributed random variable we can use the definition of the moment generating function for the normal distribution to write

$$\mathbb{E}\left[\widetilde{P}_t\right] = P_0 e^{\left[\mu+\frac{1}{2}\sigma^2\right]t}$$

Alternatively we can note that $e^{\delta t}$ has a log-normal distribution and determine the result by integration of the expected value integral using the probability density of the log-normal distribution.

Also

$$\mathbb{E}\left[\widetilde{P}_t\right] = P_0 \prod\nolimits_{i=1}^{t} \left(1+\mathbb{E}\left[r_i\right]\right)$$

and, because of our independent and identically distributed assumption,

$$\mathbb{E}\left[\widetilde{P}_t\right] = P_0 \left(1+\mathbb{E}\left[r\right]\right)^t$$

so

$$(1+\mathbb{E}[r_i]) = e^{\left[\mu+\frac{1}{2}\sigma^2\right]}$$

where by definition $\mu = \mathbb{E}[\ln(1+r_i)]$.

If we have a sample of returns, then assuming that the returns are drawn from a log-normal distribution with the same mean and variance in each time period, we would estimate $\mathbb{E}[r_i]$ with the sample arithmetic average $\overline{r}_a$, μ with the arithmetic average of $\ln(1+r_i)$ and σ^2 with the sample variance of $\ln(1+r_i)$. An approximate relationship between the arithmetic average return and the arithmetic average of the continuous compounding equivalent returns, denoted by $\overline{\delta}$, would be

$$\overline{r}_a \doteqdot e^{\overline{\delta}+\frac{1}{2}s_l^2} - 1$$

Since from the definition of the geometric average

$$(1+\overline{r}_g) = \left[\prod_{i=1}^{t}(1+r_i)\right]^{\frac{1}{t}}$$

we have

$$\ln(1+\overline{r}_g) = \frac{1}{t}\left[\sum_{i=1}^{t}\ln(1+r_i)\right]$$

or

$$\overline{\delta} = \ln(1+\overline{r}_g)$$

where $\overline{\delta}$ is the arithmetic average of the equivalent continuously compounding return. Because there is assumed to be no dependence in returns between time periods this is estimated using an arithmetic average of returns using historical data.

We then have

$$\begin{aligned}\overline{r}_a &\doteqdot (1+\overline{r}_g)e^{\frac{1}{2}s_l^2} - 1 \\ &\doteqdot \overline{r}_g + \frac{1}{2}s_l^2\end{aligned}$$

where in this case s_l^2 is the sample variance of the continuous compounding equivalent rate $\delta = \ln(1+r)$. Note that, using the method of statistical differentials, which is also called the method of propagation of error or the delta method[2],

$$\begin{aligned}Var(\ln(1+r)) &\doteqdot \left(\frac{1}{1+\mathbb{E}[r]}\right)^2 Var[r] \\ &\doteqdot Var[r]\end{aligned}$$

so that

$$\overline{r}_a \doteqdot \overline{r}_g + \frac{1}{2}s^2.$$

This is the same as the approximate relationship noted earlier between the geometric and arithmetic average of the returns for our sample drawn from a normal distribution except that we have now assumed a log-normal distribution for returns.

[2] See for example Rice (1995)[14] p149

5 Expected Returns and the CAPM

The CAPM uses expected returns and these are estimated using historical returns data. The form of the model is

$$\mathbb{E}[r] = r_f + \beta [\mathbb{E}[r_M] - r_f]$$

where $\mathbb{E}[r]$ is the discrete per period expected return on the security, $\mathbb{E}[r_M]$ is the discrete per period expected return on the market value weighted portfolio of all assets and r_f is the discrete per period risk free return. As shown above, when using historical data to estimate the expected returns, the expected per period return used in the CAPM is correctly estimated by the arithmetic average of per period returns.

There are a number of assumptions required in order to derive the CAPM and also in order to estimate the parameters for empirical testing. These issues are comprehensively dealt with in Campbell, Lo and MacKinlay (1997) [5] and also in Cochrane (2001) [6].

There are many reasons why empirical studies of the CAPM will be expected to perform poorly even if the model was a correct description of asset pricing. For instance, it is well known that if the market portfolio used in CAPM tests using realised returns is the market value weighted portfolio of all the securities being tested, then the CAPM will automatically hold. If the market portfolio used for CAPM testing is not the market value weighted portfolio then all that empirical tests will demonstrate is that the market portfolio used in the tests is not a mean-variance efficient portfolio. This is the famous "Roll Critique".

Other reasons for the failure of tests of the CAPM and other similar asset pricing models relate to assumptions of the model. There is empirical evidence that the market risk premium, a concept that does not depend an any particular asset pricing model, is not constant. The risk free rate depends on the maturity of the cash flow being valued because of the term structure of interest rates. Empirical evidence suggests that beta values also change through time. All of these issues need to be addressed in the development, testing and calibration of any asset pricing model.

Cochrane (2001) [6] discusses asset pricing based on a requirement for prices to be arbitrage-free and also using equilibrium pricing. He derives an asset pricing relationship summarised by two equations

$$p_t = \mathbb{E}(m_{t+1} x_{t+1})$$

and

$$m_{t+1} = f(\text{data, parameters})$$

where p_t is the price, x_{t+1} is the payoff, and m_{t+1} is the stochastic discount factor. The asset price is an expected present value using a stochastic discount factor, usually referred to as a deflator in the actuarial literature.

The derivation of this result does not require any particular assumptions about return distributions nor about investor preferences other than the usual prefer more to less and risk aversion assumptions. This pricing framework is derived over a single time period but this could be a single time period in a multi-period model since conditional expected values are used. It includes single time period models.

The CAPM can be derived from this pricing model using a number of assumptions. These include normal return distributions, quadratic utility, and a linear stochastic discount factor. The linear stochastic discount factor can be derived as an approximation for many models that use other utility functions or return distributions to those assumed in standard derivations of the CAPM. These issues are discussed extensively in Cochrane (2001) [6], in Chapter 9 in particular.

The Intertemporal CAPM was a modification of the CAPM developed for multiperiod asset pricing. In the Intertemporal CAPM the stochastic discount factor is linear in a number of factors including current wealth and additional factors which hedge shifts in future investment opportunities. Implementations of these multiple factor models are used in practice for portfolio selection and risk management of investment portfolios. They are the basis for modelling techniques used by quantitative fund managers and hedge funds, particularly for portfolio selection and risk management.

Empirical evidence in a large number of studies over many years suggests that a number of factors may forecast market excess returns including dividend yields, interest rates, interest rate spreads, default spreads, industry, size and book-to-market value.

5.1 Continuous Compounding and the Continuous Time CAPM

Because the CAPM was originally developed as a single time period model, various models have been developed that extend the CAPM to a multi-period model. Strong assumptions are required to apply the CAPM in a multiperiod framework without modification.

There is a version of the CAPM developed in continuous time by Merton. Merton's paper is reprinted as Chapter 11 in Merton (1992) [13]. In the Merton continuous time model, security prices are assumed to follow the dynamics

$$\frac{dP_i(t)}{P_i(t)} = \alpha_i(t)\,dt + \sigma_i(t)\,dZ_i(t) \qquad i = 1, \ldots, n$$

where $\alpha_i(t)$ is the instantaneous return on asset i, $\sigma_i(t)$ is the instantaneous volatility of asset i at time t, and $dZ_i(t)$ is the increment in a standard Wiener process. Investors maximise a lifetime utility and bequest function. Demand functions for risky securities are derived and an aggregate demand determined. Equilibrium prices are determined allowing for capital growth in share values and issuing of new shares.

Assuming a constant interest rate gives the continuous time CAPM in the form

$$\alpha_i - r = \frac{\sigma_{iM}}{\sigma_M^2}(\alpha_M - r) \qquad i = 1, \ldots, n$$

where σ_{iM} is the instantaneous covariance of the ith security return with the market portfolio return and α_M the instantaneous expected return on the market portfolio. The rates here are all instantaneous continuous compounding interest rates. The derivation is found in Merton (1992) [13].

Conditional on the price at time t we can derive a relationship between the expected return on a security and the expected return on the market portfolio over a fixed time horizon using the continuous time CAPM return model.

We can express the return as either a continuous compounding return equivalent

$$\delta_i = \ln\left[\frac{P_i(t+1)}{P_i(t)}\right]$$

or as a discrete per period return

$$1 + r_i(t) = \frac{P_i(t+1)}{P_i(t)}$$

To do this we need to solve the continuous time stochastic differential equation for the price of the asset conditional on the price at time t.

To simplify the solution assume that $\alpha_i(t)$ and $\sigma_i(t)$ are constant, in which case $P_i(t)$ will be log-normally distributed. Given

$$\frac{dP_i(t)}{P_i(t)} = \alpha_i dt + \sigma_i dZ_i(t)$$

we need to determine the form of $P_i(t+\tau)$ given $P_i(t)$. For this case the solution to the SDE is known to be

$$P_i(t+\tau) = P_i(t)\exp\left(\left(\alpha_i - \frac{1}{2}\sigma_i^2\right)\tau + \sigma_i Z_i(\tau)\right)$$

where $Z_i(\tau) = \int_t^{t+\tau} dZ_i(t)$ is normally distributed with mean zero and variance τ. Details are found in Chapter 11 of Merton (1992) [13] where the equivalent result for the total value of all the assets, the market portfolio, is also derived and given by

$$P_M(t+\tau) = P_M(t)\exp\left(\left(\alpha_M - \frac{1}{2}\sigma_M^2\right)\tau + \sigma_M X(\tau)\right)$$

where

$$X(\tau) = \int_t^{t+\tau} dX(t) = \int_t^{t+\tau} \frac{\sum_{i=1}^n w_i\sigma_i}{\sigma_M} dZ_i(t)$$

is also normally distributed with mean zero and variance τ.

For the continuous compounding returns we have

$$\delta_i = \ln\left[\frac{P_i(t+1)}{P_i(t)}\right] = \left(\alpha_i - \frac{1}{2}\sigma_i^2\right) + \sigma_i Z_i(1)$$

and

$$\delta_M = \ln\left[\frac{P_M(t+1)}{P_M(t)}\right] = \left(\alpha_M - \frac{1}{2}\sigma_M^2\right) + \sigma_M X(1)$$

We then have

$$\mathbb{E}[\delta_i] = \left(\alpha_i - \frac{1}{2}\sigma_i^2\right)$$

and using the continuous time CAPM result

$$\mathbb{E}[\delta_i] = \left(r + \frac{\sigma_{iM}}{\sigma_M^2}(\alpha_M - r) - \frac{1}{2}\sigma_i^2\right)$$

Since

$$\mathbb{E}[\delta_M] = \left(\alpha_M - \frac{1}{2}\sigma_M^2\right)$$

we obtain

$$\mathbb{E}[\delta_i] = \left(r + \frac{\sigma_{iM}}{\sigma_M^2}\left(\mathbb{E}[\delta_M] + \frac{1}{2}\sigma_M^2 - r\right) - \frac{1}{2}\sigma_i^2\right)$$

and simplifying we then obtain

$$\mathbb{E}[\delta_i] = r + \frac{\sigma_{iM}}{\sigma_M^2}(\mathbb{E}[\delta_M] - r) + \frac{1}{2}\left(\frac{\sigma_{iM}}{\sigma_M^2}\sigma_M^2 - \sigma_i^2\right)$$

which is the same as the result given in Merton (1992) [13] and also tested in Jensen (1972) [10] with his data set.

This is the CAPM that should apply over discrete time periods if the continuous time CAPM were generating the instantaneous returns. Note that under the continuous time CAPM the per period returns are log-normally distributed, whereas in the standard CAPM they are usually assumed to be normally distributed.

Note that $\mathbb{E}[\delta_i]$ is estimated using the arithmetic average of the equivalent per period continuous compounding returns and the tests in Jensen (1972) [10] are based on arithmetic averages of continuous compounding returns.

In order to derive a form of the CAPM for discrete per period expected returns assuming the continuous time CAPM holds then we need to consider

$$\begin{aligned}
\mathbb{E}[1 + r_i(t)] &= \mathbb{E}\left[\frac{P_i(t+1)}{P_i(t)}\right] \\
&= \exp(\alpha_i)
\end{aligned}$$

Using the continuous time CAPM we then have

$$\begin{aligned}
\mathbb{E}[1 + r_i(t)] &= \exp\left(\left[r + \frac{\sigma_{iM}}{\sigma_M^2}(\alpha_M - r)\right]\right) \\
&= \exp\left(\left[r + \frac{\sigma_{iM}}{\sigma_M^2}(\ln \mathbb{E}[1 + r_M(t)] - r)\right]\right)
\end{aligned}$$

and taking logs we have

$$\ln \mathbb{E}\left[1 + r_i\left(t\right)\right] = r + \frac{\sigma_{iM}}{\sigma_M^2}\left(\ln \mathbb{E}\left[1 + r_M\left(t\right)\right] - r\right)$$

Note that $\ln \mathbb{E}\left[1 + r_i\left(t\right)\right] \neq \mathbb{E}\left[\delta_i\right]$ but that $\ln \mathbb{E}\left[1 + r_i\left(t\right)\right]$ is the continuous compounding equivalent of the expected return $\mathbb{E}\left[r_i\left(t\right)\right]$.

6 Black, Jensen, and Scholes

Fitzherbert (2001) [7] uses the expected returns from the paper by Black, Jensen, and Scholes (1972) [3] to project the accumulated value of a \$1 million portfolio for 35 years from 1/1/1931, a time period that corresponds to the time period of the Black, Jensen, and Scholes (1972) [3] study. He finds that the values obtained are substantially higher than for the Standard and Poor's index over the same time period.

A number of factors need to be considered. To begin with, the Standard and Poor's index provides a single realisation over this time period of the index and does not give an expected value of the Index over a 35 year time period. As discussed earlier in the paper this is given by $P_0\left(1 + \overline{r}_g\right)^t$ and is approximately equal to $P_0\left(1 + \overline{r}_a - \frac{1}{2}s^2\right)^t$.

As we have seen from the earlier sections of this paper, the value from accumulating at the arithmetic average of the returns is an expected value, not a path realisation. Also, as noted by Fitzherbert (2001) [7] (on page 696), Black, Jensen, and Scholes (1972) [3] used an equally weighted portfolio of every security listed on the NYSE at the beginning of each month. In Black, Jensen, and Scholes (1972) [3] the equally weighted market portfolio has a monthly average excess return of 1.42%. CRSP data given in Campbell, Lo and MacKinlay (1997) [5] over the period July 3, 1962 to December 30, 1994 give a monthly average return on a value-weighted portfolio of 0.96% and a monthly average return on an equal-weighted portfolio of 1.25%.

To approximate the return on a value weighted portfolio in the Black, Jensen, and Scholes (1972) [3] data, assume a risk free monthly rate of 0.1% per month as in Fitzherbert (2001) [7] and then apply a proportionate adjustment using the CRSP ratio of returns for value weighted versus equal weighted portfolios to get an approximate equivalent for a value weighted monthly return of

$$\frac{0.96}{1.25} \times (0.1 + 1.42) = 1.167\%$$

If we compound this return over 35 years then \$1 million will accumulate to \$130.9 million which is closer to the Standards and Poor's Index accumulation than the figures given in Fitzherbert (2001) [7].

Recall from earlier in this paper, that it is valid to use the arithmetic average of historical returns to estimate the future expected value of a portfolio. If we also adjust for the fact that, since the Standard and Poor's index value is a single realisation of historical returns, this is equivalent to accumulation at a geometric average, then an equivalent interest rate for accumulation with the Black, Jensen, and Scholes (1972) [3] study data would be $0.01167 - \frac{1}{2}(0.0891)^2 = 0.77\%$. Accumulating \$1 million over 35 years at this rate gives \$25.1 million which is of the same order as for the Standard and Poor's index over this period.

Differences in the methods used to construct the portfolios in Black, Jensen, and Scholes (1972) [3] and the Standard and Poor's Index could easily account for the other differences.

Fitzherbert (2001) [7] states that "There is clearly something about this approach which is either wrong or needs further investigation." In this section of the paper we have carried out a simple further investigation, demonstrated that there is indeed something clearly wrong and that the explanation is that it arises from a misconception about of the use of geometric and arithmetic average returns.

6.1 Fitzherbert Table 6

Fitzherbert (2001) [7] adjusts the arithmetic means in Black, Jensen, and Scholes (1972) [3] to derive equivalent geometric means. However as will be clear from the earlier discussion in this paper, the geometric means are not the correct means to use to estimate expected returns in the standard CAPM. The relationship between the arithmetic and geometric average has been approximated in an earlier section of this paper for the log-normal price case and illustrated with some simulated normally distributed returns. In the published discussion of Fitzherbert (2001) [7], Greg Taylor correctly identified the error in the arguments about arithmetic averages based on Table 6 in the paper.

In the following table we give the estimates of the arithmetic mean monthly excess return and the standard deviation of the excess monthly return from Black, Jensen, and Scholes (1972) [3] Table 2 and use the approximation $\overline{r}_g \doteqdot \overline{r}_a - \frac{1}{2}s^2$ to derive an estimate of the geometric mean. These results are very close to the approximations given in Fitzherbert (2001) [7]. However the conclusions reached by Fitzherbert (2001) [7] that this demonstrates that using the geometric mean invalidates the results of the tests of Black, Jensen, and Scholes (1972) [3] is incorrect.

Portfolio	β	$\overline{r}_a$	**s**	$\overline{r}_a - \frac{1}{2}\mathbf{s}^2$	**%p.a.**
1	1.5614	0.0213	0.1445	0.0109	13.8
2	1.3838	0.0177	0.1248	0.0099	12.6
3	1.2483	0.0171	0.1126	0.0108	13.7
4	1.1625	0.0163	0.1045	0.0108	13.8
5	1.0572	0.0145	0.0950	0.0100	12.7
6	0.9229	0.0137	0.0836	0.0102	13.0
7	0.8531	0.0126	0.0772	0.0096	12.2
8	0.7534	0.0115	0.0685	0.0092	11.6
9	0.6291	0.0109	0.0586	0.0092	11.6
10	0.4992	0.0091	0.0495	0.0079	9.9
Market	1.000	0.0142	0.0891	0.0102	13.0

The betas given in this table were estimated using arithmetic averages and the standard form of the CAPM. If other definitions of return are used or if the continuous time CAPM is used, then the estimates of the relevant $\beta's$ will be different in these models since the regression variables being used have been transformed.

Even if these estimates of beta were used, then a simple linear regression of the geometric mean return and the betas in the above table produces a statistically significant positive relationship between average return and beta. The linear regression equation is

$$\overline{r}_g = 0.00751 + 0.00233\beta$$

and the p-value for the coefficient of β is 0.002 (a t-statistic of 4.51) which indicates a significant positive relationship between the geometric average return and the β in the table even though the β in the table above is not the correct β to use in testing such a relationship. In fact the form of the CAPM based on continuously compounding returns uses $\ln(1+\overline{r}_a)$ and a regression of the continuous compounding equivalent of the arithmetic average return on β in the above table gives

$$\ln(1+\overline{r}_a) = 0.00366 + 0.0106\beta$$

with a t-statistic of 21.6. The regression results for $\ln(1+\overline{r}_a)$ are almost exactly the same as those given in Jensen (1972) [10] in Table 2 on page 35 for his tests of the continuous time CAPM. There is a positive relationship between average return and beta for the data used in these studies, regardless of how you measure the average return.

The Black, Jensen, and Scholes (1972) [3] study demonstrated the linear relationship between expected return and beta based on the data set and methodology used in that study. However the results also indicated the time varying nature of the expected return relationship and that the relationship was not consistent with the traditional form of the CAPM. The results of this study do not support the standard CAPM.

The use of an arithmetic mean rather than a geometric mean return is not the reason that the CAPM empirical tests are faulted. There are many other explanations for the poor empirical performance of the CAPM. These are covered in great detail in both Cochrane (2001) [6] and Campbell, Lo and MacKinlay (1997) [5]. If the average return on the market is negative over a time period, then it is possible for there to be an empirical ex-post negative relationship between average return and β in an empirical test of the standard CAPM including such a time period, even if the actual relationship between expected return and beta generating the returns was positive.

The results of the empirical studies cited by Fitzherbert (2001) [7] do not provide empirical support for the standard CAPM, nor the continuous time CAPM, and this is found in many other published studies.

6.2 Basu and other anomalies

Fitzherbert (2001) [7] discusses the study by Basu (1977) [2] and states "While this study also shows an inverse relationship between β values and returns, this is an inverse relationship between continuously compounded return and β, not a relationship between the arithmetic average of discrete monthly return and β."

The use of continuous compounding returns or discrete compounding returns does not affect the relationship between β and return, at least not directly. In earlier sections of this paper we derived a form of the CAPM in terms of both continuous and discrete compounding returns. The relationship between β and return will hold in both cases although the form of the CAPM regression will differ.

The Basu (1977) [2] study considers portfolios with different P/E ratios. For each of these portfolios the risk and return relationship is compared. The full details of the procedure used to assess the effect of P/E ratio on risk and return is covered in the original paper and will not be discussed here in any more detail than necessary. For each of their P/E portfolios they estimate the equations

$$r_{pt} - r_{ft} = \widehat{\delta}_{pf} + \widehat{\beta}_{pf} \left[r_{mt} - r_{ft} \right]$$

where r_{pt} is the continuously compounded return on P/E portfolio p in month t; r_{mt} is the continuously compounded return on the "market portfolio" in month t; r_{ft} is the continuously compounded "risk free" return in month t; $\widehat{\delta}_{pf}$ is the estimated intercept and $\widehat{\beta}_{pf}$ is the estimated slope.

In their portfolios, as the median P/E ratio decreases, the average annual rate of return $(\overline{r}_p)$ on the portfolio increases. They also find that for portfolios with lower median P/E ratio the estimated systematic risk $\left(\widehat{\beta}_p\right)$ decreases. This leads Fitzherbert (2001) [7] to the conclusion that with continuous compounding returns, the relationship between expected return and β is negative since the higher expected return portfolios in the Basu (1977) [2] are associated with a lower β.This result will hold regardless of how returns are calculated.

Berk (1995) [4] provides an explanation as to why, even if the CAPM were the correct model generating returns, the relationship observed in Basu (1977) [2] would be expected to occur. The reason is that for higher risk portfolios, regardless of how you define risk, the expected return will be higher. In a sense expected return tells us about the risk of the portfolio. If investors regard a portfolio as riskier then they will demand a higher expected return in order to hold the portfolio. This means that if we standardise portfolios by the expected future cash flows, or expected earnings, then the riskier portfolios will have a lower price per unit of future expected cash flow in order to provide a higher expected return. Portfolios with a lower price will have a lower P/E ratio. Thus higher risk portfolios will have lower P/E ratios, regardless of how the risk is determined - it could be generated by the CAPM β. In the Basu (1977) [2] study this is what we observe for the P/E portfolios. The lower the median P/E, the higher the expected return.

The existence of the negative β relationship does not arise from the use of continuous compounding returns, there are other reasons for the results.

7 CAPM, Actuarial Education and Asset Pricing

The aim of this paper is not to review or discuss the CAPM and its use in practice. CAPM is one of many models that can be and is used in practice. Shortcomings in the CAPM have been identified and there are many models in the finance literature that are used in valuation. These include option pricing models as well as models for incorporating real options and strategy into market valuation. See for example Grenadier (2000) [9] for a collection of articles allowing for strategy and game theoretic models in financial valuation.

The CAPM is useful as a starting point for valuation provided certain assumptions hold. These include a relatively constant capital structure, no guarantee features in cash flows, no real options such as growth options, no significant default probability and the factors affecting the value are traded in similar companies/cashflows in a competitive market. The CAPM captures the risk adjustment in the discount rate and assumes a constant discount rate across time. If there are guarantees or real options or the risk of non-payment of the cash flows is significant then the CAPM is not the best method for valuing risky cash flows.

A better approach is to risk adjust the cashflows rather than the discount rates and then to present value the risk adjusted expected cash flows at the riskless interest rates determined from a current term structure of interest rates. This is the way that options and other contingent cash flows are valued.

With the CAPM, the risk free interest rates and the market risk premium are usually determined based on current market data. The market risk premium is often estimated based on historical data but the theory requires the use of current market expectations. The beta, which is used as a risk factor, is usually estimated using historical data. Estimation issues for the parameters of valuation models occur in related valuation models such as term structure models. Some of the model parameters can be calibrated to current traded market values whereas other parameters are estimated using time series historical price data. Calibration of a pricing model is an important part of the practical implementation of the model.

There are simulation studies that look at how good CAPM is as a valuation model when actual returns are generated by other models. CAPM seems to do reasonably well in these studies which are usually dividend discount models, with the market risk premium driven by factors that are assumed to affect value, such as dividend yield, book to market, interest rates, and credit spreads.

Ang and Liu (2002) [1] investigate a model for expected returns where the market risk premium is time varying and depends on a number of predictive factors. Their model allows for time varying discount rates and they consider the standard finance dividend discount model for valuing securities. They compare the mispricing from a valuation taking into account time varying betas and market risk premiums with that based on the CAPM assuming constant betas and market risk premiums. They find that the standard dividend discount model with a constant discount rate "performs extremely well... ". Studies such as this help with an understanding of when it is appropriate to use models such as the CAPM for valuation purposes.

For non-traded assets, with no market data to calibrate model parameters to, valuation will need to rely on a utility/equilibrium based model. Even so, the valuation approach would be best based on risk adjusting expected cash flows rather than using a risk adjusted interest rate. Time variation in risk premiums or in the cash flows will mean the CAPM risk adjusted discount rate approach will not be suitable. The time variation could arise from option features such as policyholder guarantees for life insurance companies.

The most common use of the CAPM in corporate finance is in determining an expected discount rate for project evaluation to be applied to expected future cash flows. For risk management and portfolio selection purposes more sophisticated models are used in practice. The more sophisticated models include factor models and many of them are discussed in Cochrane (2001) [6]. For valuation purposes it is possible to show that for many more realistic models, the CAPM is an approximation despite the rather restrictive assumptions usually required to derive the model. Cochrane (2001) [6] derives the CAPM using a number of different assumptions and also shows that the stochastic discount factor is linear.

Even if the required conditions to use the CAPM hold, there are many issues to deal with in the practical application of the CAPM. These include estimation errors for the CAPM beta. Because of changes to capital structure and the underlying business risks it is expected that betas will vary over time. There is also statistical estimation errors to consider including the need for predictive distributions for returns. Market risk premiums are also time varying and subject to similar estimation issues.

If a discounted cash flow model is going to be used with expected future cash flows discounted at risk adjusted rates, then a basis for risk adjustment such as the CAPM is required. Any practical alternative will need to be based on an assessment of historical returns data and an assumption about the level of risk and the expected return. If a competitive market model is to be used to determine values of cash flows that contain no or limited option features and where a constant capital structure can be assumed, then CAPM should be a useful starting point.

Understanding financial modelling principles and their practical application should be an integral part of the education syllabus for actuaries. The current Part I syllabus of the Institute of Actuaries (also adopted by The Institute of Actuaries of Australia) includes the following syllabus objective in Subject 109 Financial Economics:

(viii) Describe equilibrium models, such as the Capital Asset Pricing Model, discussing the principal results and assumptions and limitations of such models.

1. Describe the assumptions of the CAPM.
2. Discuss the principal results of the CAPM.
3. Discuss the limitations of the basic CAPM and some of the attempts that have been made to alter the theory to overcome these limitations.
4. Discuss the assumptions and principal results of the Arbitrage Pricing Theory model.

The syllabus for the new Financial Economics subject, CT8, includes the following syllabus objective:

(vi) Describe asset pricing models, discussing the principal results and assumptions and limitations of such models.

1. Describe the assumptions and principal of the Sharpe-Lintner-Mossin Capital Asset Pricing Model (CAPM).
2. Discuss the limitations of the basic CAPM and some of the attempts that have been made to alter the theory to overcome these limitations.
3. Discuss the assumptions and principal results of the Ross Arbitrage Pricing Theory model (APT).
4. Perform calculations using the CAPM.

The Part I syllabus objectives do not address the needs of actuarial students. They are designed so that they can be covered by a standard university finance course, allowing university actuarial programs to cover the financial economics syllabus using standard finance courses. These objectives will not allow students to apply valuation techniques to a wider range of valuation problems nor will they be aware of the major developments in valuation models that have occurred since the CAPM.

The topics in Chapters 15 and 16 of Luenberger (1998) [12] would be a much better foundation knowledge about valuation for actuarial students than the current coverage. This would be further enhanced by reference to relevant actuarial examples.

7.1 The Actuarial Education Syllabus

Finance theory has had its most success in practice in contingent claim valuation and in term structure modelling. These areas all involve relative valuation models or basic arbitrage-free models. The economic assumptions required to develop valuation models are minimal. In both of these areas the early models have been developed into sophisticated risk management models and the use of stochastic calculus and advanced financial mathematics is the main technique.

In security pricing and modelling, the success of finance theory has not been as encouraging as in derivative valuation and risk management. In this case the valuation models are not relative valuation models and rely on economic models of risk, supply, demand and equilibrium in order to derive results. This is a much more challenging area because of the need to understand the economics of the models, deal with noisy data for security returns and standard econometric assumptions not holding. Empirical testing of models is a challenging research area.

The CAPM and its derivation, along with the ideas of systematic risk, and non-systematic risk, and diversification are useful ideas. Standard finance courses all cover these concepts, and in fact these are generally the key focus of such courses at a basic level. Although it is useful for actuarial students to be exposed to these ideas in a standard finance course, this approach and coverage should not be the only basis for the actuarial education syllabus for courses in financial economics and for later actuarial courses in investment and finance. The problems that actuaries will deal with are much wider than those covered in such courses and standard finance texts. The current syllabus for the actuarial courses, including the current Financial Economics course, are very much driven by such standard texts and the content of standard finance courses.

The modern approach to asset pricing, as set out in Cochrane (2001) [6] for example, develops valuation models that provide a framework for a wide range of valuation and risk management problems. A more basic coverage is provided in Luenberger (1998) [12].

The Part I syllabus of the Institute of Actuaries does not adequately prepare future actuaries with the understanding of the financial economics that they need to practice as actuaries. The current Part II and Part III syllabuses of The Institute of Actuaries of Australia do not provide this preparation for all actuaries. The Part III subjects are an obvious place where such important educational needs can be addressed for future actuaries. Postgraduate coursework programs developed to address the needs of the actuarial profession in this area should clearly be of value to the profession.

8 Conclusions

In this paper we have demonstrated that the arithmetic average return is the correct return estimate to use when estimating per period returns from historical data. It is also the return estimate to use to project the average future value of an investment.

The CAPM can be expressed in terms of continuous compounding returns or discrete per period returns. In either case the estimate of the expected return should be based on the arithmetic average of the returns.

We have reviewed the studies referred to in Fitzherbert (2001) [7] and demonstrated that the claims made about the results of the studies arising from the incorrect averaging of returns are not correct.

The CAPM has been the subject of extensive testing and theoretical development and many developments have occurred in asset pricing since the development of the model. The actuarial education syllabus does not adequately prepare future actuaries with the foundation core technical knowledge that they need in this important area. Current actuaries would also appear likely to benefit from further education in this area as well.

A modern approach to asset pricing should be included in the syllabus for actuarial students.

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