Financial Valuation of Retirement Benefits Based on Salary

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1 Introduction

In this paper we consider a financial modelling approach to the valuation of pension benefits where the retirement benefit depends on the salary of the member in the pension plan. We demonstrate a partial differential equation (PDE) approach to modelling the dynamics of the pension fund developed from traditional actuarial mathematics and techniques used in finance. An early example of a similar approach with an application to insurance, is found in the finance literature in Shimko (1989) [8]. More general approaches to modelling life insurance contracts are found in the actuarial science literature e.g. Norberg (1996) [6] or Wolthuis (1994) [10].

The PDE model we develop in this paper can be used for the valuation of a variety of pension plan benefit structures. We consider allowing for benefits payable on death or withdrawal prior to retirement age and retirement benefits depending on final salary as well as average salary (either continuous sampling or discrete sampling). Insurance benefits are assumed to be paid from the fund on death and disability of a member. If a member withdraws from the plan, because of resignation, then he is assumed to be paid a withdrawal benefit as a transfer payment to another pension plan. The retirement benefits we consider in this paper are a fixed multiple of either final salary or average salary over a pre-specified period prior to retirement. This can be extended to the case of career average salary plans.

An allowance for an optimal early retirement benefit decision based only on the value of the benefits can be included for the case where the retirement benefit depends only on final salary. In practice such early retirement decisions will be influenced by many other factors.

Our approach to the valuation of pension benefits differs from the traditional actuarial approach often used to value these benefits. Firstly, we model the salary as a stochastic process and derive the dynamics of the benefit value. Economic considerations are then used to derive a partial differential equation (PDE) that the benefit value satisfies. Secondly, we apply a valuation principle based on an economic approach to allowing for risk similar to that used in finance theory. The valuation principle can be described as a "risk-adjusted" approach designed to determine an economic, or fair, value for the pension liabilities. The underlying assumption is that the valuation model is arbitrage-free and that benefits depending on salary, and possibly other factors, are valued consistent with the notion of no-arbitrage in finance. Our approach draws on techniques developed in finance and demonstrates how they can be applied to the valuation of pension benefits.

When the retirement benefit depends on the average salary, the retirement benefit is similar to an Asian option pay-off. Allowing for early retirement based on financial considerations can be treated in a similar manner to the American option valuation problem in finance (i.e., free boundary problem). The PDE approach in finance as in Wilmott, Dewynne and Howison [9] has been applied to the valuation of options including Asian options and American options. We demonstrate how these techniques can be applied to pension problems.

Although a large number of retirement plans are now accumulation plans, also referred to as defined contribution benefits, there still is a significant number of existing plans with retirement benefits related to salary. There also exist pension plans providing the larger of a defined benefit and a defined contribution benefit. The valuation of these "greater-of" benefits is considered in Sherris [7].

This paper is organized as follows. In Section 2, we give the general framework for the valuation of pension benefits in which the interest rate $r(t)$ is a known deterministic function and the retirement benefit paid by the pension plan is based on either the final salary of the member or the continuous average salary of the member over a specified period. Section 3 describes the case where we have discrete sampling for the average salary in the benefit formula. In Section 4 we show how the model can allow for the interest rate $r(t)$ to be random. Section 5 is devoted to the derivation of analytical expressions of the benefit values in a particular case using PDE approach and the discounted expected value of cash-flows (DEV) approach commonly used in actuarial science. Finally, in Section 6 we discuss numerical evaluation techniques. A more general method PDE method of deriving an analytical solution is considered in the Appendix.

2 Modeling Framework

This section consists of two parts. The first part covers the financial modelling approach to the valuation of pension benefits where the retirement benefit depends only on the final salary. The second part considers a model for pension benefits where the retirement benefit depends on the continuous average salary. In this section the interest rate is always assumed to be deterministic.

2.1 Financial and Actuarial Model

We denote the entry age of a member into the pension plan by x and the time since entry by t, so that at $t = 0$ the member enters the pension plan aged x. Denote the member's salary at time t for a member who entered the pension plan at age x by $S(t; x)$ which is not deterministic. We model $S(t; x)$ using

$$dS = \alpha(t, S; x)dt + \sigma(t, S)dZ, \quad S(0; x) = S_0(x) \tag{2.1}$$

where in general, the growth rate of the member's salary α can depend on time since entry, current salary, and age at entry. This allows for a salary scale to be included in the valuation where the salary increase varies by age at entry and by plan membership. We also assume that the past history of the salary is fully reflected in the present salary, which does not hold any further information. Hence, $S(t; x)$ is a Markov process.

It is also assumed that there is an instantaneous risk-free rate of interest that would be used for valuing certainty equivalent or "risk-adjusted" cash flows. The rate of interest at time t is denoted by $r(t)$ and in this section this rate is assumed to be deterministic.

The value of the benefits payable to a member of the plan at time t, when the member is aged $x + t$, will be denoted by $V(t, S; x)$.

In what follows, we first derive the mathematical model (PDE) for $V(t, S; x)$ when the retirement benefit depends only on the final salary and early retirement is allowed.

Insurance benefits are assumed to be paid from the fund on death and disability of a member. If a member withdraws from the plan, because of resignation, then he is assumed to be paid a withdrawal benefit as a transfer payment to another pension plan.

The pension plan can be modelled as a continuous-time Markov model. Wolthuis [10] and Norberg [6] provide the specification of a continuous-time Markov model in actuarial life insurance mathematics. We consider the states of active (a), dead (d), and withdrawn or resigned (w). Retirement is assumed to occur at the final age of service in the pension fund. The transition intensities from active membership to the dead or withdrawn states are denoted by $\mu_d(t;x)$ and $\mu_w(t;x)$ respectively. These are referred to as forces of decrement in traditional actuarial terminology. Thus the force of decrement of death or disability at time t for a member aged x at entry, when he is aged $x+t$, is $\mu_d(t;x)$ and the force of withdrawal is $\mu_w(t;x)$.

The sums-at-risk for these decrements are denoted by $B_i(t,S;x)$ for $i = d,w$. The sum at risk is the amount payable under the benefits of the pension plan less the value of the benefits. Shimko [8] refers to these as "terminating" cash flows. These are paid when an event occurs resulting in the payment of a benefit cash flow after which the value of benefits to the member representing their entitlements in the pension fund will be zero.

In traditional actuarial mathematics, assuming that the benefits paid are also deterministic, Thiele's differential equation (refer to Bowers [2]) gives the change in the value of the benefits, ignoring any contributions into the fund, as

$$dV = r(t)Vdt - \sum_{i=d,w} \mu_i(t;x)\left[A_i(t;x) - V\right]dt$$

where $A_i(t;x)$ is the deterministic benefit paid on exit from the pension fund at time t due to decrement i. The expression $[A_i(t;x) - V]$ is the sum-at-risk for decrement i. Note that the $\sum_{i=d,w} \mu_i(t;x)\left[A_i(t;x) - V\right]dt$ is an expected value of the net cash flow payable from the fund. This is more readily seen when the differential equation is derived in the context of a continuous-time Markov model.

We can rewrite this as

$$dV = \left[r(t) + \sum_{i=d,w} \mu_i(t;x)\right]Vdt - \sum_{i=d,w} \mu_i(t;x)A_i(t;x)dt$$

From this expression we can see that the same result would hold for a fund with no decrements paying fixed deterministic cash flows between time t and $t+dt$

of $\sum_{i=d,w} \mu_i(t;x)\,[A_i(t;x)]dt$ and earning an interest rate of $r(t) + \sum_{i=d,w} \mu_i(t;x)$.

Returning to the case where benefit values depend on salary, using Ito's lemma for $V(t,S;x)$ and the model (2.1) for salary, and ignoring any cash flows for benefit payments from the fund, we obtain

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \left(\frac{\partial V}{\partial t} + \alpha(t,S;x)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma(t,S)^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma(t,S)\frac{\partial V}{\partial S}dZ \quad (2.2) \end{aligned}$$

Now we assume that a price of risk for salary exists and is given by $\lambda(t,S;x)$ and define

$$\theta(t,S;x) = \alpha(t,S;x) - \lambda(t,S;x)\sigma(t,S). \quad (2.3)$$

Then it follows from (2.1) that

$$dS = \theta(t,S;x)dt + \sigma(t,S)(dZ + \lambda(t,S;x)dt)$$

or

$$dS = \theta(t,S;x)dt + \sigma(t,S)dZ^* \quad (2.4)$$

where

$$dZ^* = dZ + \lambda(t,S;x)dt \quad (2.5)$$

We assume that there exists a probability measure for our salary model such that under this new measure Q,

$$E^Q\left[dZ^*\right] = 0 \quad (2.6)$$

and as a result, the process for S under this change of measure has been "risk-adjusted". This assumption is our valuation principle. The valuation principle is based on the economic notion that the risk-adjusted expected change in the benefit value, after allowing for benefit cash flows from the fund, should equal the risk free interest rate.

Applying this we then have

$$E^Q\left[dV\right] = rVdt - \sum_{i=d,w} \mu_i(t;x)B_i(t,S;x)dt. \quad (2.7)$$

where r is the instantaneous known riskless rate of interest, which is deterministic in this section of the paper.

Combining (2.2) , (2.4) and (2.6) with (2.7), we have the following valuation equation for V:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma(t,S)^2\frac{\partial^2 V}{\partial S^2}+\theta(t,S;x)\frac{\partial V}{\partial S}-rV+\sum_{i=d,w}\mu_i(t;x)B_i(t,S;x)=0. \quad (2.8)$$

To solve this PDE (2.8), we need boundary conditions and a final condition.

In this model we assume that the retirement benefits are expressed as lump sum values at the retirement date, and for age retirement (assumed at a fixed age of 65), the benefit paid is a multiple a of final salary based on the number of years membership in this plan. This multiple reflects the value of the life annuity based on pensioner mortality and the deterministic interest rate. It does not vary by salary at retirement.

In Section 4, we relax the assumptions of deterministic interest rates. We then must value the pension payments allowing for varying interest rates at retirement.

The sum at risk for retirement is

$$B_r(T,S;x)=a(T)S(T)-V(T,S;x), \quad (2.9)$$

where $T=65-x$. This gives the final condition for V:

$$\text{at } \ t=T=65-x, \quad V(T,S;x)=a(T)S(T). \quad (2.10)$$

On early retirement, which we assume can occur in the last m years prior to age 65, the retirement benefit is discounted by a factor of b for every year prior to age 65. Thus, for $65-x-m\leq t\leq 65-x$, we have the sum at risk is equal to the early retirement benefit less the value of the benefits

$$B_r(t,S;x)=(1-b(T-t))a(T)S(t)-V(t,S;x) \quad (2.11)$$

We assume that members of the plan elect to retire early when the value of the benefit that they receive from the fund on early retirement exceeds the value of the benefit if they remain in the fund. This is only one criteria for determining whether it is optimal for the member to retire. Other approaches can take into account the tradeoff between leisure and work as well as direct financial benefit amounts as in Kingston (1998) [4]. Under our assumptions the retirement decision is modelled in a similar way to early exercise of an option.

For the m years prior to age 65, there will be two free boundary conditions that we need to determine as part of the solution to the PDE (2.8). We denote the free boundary by $S = S_f(t)$ for $65 - x - m \leq t \leq 65 - x$. Therefore, in the same way as deriving the free boundary conditions for the American option, in the present situation we obtain two boundary conditions on the free boundary with one being

$$\text{on } S = S_f(t): \quad V(t, S; x) = (1 - b(T - t))a(T)S(t) \tag{2.12}$$

and another so-called smooth pasting condition. To completely solve the PDE (2.7), we also need the boundary conditions:

$$S = 0: \quad V = f(t), \tag{2.13}$$

$$S \to \infty: \quad \left|\frac{\partial V}{\partial S}\right| < \infty. \tag{2.14}$$

Therefore, equation (2.8) together with the final condition (2.10), two free boundary conditions, and boundary conditions (2.13)–(2.14) consists of the mathematical model for the financial value of pension benefit, where the retirement benefits are based on a member's final salary, and early retirement is allowed.

2.2 Continuous Average Salary and No Early Retirement

In what follows we consider the pension plan where the retirement benefit depends on the continuous average of the last n year's salary, and early retirement is not allowed. The valuation of such a pension plan is similar to the pricing for an Asian option with European style exercise.

Following the general framework for the valuation of an Asian option, covered in Wilmott, Dewynne & Howison [9], we introduce the new variable

$$I(t; x) = \int_0^t g(\tau, S(\tau; x))d\tau, \tag{2.15}$$

where

$$g(t, S) = \begin{cases} 0, & t < T - n, \\ h(S), & T - n \leq t \leq T. \end{cases} \tag{2.16}$$

In the above, $h(S)$ is a given function. Since $S(t)$ is a Markov process, we may treat t, S and I as independent variables. We note that

$$\begin{aligned} dI &= I(t+dt) - I(t) \\ &= \left[\int_0^{t+dt} g(\tau, S(\tau; x))d\tau - \int_0^t g(\tau, S(\tau; x))d\tau\right] \\ &= \int_t^{t+dt} g(\tau, S)d\tau \\ &= g(t, S)dt. \end{aligned} \tag{2.17}$$

Applying Ito's lemma to $V = V(t, S, I; x)$, and ignoring benefit cash flows, we get

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial I}dI + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \left(\frac{\partial V}{\partial t} + \alpha(t, S; x)\frac{\partial V}{\partial S} + g(t, S)\frac{\partial V}{\partial I} + \frac{1}{2}\sigma(t, S)^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma(t, S)\frac{\partial V}{\partial S}dZ \end{aligned} \tag{2.18}$$

In a similar way to the derivation for (2.8), we obtain

$$\frac{\partial V}{\partial t} + \theta(t, S; x)\frac{\partial V}{\partial S} + g(t, S)\frac{\partial V}{\partial I} + \frac{1}{2}\sigma(t, S)^2\frac{\partial^2 V}{\partial S^2} - rV + \sum_{i=d,w} \mu_i(x, t)B_i(t, S; x) = 0 \tag{2.19}$$

where we have used the fact that the process for S is "risk-adjusted".

For age retirement at age 65 we assume that the benefit paid is a multiple of the final average salary based on the number of years membership in the plan. Thus, the final condition for equation (2.19) is:

$$\text{at} \quad t = T = 65 - x, \quad V(T, S, I; x) = a(T)\frac{I(T)}{n}. \tag{2.20}$$

There is no free boundary, and the other boundary conditions are:

$$S = 0: \quad V = f(t, I), \tag{2.21}$$

$$S \to \infty: \quad \left|\frac{\partial V}{\partial S}\right| < \infty, \tag{2.22}$$

$$I = 0: \quad V = 0, \tag{2.23}$$

$$I \to \infty: \quad \left|\frac{\partial V}{\partial I}\right| < \infty. \tag{2.24}$$

3 Discrete Sampling for Average Salary

Pension plans providing defined benefit pensions use a variety of different methods for determining the average salary. Many plans base the pension on the total salary paid over the final n year's prior to retirement. In this case the average salary can be determined as a continuous average as in the previous section. Other plans base the final benefit on the average of the member's salary at evenly spaced dates prior to retirement. For example, a plan may base the benefit payment on the average of the member's salary determined as at the end of each of the n calendar years prior to retirement. In this case the final average salary is based on a discrete average. This case is similar to an Asian option where the averaging is taken at discrete dates.(Wilmott, Dewynne & Howison [9])

Assume that the salary average is based on sampling the salary at discrete time points t_i for $i = 1, ..., n$. We use t_i^- to denote the time point just before the ith salary sample date used in the averaging and t_i^+ to denote the time point just after. Because of the continuity of $V(t, S, I; x)$, we then have that the benefit value satisfies the following jump condition at each sample date

$$V(t_i^-, S, I; x) = V(t_i^+, S, I + S; x). \tag{3.1}$$

Between sample dates the benefit satisfies

$$\frac{\partial V}{\partial t} + \theta(t, S; x)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma(t, S)^2\frac{\partial^2 V}{\partial S^2} - rV + \sum_{i=d,w} \mu_i(t; x)B_i(t, S; x) = 0. \tag{3.2}$$

Valuation of the pension plan would then proceed in a similar manner to the techniques used for discrete Asian options.

4 Random Interest Rates

In the previous sections we assumed that interest rates were deterministic. Benefit values paid on retirement will depend on interest rates, and the benefit cash flows extend over very long time periods. The financial risk in the value of the retirement benefits arising from variable interest rates will be significant. Therefore, it is important to allow for random interest rates in the valuation.

We assume that a money market account exists and earns the instantaneous short rate of interest $r(t)$ where

$$dr = \mu(r,t)dt + \nu(r,t)dZ_r \tag{4.1}$$

where we assue that the instantaneous correlation between the Weiner increment for the salary and the interest rate is $\rho(t)$.

From Ito's Lemma, the benefit value $V(t, S, I, r; x)$ will satisfy the following stochastic differential equation, ignoring benefit cash flows,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial I}dI + \frac{\partial V}{\partial r}dr \\ &\quad + \frac{1}{2}\left\{ \frac{\partial^2 V}{\partial S^2}(dS)^2 + 2\frac{\partial^2 V}{\partial S \partial r}(dSdr) + \frac{\partial^2 V}{\partial r^2}(dr)^2 \right\} \end{aligned} \tag{4.2}$$

We then have

$$\begin{aligned} dV &= \left\{ \begin{array}{l} \frac{\partial V}{\partial t} + \alpha(t,S;x)\frac{\partial V}{\partial S} + g(t,S)\frac{\partial V}{\partial I} + \mu(t,r)\frac{\partial V}{\partial r} \\ + \frac{1}{2}\sigma(t,S)^2\frac{\partial^2 V}{\partial S^2} + \rho(t)\sigma(t,S)\nu(t,r)\frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}\nu(r,t)^2\frac{\partial^2 V}{\partial r^2} \end{array} \right\} dt \\ &\quad + \sigma(t,S)\frac{\partial V}{\partial S}dZ + \nu(r,t)\frac{\partial V}{\partial r}dZ_r. \end{aligned} \tag{4.3}$$

Now assume that the price of risk for interest rate risk is given by $\gamma(t,r)$ and define

$$\delta(t,r) = \mu(t,r) - \gamma(t,r)\nu(t,r). \tag{4.4}$$

We have

$$dr = \delta(t,r)dt + \nu(t,r)(dZ_r + \gamma(t,r)dt), \tag{4.5}$$

or

$$dr = \delta(t,r)dt + \nu(t,r)dZ_r{}^* \tag{4.6}$$

where

$$dZ_r{}^* = dZ_r + \gamma(t,r)dt. \tag{4.7}$$

In the case of random interest rates we assume that there exists a probability measure for our bivariate distribution of salary and interest rates so that under this new measure Q,

$$E^Q\left[dZ^*\right] = 0 \;\text{ and }\; E^Q\left[dZ_r^*\right] = 0. \tag{4.8}$$

The expected change in value of the benefits under the change of measure, after allowing for the benefit cash flows, should equal the risk free interest rate since we have "risk-adjusted" the benefit value under this change of measure. Thus,

$$E^Q[dV] = rVdt - \sum_{i=d,w} \mu_i(t;x) B_i(t,S,I;x) dt \tag{4.9}$$

where r is the instantaneous riskless rate of interest.

The pension benefit value then satisfies the following partial differential equation

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \theta(t,S;x)\frac{\partial V}{\partial S} + g(t,S)\frac{\partial V}{\partial I} + \delta(t,r)\frac{\partial V}{\partial r} \\
&+\frac{1}{2}\sigma(t,S)^2\frac{\partial^2 V}{\partial S^2} + \rho(t)\sigma(t,S)\nu(t,r)\frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}\nu(t,r)^2\frac{\partial^2 V}{\partial r^2} \\
&-rV + \sum_{i=d,w} \mu_i(t;x) B_i(t,S,I;x) = 0.
\end{aligned} \tag{4.10}$$

Results from the term structure literature, as in Baxter and Rennie [1] or Musiela and Rutkowski [5], allow for interest rate contingent cash flows. In the pension case the value of the cash flows is contingent on both the interest rate and salary (growth rate)

5 Analytical Solutions

5.1 Log-normal Salary and Benefits Based on Final Salary

In order to derive analytical solutions, we make some simplifying assumptions. The function α will be assumed to be of the form $\alpha(x+t)S$ so that the growth rate in the salary will be a percentage of current salary and this percentage depends only on current age. The percentage is determined from the salary scale used for the actuarial valuation of the plan. We will assume that $\sigma(S) = \sigma S$ where σ is a constant, so that salary will have a log-normal distribution. The higher the level of salary, the higher the variance of changes in salary. Then we have

$$dS = \alpha(x+t)Sdt + \sigma S dZ \quad S(t_0, x) = S_0(x) \tag{5.1}$$

where $t_0 \geq 0$ is initial time and $S_0(x)$ is known. Now assume that the price of risk for salary is given by $\lambda(t;x)$ and define

$$\theta(t;x) = \alpha(x+t) - \lambda(t;x)\sigma. \tag{5.2}$$

Thus

$$dS = \theta(t;x)Sdt + \sigma S dZ^* \tag{5.3}$$

where, under the probability measure Q, dZ^* is an increment of a standard Weiner process. Thus, for $t_0 \leq t < T$ we have

$$S(t;x) = S_0(x)\exp\left(\int_{t_0}^{t}\theta(\tau;x)d\tau - \frac{1}{2}\sigma^2(t-t_0) + \sigma(Z^*(t) - Z^*(t_0))\right) \tag{5.4}$$

and

$$E^Q\left[S(t;x)|S(t_0;x)\right] = S_0(x)\exp\left(\int_{t_0}^{t}\theta(\tau;x)d\tau\right). \tag{5.5}$$

Note that the salary at time t has a log-normal distribution with

$$E^Q\left[\ln S(t_0;x)|\ln S(t_0;x)\right] = \ln S_0(x) + \int_{t_0}^{t}\theta(\tau;x)dt - \frac{1}{2}\sigma^2(t-t_0)$$

and

$$Var^Q\left[\ln S(t;x)|\, S(t_0;x)\right] = \sigma^2(t-t_0)$$

We will initially assume that the pension plan pays retirement benefits based on a member's final salary with no averaging. The case of geometric averaging with a log-normal salary assumption will be considered later. Insurance benefits are assumed to be paid from the fund on death and disability of a member based on a multiple of the salary at the date of death. If a member withdraws from the plan, because of resignation, then they are assumed to receive the value of their benefits at the date of withdrawal as a transfer payment to another pension plan.

The benefit assumptions are as follows.

(i) The death and disability benefit is a constant multiple of salary, kS. Thus the sum at risk at death is,

$$B_d(t,S;x) = kS - V(t,S;x). \tag{5.6}$$

(ii) The withdrawal benefit is equal to the value of the benefits in the fund at withdrawal. Thus the sum at risk on withdrawal is

$$B_w(t,S;x) = V(t,S;x) - V(t,S;x) = 0. \tag{5.7}$$

(iii) The benefit paid for retirement is a multiple of final salary based on the number of years membership in the plan. The multiple used reflects the value of the pension annuity paid at retirement. Thus,

$$B_r(T,S;x) = a(T)S(T) - V(T,S;x) \tag{5.8}$$

where

$$T = 65 - x \tag{5.9}$$

(iv) Retirement does not occur prior to age 65.

5.2 PDE Approach

Under these simplifying assumptions, the benefit values satisfy the following PDE and the final condition:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \theta(t;x)S\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} - \left(r(t) + \mu_d(t;x)\right)V + \mu_d(t;x)kS = 0, \\ \text{at} \quad t = T = 65 - x: \quad V = a(T)S(T). \end{cases} \tag{5.10}$$

It turns out from (5.1) that when S reaches zero it remains at zero. Thus the boundary condition on $S = 0$ now is

$$\text{on} \quad S = 0: \quad V = 0. \tag{5.11}$$

In what follows, we derive the analytical solution to (5.10)– (5.11). Using the existence and uniqueness of problem (5.10)– (5.11) and noticing the special form of (5.10)– (5.11), we guess that the solution $V(t,S;x)$ is proportional to S, i.e.,

$$V(t,S;x) = C(t;x)S \tag{5.12}$$

where $C(t;x)$ will be specified later. Substituting (5.12) into (5.10), we have

$$\begin{cases} \dfrac{dC(t;x)}{dt} + (\theta(t;x) - (r(t) + \mu_d(t;x))C(t;x) + \mu_d(t;x)k = 0, \\ \text{at} \quad t = T = 65 - x: \quad C(T;x) = a(T). \end{cases} \tag{5.13}$$

The solution $C(t;x)$ is:

$$\begin{aligned} C(t;x) &= a(T)\exp\left(\int_t^T (\theta(\tau;x) - (r(\tau) + \mu_d(\tau;x)))d\tau\right) \\ &+ \int_t^T \mu_d(\tau_1;x)k\exp\left(\int_t^{\tau_1} (\theta(\tau;x) - (r(\tau) + \mu_d(\tau;x)))d\tau\right)d\tau_1. \end{aligned} \tag{5.14}$$

The solution to (5.10)–(5.11) turns out to be

$$
\begin{aligned}
& V(t,S;x) \\
= \;& a(T)S(t;x)\exp\left(\int_t^T (\theta(\tau;x)-(r(\tau)+\mu_d(\tau;x)))\,d\tau\right) \\
+ \;& S(t;x)\int_t^T \mu_d(\tau_1,x)k\exp\left(\int_t^{\tau_1}(\theta(\tau;x)-(r(\tau)+\mu_d(\tau;x)))d\tau\right)d\tau_1.
\end{aligned}
\tag{5.15}
$$

Remark 1 *The PDE approach for deriving the analytical solution to the more general death and retirement benefit is considered in the Appendix.*

5.3 Expected Discounted Value Approach

The traditional actuarial approach to valuation of pension benefits is to determine "discounted expected values" of future cash flows for benefits payable in a pension plan. The discounting has to take into account both interest and decrements due to mortality and withdrawal. Since interest rates are usually assumed to be deterministic in the traditional actuarial approach it is necessary to modify this approach if interest rates are stochastic. In this case we use an "expected discounted value" approach where cash flows are discounted at the random short interest rates before taking expected values. Expected values are taken with respect to a "risk-neutral" or "risk-adjusted" probability measure.

In what follows, we derive the analytical solution to the problem considered in Section 5.1 using the expected discounted value approach (EDV).

In actuarial mathematics, as in Bowers et al [2], the age-at-death or withdrawal from a pension fund is a random variable. Consider the case of a death benefit. Let X be the age-at-death random variable. The survivorship function is

$$s(x) = 1 - F_X(x)\,, \tag{5.16}$$

where $F_X(x)$ is the distribution function of X. The force of mortality, or hazard rate, is defined as

$$\mu(x) = -\frac{s'(x)}{s(x)}. \tag{5.17}$$

We also have

$$s(x) = \exp\left(-\int_0^x \mu(\tau)d\tau\right). \tag{5.18}$$

Now consider the random variable $T(x)$ the future life time of a life aged x. We have

$$\Pr\left(T(x) > t | T(x) > t_0\right) = \frac{s(x+t)/s(x)}{s(x+t_0)/s(x)} = \exp\left(-\int_{t_0}^t \mu(x+\tau)d\tau\right), \tag{5.19}$$

so that the conditional probability distribution function of $T(x)$ is

$$F_{T(x)|T(x)>t_0}(t) = 1 - \frac{s(x+t)/s(x)}{s(x+t_0)/s(x)} = 1 - \exp\left(-\int_{t_0}^t \mu(x+\tau)d\tau\right). \tag{5.20}$$

and the conditional probability density function is

$$f_{T(x)|T(x)>t_0}(t) = \mu(x+t)\exp\left(-\int_{t_0}^t \mu(x+\tau)d\tau\right) \tag{5.21}$$

Finally, consider a fixed amount of B payable on the death of the life aged x, provided death occurs prior to age $T = 65 - x$, and with deterministic interest rate $r(\tau)$ at time τ. The expected discounted value of the payment at time $t = t_0$ is

$$\begin{aligned} & E\left[B\exp\left(-\int_{t_0}^{T(x)} r(s)ds\right)\middle| T(x) > t_0\right] \\ = \ & B\int_{t_0}^T \exp\left(-\int_{t_0}^t r(\tau)d\tau\right)\mu(x+t)\exp\left(-\int_{t_0}^t \mu(x+\tau)d\tau\right)dt \\ = \ & B\int_{t_0}^T \mu(x+t)\exp\left(-\int_{t_0}^t (r(\tau)+\mu(x+\tau))d\tau\right)dt. \end{aligned} \tag{5.22}$$

To illustrate how this approach is applied to the pension fund case, we consider the pension fund with benefits on death and disability at time t as a constant multiple of salary, $kS(t;x)$, and retirement benefits of a constant multiple of salary of $aS(T;x)$ at age 65. We assume interest rates are deterministic and that salary has a log-normal distribution. We assume that the random variable $T(x)$ and stochastic process $S(t;x)$ are independent.

Using the double expectation theorem, we have

$$\begin{aligned} V(t_0,S;x) &= E^Q \text{ discounted future cash flows}|T(x)>t_0] \\ &= E^Q[\, E^{T(x)|T(x)>t_0}[\text{ discounted future cash flows}|S(t)]] \end{aligned} \tag{5.23}$$

Combining (5.19), (5.22) and (5.23), noticing that $\mu(x+t)=\mu_d(t;x)$, we get

$$\begin{aligned} &V(t_0,S;x) \\ &= E^Q\left[E^{T(x)|T(x)>t_0}\left[\begin{array}{l} kS(T(x);x)\exp\left(-\int_{t_0}^{T(x)} r(\tau)d\tau\right) \\ +aS(T;x)I_{\{T(x)>T|T(x)>t_0\}}\exp\left(-\int_{t_0}^{T} r(\tau)d\tau\right) \end{array} \middle| S(t) \right]\right] \\ &= E^Q\left[\begin{array}{l} \int_{t_0}^{T} kS(t;x)\mu_d(t;x)\exp\left(-\int_{t_0}^{t}(r(\tau)+\mu_d(\tau;x))d\tau\right)dt \\ +aS(T;x)\exp\left(-\int_{t_0}^{T}(r(\tau)+\mu_d(\tau;x))\,d\tau\right) \end{array}\right]. \end{aligned} \tag{5.24}$$

On a "risk-adjusted" basis we use the salary inflation rate adjusted for the price of risk with $\theta(t;x)=\alpha(x+t)-\lambda(t;x)\sigma$. Thus,

$$\begin{aligned} &V(t_0,S;x) \\ &= E^Q\left[\begin{array}{l} \int_{t_0}^{T} k\mu_d(t;x)\exp\left(-\int_{t_0}^{t}(r(\tau)+\mu_d(\tau;x))d\tau\right)\times \\ S(t_0;x)\exp\left(\int_{t_0}^{t}\theta(\tau;x)d\tau-\frac{1}{2}\sigma^2(t-t_0)+\sigma(Z^*(t)-Z^*(t_0))\right)dt \\ +a\exp\left(-\int_{t_0}^{T}(r(\tau)+\mu_d(\tau;x))d\tau\right)\times \\ S(t_0;x)\exp\left(\int_{t_0}^{T}\theta(\tau;x)d\tau-\frac{1}{2}\sigma^2(T-t_0)+\sigma(Z^*(T)-Z^*(t_0))\right) \end{array}\right] \end{aligned} \tag{5.25}$$

where $Z^*(T)$ is a standard Weiner process under the "risk-adjusted" measure. The value of benefits is therefore

$$V(t_0,S;x) = kS(t_0;x)\int_{t_0}^{T}\mu_d(t;x)\exp\left(\int_{t_0}^{t}(\theta(\tau;x)-(r(\tau)+\mu_d(\tau;x)))d\tau\right)dt$$

$$+aS(t_0;x)\exp\left(\int_{t_0}^{T}(\theta(\tau;x)-(r(\tau)+\mu_d(\tau;x)))d\tau\right). \qquad (5.26)$$

which is the same as in (5.15).

We should mention that the EDV approach is more intutive for this simpler case. The PDE approach provides an alternative approach for solving more general cases (see Appendix). If numerical techniques are required then the PDE approach has been implemented numerically for a range of similar problems in the financial and mathematical literature.

Note that we needed the price of risk for salary in order to evaluate the expectation, and also to solve the PDE. The parameters of the process for salary inflation can be determined using empirical data and the salary scales used for the pension valuation. The process for determining the expected salary growth rates are well established in the actuarial literature. However, determining the price of risk for salary growth is a greater challenge. One approach would be to consider salary growth at an economy wide level and to assume that only the risk at this level should be priced in the value of pension benefits. A relationship between salary growth and economy wide productivity could be used to estimate a price of salary risk based on securities whose cash flows depended on productivity. This is an area worthy of further research.

5.4 Log-normal Salary and Benefits Based on Continuous Geometric Average Salary

Assume that the average salary used is a continuous geometric average and define

$$I(t;x)=\left\{\int_0^t \ln S(\tau;x)d\tau\right\}. \qquad (5.27)$$

In this case, as shown by Kemna and Vorst (1990) [3], $I(t;x)$ has a log-normal distribution with

$$E^Q\left[I(t;x)\right]=\ln S_0(x)+\frac{1}{2}t\left[\int_0^t \alpha(x+\tau)d\tau-\frac{1}{2}\sigma^2 t\right]$$

and

$$Var^Q\left[I(t;x)\right]=\frac{1}{3}\sigma^2 t^3.$$

For this special continuous average, it would be possible to obtain an analytical solution using the approach of either PDE or EDV similar to that used in Sections 5.2 and 5.3. Note that for more complex forms of averaging for the salary it will be necessary to use numerical techniques to determine numerical values for the benefit payments in the pension plan.

6 Numerical Techniques

In practice it will be necessary to carry out valuation for all of the possible ages at entry and periods of membership in the pension plan in order to value the payments from the fund. The analytical formulae can be used to value simple cases but for many benefit payments, especially those based on discrete averaging or arithmetic averaging of the salary, it will be necessary to compute the values numerically. Solving the PDE is one approach. If interest rates are allowed to be stochastic then this will require solution of a two-dimensional PDE. The alternative approach would be to evaluate the "expected discounted value" solution using simulation and variance reduction. For example the use of quasi-random numbers could make this an extremely efficient method for valuing these benefits. The issues in numerical evaluation of benefit values are left for future research.

7 Conclusion

This paper has set out a PDE approach to the valuation of pension benefits based on salary. The approach incorporates the forces of decrement of traditional actuarial mathematics and a valuation principle based on the arbitrage-free approach of financial economics. We develop analytical formulae for simple cases by solving the PDE and by modifying the traditional discounted expected value approach found in the actuarial literature.

An important factor required is the price of risk for salary inflation. How to determine this in practice is a difficult issue worthy of further investigation.

8 Appendix

In this appendix we derive the analytical solution for more general death and retirement benefits. We assume that

(i) $S(t;x)$ is log-normal distrubution;

(ii) The death and disability benefit is a function of S. Thus,

$$B_d(t,S;x) = K(t,S;x) - V(t,S;x); \tag{8.1}$$

(iii) The withdrawal benefit is equal to the value of the benefits in the fund at withdrawal. Thus the sum at risk on withdrawal is

$$B_w(t,S;x) = V(t,S;x) - V(t,S;x) = 0. \tag{8.2}$$

(iv) The benefit paid for retirement is a given function of final salary $A(S)$ Thus,

$$B_r(T,S;x) = A(S) - V(T,S;x) \tag{8.3}$$

where

$$T = 65 - x \tag{8.4}$$

(iv) Retirement does not occur prior to age 65.

Under these simplifying assumptions, the benefit values satisfy the PDE and the final condition as follows.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \theta(t;x)S\dfrac{\partial V}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} - (r(t) + \mu_d(t;x))V + \mu_d(t;x)K(t,S;x) = 0, \\ \text{at} \quad t = T = 65 - x: \quad V = A(S). \end{cases} \tag{8.5}$$

It turns out from (5.1) that the boundary condition on $S=0$ now is

$$\text{on} \quad S = 0: \quad V = 0. \tag{8.6}$$

In what follows, we derive the analytical solution to (8.5)– (8.6). Since this is a linear equation, the solution to (8.5) can be expressed as the sum of V_1 and V_2 with V_1 satisfying

$$\begin{cases} \dfrac{\partial V_1}{\partial t} + \theta(t;x)S\dfrac{\partial V_1}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V_1}{\partial S^2} - (r(t) + \mu_d(t;x))\, V_1 = 0, \\ \text{at} \quad t = T: \quad V_1 = A(S) \\ \text{on} \quad S = 0: \quad V_1 = 0 \end{cases} \tag{8.7}$$

and V_2 satisfying

$$\begin{cases} \dfrac{\partial V_2}{\partial t} + \theta(t;x) S \dfrac{\partial V_2}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V_2}{\partial S^2} - (r(t) + \mu_d(t;x)) V_2 + \mu_d(t;x) K(t,S;x) = 0, \\ \text{at} \quad t = T: \quad V_2 = 0 \\ on \quad S = 0: \quad V_2 = 0. \end{cases} \tag{8.8}$$

To solve (8.7), we make the following substitutions:

$$\overline{S} = S e^{\xi(t)} \tag{8.9}$$

$$\overline{V} = V_1 e^{\zeta(t)} \tag{8.10}$$

$$\bar{t} = \eta(t) \tag{8.11}$$

where $\xi(t)$, $\zeta(t)$ and $\eta(t)$ are to be chosen carefully so as to eliminate all time dependent coefficients in (8.7). In these new variables, (8.7) becomes

$$\eta'(t)\frac{\partial \overline{V}}{\partial \bar{t}} + (\theta(t;x) + \xi'(t))\overline{S}\frac{\partial \overline{V}}{\partial \overline{S}} + \frac{1}{2}\sigma^2 \overline{S}^2 \frac{\partial^2 \overline{V}}{\partial \overline{S}^2} - (r(t) + \mu_d(t;x) + \zeta'(t))\overline{V} = 0, \tag{8.12}$$

where $\eta'(t) = d\eta/dt$ etc. Now eliminate the coefficients of $\overline{V}$ by choosing

$$\zeta(t) = \int_t^T (r(\tau) + \mu_d(\tau;x))d\tau, \tag{8.13}$$

and remove the remaining time dependence by setting

$$\xi(t) = \int_t^T \theta(\tau;x)d\tau - \frac{1}{2}\sigma^2(T-t), \tag{8.14}$$

$$\eta(t) = \frac{1}{2}\sigma^2(T-t). \tag{8.15}$$

With these choices (8.7) becomes

$$\frac{\partial \overline{V}}{\partial \bar{t}} = \overline{S}^2 \frac{\partial^2 \overline{V}}{\partial \overline{S}^2} + \overline{S}\frac{\partial \overline{V}}{\partial \overline{S}}, \quad 0 < \overline{S} < \infty \tag{8.16}$$

which has coefficients independent of time. The final condition and boundary condition now are:

$$\text{at} \quad \bar{t} = 0: \quad \overline{V} = A(\overline{S}) \tag{8.17}$$

$$on \quad \overline{S} = 0: \quad \overline{V} = 0. \tag{8.18}$$

We now make the following substitution:

$$\overline{S} = e^y. \tag{8.19}$$

Then $\overline{V}(\bar{t}, y)$ is the solution to the following heat equaion:

$$\begin{cases} \dfrac{\partial \overline{V}}{\partial \bar{t}} = \dfrac{\partial^2 \overline{V}}{\partial y^2}, & -\infty < y < \infty, \\ \bar{t} = 0: \quad \overline{V} = A(e^y). \end{cases} \tag{8.20}$$

Therefore,

$$\overline{V}(\bar{t}, y) = \frac{1}{2\sqrt{\pi \bar{t}}} \int_{-\infty}^{\infty} A(e^z) e^{-\dfrac{(z-y)^2}{4\bar{t}}} dz. \tag{8.21}$$

It turns out that

$$\overline{V}(\bar{t}, \overline{S}) = \frac{1}{2\sqrt{\pi \bar{t}}} \int_0^{\infty} \frac{A(S')}{S'} e^{-\dfrac{(\log S' - \log \overline{S})^2}{4\bar{t}}} dS'. \tag{8.22}$$

Then, the analytical solution to (8.5) is:

$$\begin{aligned} V_1(t, S; x) &= \frac{e^{-\int_t^T (r(\tau) + \mu_d(\tau; x)) d\tau}}{\sigma\sqrt{2\pi(T-t)}} \\ &\times \int_0^{\infty} \frac{A(S')}{S'} e^{\dfrac{-(\log(\frac{S'}{S}) - \int_t^T \theta(\tau; x) d\tau + \frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}} dS'. \end{aligned} \tag{8.23}$$

To get an analytical solution to (8.8), we first prove the following Duhamel principle:

Lemma 1 *Suppose that $W(t, S; x, \tau_1)$ is a solution to the following homogenuous equation*

$$\begin{cases} \dfrac{\partial W}{\partial t} + \theta(t; x) S \dfrac{\partial W}{\partial S} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 W}{\partial S^2} - (r(t) + \mu_d(t; x)) W = 0, \ t < \tau_1 \\ at \quad t = \tau_1: \quad W(t, S; x, t) = \mu_d(t; x) K(t, S; x) \\ on \quad S = 0: \quad W(t, S; x, \tau_1) = 0. \end{cases} \tag{8.24}$$

Then

$$V_2(t, S; x) \triangleq \int_t^T W(t, S; x, \tau_1) d\tau_1 \tag{8.25}$$

is a solution to (8.8).

Proof. It turns out from (8.25) that

$$\frac{\partial V_2}{\partial t} = \int_t^T \frac{\partial W}{\partial t} d\tau_1 - W(t,S;x,t) = \int_t^T \frac{\partial W}{\partial t} d\tau_1 - \mu_d(t;x)K(t,S;x) \tag{8.26}$$

$$\frac{\partial V_2}{\partial S} = \int_t^T \frac{\partial W}{\partial S} d\tau_1, \tag{8.27}$$

$$\frac{\partial^2 V_2}{\partial S^2} = \int_t^T \frac{\partial^2 W}{\partial S^2} d\tau_1. \tag{8.28}$$

Now it is easy to verify that V_2 defined by (8.25) is a solution to (8.8). □

Noticing that problem (8.24) is similar to (8.7), we have

$$W(t,S;x,\tau_1) = \frac{e^{-\int_t^{\tau_1} (r(\tau)+\mu_d(\tau;x))d\tau}}{\sigma\sqrt{2\pi(\tau_1 - t)}} \times \int_0^\infty \mu_d(\tau_1;x)K(\tau_1,S;x) e^{\frac{-\left(\log(S'/S) - \int_t^{\tau_1}\theta(\tau;x)d\tau + \frac{1}{2}\sigma^2(\tau_1 - t)\right)^2}{2\sigma^2(\tau_1 - t)}} dS' \tag{8.29}$$

Thus, the analytical solution to (8.5)–(8.6) is

$$V(t,S;x) = V_1(t,S;x) + \int_t^T W(t,S;x,\tau_1)d\tau_1. \tag{8.30}$$

Since S' has a log-normal distribution we note that

$$\begin{aligned}
&\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty e^{\frac{-\left[log(S') - \left(\log S + \int_t^T \theta(\tau;x)d\tau - \frac{1}{2}\sigma^2(T-t)\right)\right]^2}{2\sigma^2(T-t)}} dS' \\
&= \frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_0^\infty S' e^{\frac{-\left[\log S' - \left(\log S + \int_t^T \theta(\tau;x)d\tau - \frac{1}{2}\sigma^2(T-t)\right)\right]^2}{2\sigma^2(T-t)}} \frac{dS'}{S'} \\
&= \exp\left(\log S + \int_t^T \theta(\tau;x)d\tau - \frac{1}{2}\sigma^2(T-t) + \frac{1}{2}\sigma^2(T-t)\right) \\
&= S(t)\exp\left(\int_t^T \theta(\tau;x)d\tau\right).
\end{aligned} \tag{8.31}$$

Thus, in the simpler case of $K(t,S;x) = kS$ and $A(S) = aS$, we have

$$V_1(t,S;x) = aS(t)\, e^{\int_t^T (\theta(\tau;x) - (r(\tau) + \mu_d(\tau;x)))d\tau} \tag{8.32}$$

and

$$W(t,S;x,\tau_1) = \mu_d(\tau_1;x)kSe^{\int_t^{\tau_1} [\theta(\tau;x) - (r(\tau) + \mu_d(\tau;x))]\, d\tau}. \tag{8.33}$$

It is clear that $V(t,S;x) = V_1(t,S;x) + V_2(t,S;x)$ is the same solution we get in Sections 5.2 and 5.3.

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