BUY AND HOLD STRATEGIES IN OPTIMAL PORTFOLIO SELECTION PROBLEMS: COMONOTONIC APPROXIMATIONS

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1. Investment strategies.
2. Buy and hold strategy. Terminal wealth.
3. Upper and lower bounds for the terminal wealth.
4. Optimal portfolio.
1. INVESTMENT STRATEGIES

There are \((m+1)\) securities. One of them is riskfree: 

\[
\frac{dP^0(t)}{P^0(t)} = rdt.
\]

There are \(m\) risky assets: 

\[
\frac{dP^i(t)}{P^i(t)} = \mu_i dt + \sum_{j=1}^{d} \bar{\sigma}_{ij} dW^j(t).
\]

By defining \(B^i(t) = \frac{1}{\sigma_i} \sum_{j=1}^{d} \bar{\sigma}_{ij} W^j(t)\), 

\[
\frac{dP^i(t)}{P^i(t)} = \mu_i dt + \sigma_i dB^i(t), \quad i = 1, \ldots, m.
\]
From the solution to this equation,

\[ P^i(t) = p_i \exp \left[ \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) t + \sigma_i B^i(t) \right], \]

we obtain that the random yearly returns of asset \( i \) in year \( k \), \( Y^i_k \), are independent and have identical normal distributions with

\[
E[Y^i_k] = \mu_i - \frac{1}{2} \sigma_i^2,
\]

\[
\text{Var}[Y^i_k] = \sigma_i^2, \text{ and}
\]

\[
\text{Cov}[Y^i_k, Y^j_l] = \begin{cases} 
0 & \text{if } k \neq l, \\
\sigma_{ij} & \text{if } k = l.
\end{cases}
\]
Let $\Pi(t) = (\Pi_0(t), \Pi_1(t), \ldots, \Pi_m(t))$ denote the vector describing the proportions of wealth invested in each asset at time $t$.

In general, a vector $\Pi(t)$ will define an investment strategy.

If one unit of a security is constructed according to the investment strategy $\Pi(t)$, let $P(t)$ be the price of that unit at time $t$. Then,

$$\frac{dP(t)}{P(t)} = \sum_{i=0}^{m} \Pi_i(t) \frac{dP^i(t)}{P^i(t)} = \left[ \sum_{i=1}^{m} \Pi_i(t)(\mu_i - r) + r \right] dt + \sum_{i=1}^{m} \Pi_i(t) \sigma_i dB^i(t).$$

If $\Pi(t)$ is prefixed, $P(t)$ can be obtained by solving the stochastic differential equation above (constantly rebalanced portfolio).
2. BUY AND HOLD STRATEGY. TERMINAL WEALTH

- The new amounts of money $\alpha(t)$ are invested at time $t = 0, 1, \ldots, n - 1$ in some prefixed proportions $\bar{\pi}(t) = (\bar{\pi}_0(t), \bar{\pi}_1(t), \ldots, \bar{\pi}_m(t))$.

- Fractions $\bar{\pi}_i(t)$ are always the same. Denoting $\bar{\pi}_i(0) = \pi_i$, then $(\bar{\pi}_0(t), \ldots, \bar{\pi}_m(t)) = (\pi_0, \ldots, \pi_m)$, for every $t = 0, 1, \ldots, n - 1$.

- New quantities are invested once in a period of time (typically, once in a year), i.e.,

$$\alpha(t) = \begin{cases} 
\alpha_i & \text{if } t = i, \text{ for } i = 0, 1, \ldots, n - 1, \\
0 & \text{otherwise}.
\end{cases}$$

- The decision maker follows a buy and hold strategy, i.e., no securities are sold.

Buy & Hold Strategies

Coomonotonic Approximations
Objective: To compute the terminal wealth $W_n(\pi)$ for a given buy and hold strategy $\pi = (\pi_0, \pi_1, \ldots, \pi_m)$.

Let $Z^i_j$ be the sum of returns of 1 unit of capital invested at time $t = j$ of asset $i$ from time $t = j$ to the final time $t = n$,

$$Z^i_j = \sum_{k=j+1}^{n} Y^i_k.$$

The terminal wealth invested in asset $i$ is

$$W^i_n(\pi) = \sum_{j=0}^{n-1} \pi_i \alpha_j e^{Z^i_j},$$

whereas the terminal wealth will be given by

$$W(\pi) = \sum_{i=0}^{m} W^i(\pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{Z^i_j}.$$
3. UPPER AND LOWER BOUNDS FOR THE TERMINAL WEALTH

Let $X = (X_1, X_2, \ldots, X_n)$ and let $S = X_1 + X_2 + \cdots + X_n$. It can be shown that

$$S^l \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} S^c,$$

where $S^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$ and $S^l = \sum_{i=1}^{n} E[X_i | \Lambda]$. 

Buy & Hold Strategies

Ccomonotonic Approximations
If \( S = \sum_{i=1}^{n} \bar{\alpha}_i e^{\bar{Z}_i} \) with \( \bar{\alpha}_i \geq 0, \)

\[
S^c = \sum_{i=1}^{n} F^{-1}_{\bar{\alpha}_i e^{\bar{Z}_i}}(U) = \sum_{i=1}^{n} \bar{\alpha}_i e^{E[\bar{Z}_i]+\sigma_{\bar{Z}_i} \Phi^{-1}(U)}.
\]

For a given \( \Lambda = \sum_{j=1}^{n} \gamma_j \bar{Z}_j, \)

\[
S^l = \sum_{i=1}^{n} \bar{\alpha}_i E[e^{\bar{Z}_i} \mid \Lambda] = \sum_{i=1}^{n} \bar{\alpha}_i e^{E[\bar{Z}_i]+\frac{1}{2}(1-r_i^2)\sigma_{\bar{Z}_i}^2 + r_i \sigma_{\bar{Z}_i} \Phi^{-1}(U)}.
\]

We need values of \( \gamma_j \) that minimize of the “distance” between \( S \) and \( S^l. \)
‘Maximal Variance’ lower bound approach. As we have that \( \text{Var}[S] = \text{Var}[S^l] + \mathbb{E}[\text{Var}[S | \Lambda]] \), it seems reasonable to choose the coefficients \( \gamma_j \) such that the variance of \( S^l \) is maximized:

\[
\gamma_k = \bar{\alpha}_k \, e^{\mathbb{E} \left[ \bar{Z}_k \right] + \frac{1}{2} \sigma^2_{\bar{Z}_k}}.
\]

‘Taylor-based’ lower bound approach. \( \Lambda \) is a linear transformation of a first order approximation to \( S \):

\[
\gamma_k = \bar{\alpha}_k \, e^{\mathbb{E} \left[ \bar{Z}_k \right]}.
\]
Comonotonic Upper Bound B&H strategy:

\[ W^c(\pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2}\sigma_i^2)} + \sqrt{n-j}\sigma_i\Phi^{-1}(U). \]

Note that \( W^c(\pi) \) is a linear combination of fractions \( \pi_i \), \( i = 0, \ldots, m \).

Comonotonic Lower Bound B&H strategy:

\[ W^l(\pi) = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2}r_{ij}^2\sigma_i^2)} + r_{ij}\sqrt{n-j}\sigma_i\Phi^{-1}(U) \]

where the correlation coefficients \( r_{ij} \) are given by

\[ r_{ij}^{MV} = \frac{\sum_{k=0}^{m} \sum_{l=0}^{n-1} \pi_k \alpha_l (n - \max(j, l))\sigma_{ik}e^{(n-l)\mu_k}}{\sigma_i \left[ (n-j)\sum_{s,k=0}^{m} \sum_{t,l=0}^{n-1} \pi_s \pi_k \alpha_t \alpha_l (n - \max(t, l))\sigma_{sk}e^{(n-t)\mu_s + (n-l)\mu_k} \right]^{1/2}}. \]
and

\[ r_{ij} = \frac{\sum_{k=0}^{m} \sum_{l=0}^{n-1} \pi_k \alpha_l (n - \max(j, l)) \sigma_{ik} e^{(n-l)\left[\mu_k - \frac{1}{2} \sigma_k^2\right]}}{\sigma_i (n - j)^{1/2}} \cdot \\
\left[ \sum_{s,k=0}^{m} \sum_{t,l=0}^{n-1} \pi_s \pi_k \alpha_t \alpha_l (n - \max(t, l)) \sigma_{sk} e^{(n-t)\left[\mu_s - \frac{1}{2} \sigma_s^2\right] + (n-l)\left[\mu_k - \frac{1}{2} \sigma_k^2\right]} \right]^{-1/2} \]
Numerical illustration: 2 risky, 1 risk-free. $\mu_1 = 0.06$, $\mu_2 = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.2$, Pearson’s correlation: 0.5, $r = 0.03$.

Every period $\alpha_i = 1$, invested in proportions: 19% risk-free asset, 45% first risky asset, 36% in the second risky asset. This amount is invested for $i = 0, \ldots, 19$, whereas in $i = 20$ the invested amount is $\alpha_{20} = 0$. The simulated results were obtained with 500,000 random paths.
<table>
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<tr>
<th>$p$</th>
<th>$MC$</th>
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<th>$LB_T$</th>
<th>$UB$</th>
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<td>123.99</td>
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<td>-0.93%</td>
<td>+22.08%</td>
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Buy & Hold Strategies

Comonotonic Approximations
4. OPTIMAL PORTFOLIO

Possible criteria: maximizing an expected utility, Yaari’s dual theory of choice under risk:

\[
\max_{\pi} \rho_f[W_n(\pi)] = \max_{\pi} \int_0^\infty f(\Pr(W_n(\pi) > x))dx,
\]

risk measures (some of them correspond to distorted expectations \( \rho_f[W_n(\pi)] \) for appropriate choices of the distortion function \( f \)).
Value at Risk at level $p$:

$Q_p[X] = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}$. If $F_X$ is an strictly increasing function, then it coincides with the related risk measure

$Q_p^+[X] = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}$,  $p \in (0, 1)$.

Additive for sums of comonotonic risks.

Conditional Left Tail Expectation at level $p$ ($CLTE_p[X]$):

$$CLTE_p[X] = E \left[ X \mid X < Q_p^+[X] \right] , \quad p \in (0, 1).$$
For the upper and lower bounds in B&H strategy:

\[
Q_p[W^c(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2} \sigma_i^2) + \sqrt{n-j} \sigma_i \Phi^{-1}(p)},
\]

\[
Q_p[W^l(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{(n-j)(\mu_i - \frac{1}{2} r_{ij}^2 \sigma_i^2) + r_{ij} \sqrt{n-j} \sigma_i \Phi^{-1}(p)},
\]

\[
CLTE_p[W^c(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{\mu_i(n-j)} \frac{1 - \Phi(\sqrt{n-j} \sigma_i - \Phi^{-1}(p))}{p},
\]

\[
CLTE_p[W^l(\pi)] = \sum_{i=0}^{m} \sum_{j=0}^{n-1} \pi_i \alpha_j e^{\mu_i(n-j)} \frac{1 - \Phi(\sqrt{n-j} r_{ij} \sigma_i - \Phi^{-1}(p))}{p}.
\]
Maximizing the Value at Risk: for a given probability $p$ and a given investment strategy, let $K_p(\pi)$ be the $p$-target capital defined as the $(1 - p)$-th order ‘+’-quantile of terminal wealth, $K_p(\pi) = Q_{1-p}^+[W(\pi)]$.

For the optimal case:

$$K_p^* = \max_{\pi} Q_{1-p}^+[W(\pi)].$$

Alternatives:

$$K_p^{c*} = \max_{\pi} Q_{1-p}^+[W^c(\pi)]$$ or $$K_p^{l*} = \max_{\pi} Q_{1-p}^+[W^l(\pi)].$$
<table>
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</tr>
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<tr>
<td>$K^*$</td>
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<tr>
<td>$K^*$</td>
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<td>25.805</td>
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Buy & Hold Strategies  Comonotonic Approximations
Maximizing the CLTE: \( \max_{\pi} CLTE_{1-p}[W(\pi)] \). This optimization problem describes decisions of risk averse investors.

The \( CLTE_{1-p} \) has the following nice property (lacking with the VaR):

\[
CLTE_{1-p}[W^c(\pi)] \leq CLTE_{1-p}[W(\pi)] \leq CLTE_{1-p}[W^l(\pi)].
\]

Alternative: \( \max_{\pi} CLTE_{1-p}[W^l(\pi)] \).
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