

A stochastic control model for individual asset-liability management*

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Abstract

In the tradition of Merton (1969, 1971) we seek to describe the optimal behaviour of an individual through his lifetime. Our model is based on Richard (1975), which includes optimal insurance and annuity demand. We extend that work by modelling labour income as a stochastic process, explicitly recognising the market incompleteness posed by salaries, as opposed to the deterministic income flows assumed in Richard. A closed-form solution is not available for this finite horizon problem. We adopt the Markov chain technique of Kushner & Dupuis (2001) to solve the model. Our solution provides support for hump shaped consumption, age-phased investment and optimal life insurance rules related to income levels.

Keywords: Asset-liability modelling, optimal portfolio selection, financial planning, life insurance, annuities, stochastic control.

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1 Introduction

For the vast majority of workers human capital is an extremely significant component of their total wealth. The income process is thus highly influential on individual decision making. The Richard (1975) model assumes income is deterministic. Yet the salary flows that make up human capital are not deterministic. Given the importance of these flows a relevant model of life cycle financial planning should allow for the stochastic nature of income.

In this paper we extend the Richard model to allow for stochastic income. How is an agent's behaviour affected by the random changes in salary? In a world of fluctuating income, how will an individual consume, invest, insure and annuitise? While solutions of the optimal consumption and investment problem with random income exist in discrete time, none involve consideration of optimal life insurance and annuity demand—the contribution of Richard. In addition, the solution methodology adopted here, starting with the problem couched in continuous time, is novel.

Bodie, Merton & Samuelson (1992) flagged the importance of considering random income in the formulation of the optimal consumption and investment problem. Since their initial work a number of papers have appeared which present solutions of the problem. Recent work includes that of Viciara (2001), Campbell & Viciara (2002, Chapter 7) and Koo (1998).

The model we develop, based on an extension of Richard to allow for stochastic income, is detailed in section 2. Its parameterisation is discussed in section 3. The numerical solution of this problem follows in section 4. We use the Markov chain technique of Kushner & Dupuis (2001), extended to allow for two state variables, wealth and income. Computational details are given in section 5.

The results from the solution of the extended model are discussed in section 6. The expected consumption path over the life cycle is now humped shaped to age 65, then increasing and convex. Gourinchas & Parker (2002) also found hump shaped consumption in a model with stochastic income and attributed this result to initial precautionary saving (buffer stock consumption) with life cycle saving at later ages to save for retirement and bequest motives. We speculate that this explanation for consumption also holds for our model.

Compared with the Richard model we find the extreme investment in risky assets while young tempered to lower levels, while exhibiting age phasing over the life cycle. Also, expected life insurance demand is higher later in working life, while annuity demand is lower.

Section 7 concludes the paper.

2 The model

We now turn to the model we will explore throughout the rest of this chapter. As we have done earlier (Purcal & Piggott 2001*a*, Purcal & Piggott 2001*b*), we work with the Richard model, a continuous-time finite-horizon model of an optimising agent.

Our objective is to describe the optimal behaviour of an individual throughout his lifetime in the presence of stochastic income, as opposed to the deterministic income flows assumed in the Richard (1975) model. A closed-form solution is not available for this problem and we have to use numerical methods. As we have done earlier we will adopt the Markov chain technique of Kushner & Dupuis (2001) to solve the model. This gives us an opportunity to explore the properties of this solution method with multiple state variables—a much more involved problem.

Campbell & Viciera have recently presented a solution to the lifetime optimal consumption and investment problem with stochastic income for a finite horizon discrete time model. Our modelling below complements this work. It differs though as it stems from a continuous time framework and uses an entirely different solution approach. It also addresses the question of optimal life insurance and annuity demand, not touched upon by Campbell & Viciera. With respect to the continuous time literature, Koo pointed out that a solution to the finite horizon consumption and investment problem with stochastic income was an open question. The following work provides a solution to this question, albeit a numeric one.

2.1 The Richard model

Richard models a multi-period utility maximizing investor with objective¹

$$\max E\left[\int_{\tau}^T U(C(t), t)dt + B(Z(T), T)\right], \quad (1)$$

where T is the investor's uncertain time of death, and U , C , Z and B are the investor's utility, consumption, legacy at death and utility from bequest. The investor is able to choose between two securities, one risky and one risk-free, with the price of the risky asset, Q , following geometric Brownian motion

$$\frac{dQ(t)}{Q(t)} = \alpha dt + \sigma dq(t), \quad (2)$$

where $dq(t)$ is a Wiener increment.

¹For more extensive discussion of the Richard model see Purcal & Piggott (2001*a*).

The investor's change in wealth is given by the stochastic differential equation

$$dW(t) = -C(t)dt - P(t)dt + Y(t)dt + rW(t)dt + (\alpha - r)\pi(t)W(t)dt + \sigma\pi(t)Wdq(t), \quad (3)$$

where $P(t)$, $Y(t)$, $W(t)$ are, respectively, the investor's life insurance premium paid, income (assumed to be non-stochastic), and wealth at time t . From equation (2), the mean return on risky investment is α , with standard deviation σ , while the risk-free investment returns r ; the investor places a proportion π of wealth in the risky asset.

Richard's model necessarily incorporates the probability of death of an investor. Let the investor's age-at-death, X , a continuous random variable, have a cumulative distribution function given by $F(x)$ and probability density function of $f(x)$. Consequently, $S(x) = 1 - F(x)$ gives the probability that the investor lives to age x . The function $S(x)$ is known as the survival function. The conditional probability density function (the probability the investor dies at exact age x , having survived to that age) is given by $f(x)/S(x)$, and is known as the *force of mortality* by demographers and actuaries, or as the *hazard rate* or *intensity rate* by reliability theorists (Elandt-Johnson & Johnson 1980).

The investor buys instantaneous term life insurance to the amount of $Z(t) - W(t)$. For this, a premium of $P(t)$ is paid. If we denote the force of mortality by $\mu(t)$, then the amount of premium paid for actuarially fair insurance will be

$$P(t) = \mu(t)(Z(t) - W(t)). \quad (4)$$

The investor's problem is to solve equation (1), subject to budget constraint (3) and initial wealth condition $W(0) = W_0$, by optimal choice of controls C , π and Z . U is assumed to be strictly concave in C and B is assumed strictly concave in Z . In fact, Richard takes these functions as

$$U(C(t), t) = h(t) \frac{C^\gamma(t)}{\gamma}, \quad \gamma < 1, h > 0, C > 0 \quad (5)$$

$$B(Z(t), t) = m(t) \frac{Z^\gamma(t)}{\gamma}, \quad \gamma < 1, h > 0, Z > 0. \quad (6)$$

He is able to determine analytic solutions to his model with these assumptions.

2.2 Extending the Richard model

We extend the Richard model by no longer assuming the income process, $Y(t)$, is deterministic. Koo and others (Viciera 2001, Campbell & Viciera 2002) have

modelled the stochastic salary process as having an expected exponential path. In continuous-time, we have

$$dY/Y = \hat{\alpha}dt + \hat{\sigma}d\hat{q}, \quad (7)$$

where $\hat{\alpha}$ represents the expected growth rate of labour income and $\hat{\sigma}$ its volatility. We can represent our modelling of the resulting two controlled processes as follows:

$$\begin{pmatrix} dW \\ dY \end{pmatrix} = \begin{pmatrix} -C - P + Y + rW + (\alpha - r)\pi W \\ \hat{\alpha}Y \end{pmatrix} dt + \begin{pmatrix} \sigma\tilde{\pi}W & 0 \\ 0 & \hat{\sigma}Y \end{pmatrix} \begin{pmatrix} dq \\ d\hat{q} \end{pmatrix}, \quad (8)$$

where the first row is equation (3) of the Richard model and the second is equation (7) above. Note that in this model it is financial wealth, W , which is our state variable. Earlier when working with the Richard model we used \tilde{W} , or adjusted wealth, as a state variable.

In (8) we have assumed that the two Brownian motions, $\hat{q}(t)$ and $q(t)$, are not correlated. Empirical evidence exists to support such a view, and this will be discussed in the parameterisation section 3 below. This assumption also has technical merits.

By assuming idiosyncratic income we are able to use the finite difference approach used earlier in implementing the Markov chain solution technique of Kushner & Dupuis. As Kushner & Dupuis (pp. 108–13) point out, for a finite difference approach to work the off-diagonal terms of the covariance matrix must not be large with respect to the diagonal terms. If this condition doesn't hold the approximating Markov chain must be directly constructed (Kushner & Dupuis, pp. 113–22). Such a scheme is considerably more involved, and we leave it as a task for future research.

With the addition of equation (8) to the Richard model we have moved from a complete market model to an incomplete market model. With $Y(t)$ as a deterministic process income could be perfectly replicated by traded assets. As a stochastic process, however, $Y(t)$ can only be imperfectly hedged.

3 Parameterisation

In solving this extended version of the Richard model we have used the Japanese parameter values employed in our earlier work (Purcal & Piggott 2001a). There we advanced arguments that appropriate forms for $h(t)$ and $m(t)$ in (5) and (6) are $e^{-\rho t}$ and $e^{-\rho t} \phi(t)^{1-\gamma}$ respectively, where $\phi(t) = \int_t^\omega \exp(-r(\theta - t))d\theta$. This

Table 1 Economic and financial data for Japan.

	Wages ¹	Prices ²	Nikkei(Real) ³	Nikkei(Nom) ⁴	Bill Rate ⁵
Mean	5.9%	3.9%	4.1%	8.2%	4.7%
Volatility	2.2%	2.5%	26.5%	25.4%	0.9%
Range	(-1.9%, 29.1%)	(-1.1%, 24.7%)	(-40.9%, 95.1%)	(-38.6%, 106.6%)	(0.5%, 12.2%)

Sources:

1. CEIC database, Japanese Nominal Wage Index, All Industries, Monthly, Jan.1970-Jun.2000.
2. CEIC database, Japanese Consumer Price Index, Monthly, Jan.1970-Jun.2000.
3. Gary Burtless, The Brookings Institution, Washington, DC (2001), Real Total Return of Japan Stock, 1970-2000.
4. Gary Burtless, The Brookings Institution, Washington, DC (2001), Real Total Return of Japan Stock, 1970-2000.
5. CEIC database, Japanese Three Month Bank of Japan Bill Rate, Monthly, Jan.1970-Jun.2000.

Table 2 Parameters used in the numerical solution of the model.

α	0.025
r	0.005
ρ	0.005
σ	0.2
<i>Mortality</i>	JLT18 (male)
γ	-0.5
ω	110
$Y (=W(0))$	¥ 4 375 686

implies the investor wishes to provide a family income benefit—a term certain annuity to his surviving spouse, which pays $2/3C^*(t)$ from the date of death to the limiting age of the mortality table.

The average annual real salary growth rate for Japan was 1.92% p.a. The parameter $\hat{\alpha}$ was consequently set to this value. The standard deviation of annual salary growth rates was 2.2% and this was the value taken for $\hat{\sigma}$. These economic and financial data we used to parameterise the model are summarised in table 1. In table 2 we set out the other parameters we adopt for the model. The values we adopt reflect real values of asset accumulation, hence α , the rate of return on the risky asset, is chosen as the real rate of return on the Nikkei: $(1.064/1.039) - 1 \approx 0.025$. The safe rate, r , is similarly chosen. We adopt Japanese male population mortality given by the Ministry of Health and Welfare (1995) Japanese Life Table, number 18, excluding the effects of the Kōbe earthquake. A value of γ of -0.5 reflects an individual who is somewhat risk averse. We set the individual's yearly earnings to 12 times the average monthly cash earnings of regular employees for

calendar year 1999: $12 \times 364\,638 = 4\,375\,656$ yen.²

We assume this amount remains constant over the employee's working life. The employee works from age 30 to 65, at which point he must retire.

What is new here is the extension of the model with equation (7), and it is to the parameterisation of this equation that we now turn.

A number of authors have refined the modelling of an exponential salary process to allow for time varying coefficients (Campbell & Viciera 2002, Gourinchas & Parker 2002). Such an extension is also appropriate for the case of Japan. The age-earnings profile there also exhibits a hump shape, rather than steady exponential growth (Hart & Kawasaki 1999, Chapter 9). Such an extension could be readily incorporated into our modelling and solution technique. In the interests of simplicity we have, however, left this as a task for future research.

Another refinement to the modelling of lifetime earnings is the inclusion of correlation between the risky investment and salary processes. In Campbell & Viciera (2002, Chapter 7) the authors, using U.S. Panel Study of Income Dynamics (PSID) and Center for Research in Security Prices (CRSP) data, found correlations between thirty and fifty per cent between the income and risky assets processes. As mentioned in subsection 2.2 above we have assumed zero correlation between these processes in our modelling of Japanese lifetime earnings. In fact, for the period January 1971 to June 2000 we found only a very small degree of correlation, 2.2%, between trailing Nikkei returns and all industry wage growth. The two series also exhibited no statistically significant linear relationship over that period. Various empirical studies support such a view (Davis & Willen 2000) for the case of Japan.

4 Solving the model

The HJB equation for our maximum problem (1) together with the controlled processes in (8) is as follows:

$$\begin{aligned}
 0 = \max_{C,Z,\pi} & \left\{ \mu(t)\phi(t)B(Z(t)) + U(C(t)) - \mu(t)J - \rho J + J_t \right. \\
 & + [\pi\alpha W + (1 - \pi)rW - C - P + Y]J_W + \hat{\alpha}YJ_Y \\
 & \left. + \frac{1}{2}\sigma^2\pi^2W^2J_{WW} + \frac{1}{2}J_{YY}\hat{\sigma}^2Y^2 \right\}. \quad (9)
 \end{aligned}$$

We will approximate the solution of this using finite difference techniques as we did in Purcal & Piggott (2001*b*).

²These values are from the Japanese Institute of Labour webpage, located at <http://www.jil.go.jp/estatis/e0301.htm>

Following our earlier approach, we approximate the partial derivatives in (9) as follows:

$$\begin{aligned} f_t(W, Y, t) &\rightarrow \frac{f(W, Y, t + \delta) - f(W, Y, t)}{\delta}, \\ f_W(W, Y, t) &\rightarrow \frac{f(W + h, Y, t + \delta) - f(W, Y, t + \delta)}{h} \quad \text{for } d^+, \end{aligned}$$

where d^+ represents the group of positive coefficients of J_W in (9), i.e. $d^+ = rW + (\alpha - r)\pi W + Y$, and

$$f_W(W, Y, t) \rightarrow \frac{f(W, Y, t + \delta) - f(W - h, Y, t + \delta)}{h} \quad \text{for } d^-,$$

where d^- represents the group of negative coefficients of J_W in (9), i.e. $d^- = C + P$, and

$$\begin{aligned} f_{WW}(W, Y, t) &\rightarrow \frac{f(W + h, Y, t + \delta) + f(W - h, Y, t + \delta) - 2f(W, Y, t + \delta)}{h^2}, \\ f_Y(W, Y, t) &\rightarrow \frac{f(W, Y + h, t + \delta) - f(W, Y, t + \delta)}{h}, \\ f_{YY}(W, Y, t) &\rightarrow \frac{f(W, Y + h, t + \delta) + f(W, Y - h, t + \delta) - 2f(W, Y, t + \delta)}{h^2}, \end{aligned}$$

and

$$\begin{aligned} f_{WY}(W, Y, t) &\rightarrow \frac{2f(W, Y, t) + f(W + h, Y + h, t + \delta) + f(W - h, Y - h, t + \delta)}{2h^2} \\ &\quad - \frac{f(W + h, Y, t + \delta) + f(W - h, Y, t + \delta)}{2h^2} \\ &\quad - \frac{f(W, Y + h, t + \delta) + f(W, Y - h, t + \delta)}{2h^2}. \end{aligned}$$

Using the approximations above, we can re-write (9) as follows, where $V(\cdot, \cdot, \cdot)$ represents the solution to the resulting finite difference equation:

$$\begin{aligned} 0 = \max_{C, Z, \tilde{\pi}} &\left\{ \mu(t)\phi(t)\tilde{B}(Z(t)) + \tilde{U}(C(t)) - (\mu(t) + \rho)V(W, Y, t) \right. \\ &+ \frac{V(W, Y, t + \delta) - V(W, Y, t)}{\delta} \\ &+ \frac{V(W + h, Y, t + \delta) - V(W, Y, t + \delta)}{h}d^+ \\ &\left. - \frac{V(W, Y, t + \delta) - V(W - h, Y, t + \delta)}{h}d^- \right\} \end{aligned}$$

$$\begin{aligned}
& + \hat{\alpha} Y \frac{V(W, Y + h, t + \delta) - V(W, Y, t + \delta)}{h} \\
& + \frac{1}{2} (\sigma \pi W)^2 \frac{V(W + h, Y, t + \delta) + V(W - h, Y, t + \delta) - 2V(W, Y, t + \delta)}{h^2} \\
& + \frac{1}{2} \hat{\sigma}^2 Y^2 \frac{V(W, Y + h, t + \delta) - 2V(W, Y, t + \delta) + V(W, Y - h, t + \delta)}{h^2} \Big\}.
\end{aligned} \tag{10}$$

Equation (10) will be used for two purposes. Firstly, we will use it to determine the optimal values of the three control variables. Secondly, it will be carefully rearranged to give a recursive Markov chain optimisation problem. Such a form is very useful in the numerical solution of our original problem (9). In addition we can use this form to demonstrate that the Markov chain representation of the problem is locally consistent in the sense of Kushner & Dupuis (2001). Kushner (1977) proved such locally consistent finite difference schemes converge to the true solution.

4.1 Optimal controls

We solve (10) for the three controls by differentiating with respect to each, and setting the results equal to zero. This produces the following results:

$$C^* = \left[\frac{V(W, Y, t + \delta) - V(W - h, Y, t + \delta)}{h} \right]^{\frac{1}{\gamma-1}}, \tag{11}$$

$$Z^* = \phi(t) \left[\frac{V(W, Y, t + \delta) - V(W - h, Y, t + \delta)}{h} \right]^{\frac{1}{\gamma-1}}, \tag{12}$$

and

$$\begin{aligned}
\pi^* &= \frac{h(\alpha - r)W}{(\sigma W)^2} \\
&\quad \times \frac{V(W, Y, t + \delta) - V(W + h, Y, t + \delta)}{V(W + h, Y, t + \delta) - 2V(W, Y, t + \delta) + V(W - h, Y, t + \delta)},
\end{aligned} \tag{13}$$

which are identical to our results in the one-dimensional case.

4.2 Markov chain representation

We can obtain a recursive Markov chain representation of (10). We proceed as follows. Rearrange (10):

$$0 = \max_{C, Z, \bar{\pi}} \left\{ \mu(t) \phi(t) \tilde{B}(Z(t)) + \tilde{U}(C(t)) \right\}$$

$$\begin{aligned}
& - \left(\mu(t) + \rho + \frac{1}{\delta} \right) V(W, Y, t) \\
& + \left(\frac{1}{\delta} - \frac{d^+}{h} - \frac{d^-}{h} - \frac{(\sigma\pi W)^2}{h^2} - \frac{\hat{\alpha}Y}{h} - \frac{(\hat{\sigma}Y)^2}{h^2} \right) \\
& \quad \times V(W, Y, t + \delta) \\
& + \left(\frac{d^+}{h} + \frac{(\sigma\pi W)^2}{2h^2} \right) V(W + h, Y, t + \delta) \\
& + \left(\frac{d^-}{h} + \frac{(\sigma\pi W)^2}{2h^2} \right) V(W - h, Y, t + \delta) \\
& + \left(\frac{\hat{\alpha}Y}{h} - \frac{(\hat{\sigma}Y)^2}{2h^2} \right) V(W, Y + h, t + \delta) \\
& + \left(\frac{(\hat{\sigma}Y)^2}{2h^2} \right) V(W, Y - h, t + \delta) \}. \tag{14}
\end{aligned}$$

We can express (14) more elegantly as follows:

$$\begin{aligned}
& V(W, Y, t) \\
& = \max_{C, Z, \tilde{\pi}} \left\{ \frac{1}{1 + \delta\mu + \delta\rho} \left[\delta \left\{ \mu(t)\phi(t)\tilde{B}(Z(t)) + \tilde{U}(C(t)) \right\} \right. \right. \\
& \quad \left. \left. + \sum_{\theta=-1}^1 \sum_{\phi=-1}^1 \hat{p}(W + \theta h, Y + \phi h) \right. \right. \\
& \quad \left. \left. \times V(W + \theta h, Y + \phi h, t + \delta) \right] \right\} \tag{15}
\end{aligned}$$

where the $\hat{p}(\cdot, \cdot)$ may be interpreted as transition probabilities of a Markov chain. The notation $\hat{p}(\cdot, \cdot)$ refers to the probability of moving from the current state (W, Y) at time t to the state (\cdot, \cdot) at time $t + \delta$. The probabilities in (15) take the following values:

$$\hat{p}(W - h, Y - h) = 0, \tag{16}$$

$$\hat{p}(W, Y - h) = \frac{\delta}{h^2} \left(\frac{1}{2}(\hat{\sigma}Y)^2 \right), \tag{17}$$

$$\hat{p}(W + h, Y - h) = 0, \tag{18}$$

$$\hat{p}(W - h, Y) = \frac{\delta}{h^2} \left(hd^- + \frac{1}{2}(\sigma\pi W)^2 \right), \tag{19}$$

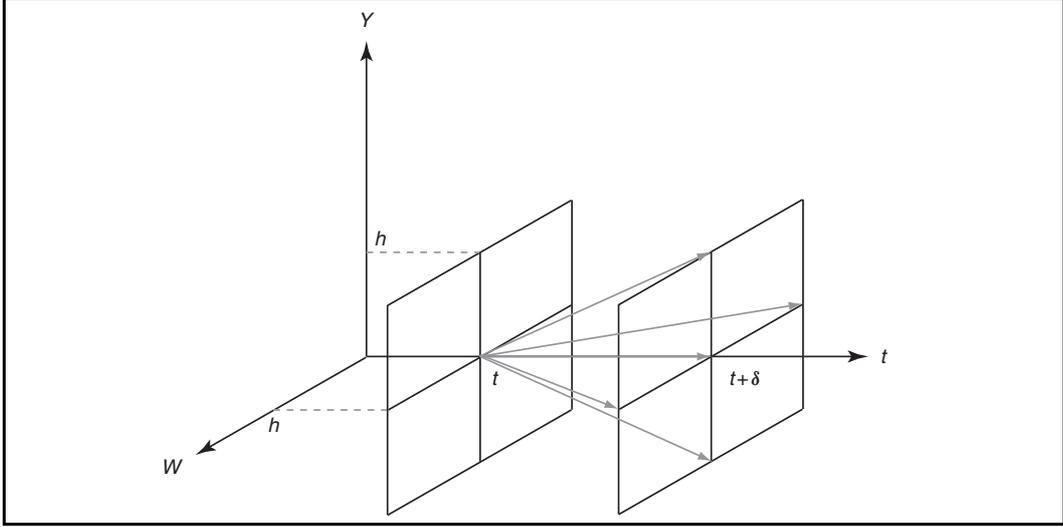


Figure 1: Markov chain transitions.

$$\hat{p}(W, Y) = 1 - \frac{\delta}{h^2} \left(hd^+ + hd^- + (\sigma\pi W)^2 + h\hat{\alpha}Y + (\hat{\sigma}Y)^2 \right), \quad (20)$$

$$\hat{p}(W + h, Y) = \frac{\delta}{h^2} \left(hd^+ + \frac{1}{2}(\sigma\pi W)^2 \right), \quad (21)$$

$$\hat{p}(W - h, Y + h) = 0, \quad (22)$$

$$\hat{p}(W, Y + h) = \frac{\delta}{h^2} \left(h\hat{\alpha}Y + \frac{1}{2}(\hat{\sigma}Y)^2 \right), \quad (23)$$

and

$$\hat{p}(W + h, Y + h) = 0. \quad (24)$$

Thus the scheme allows only nine possible local movements from the initial position (W, Y) . Of the nine possible movements, only five are permitted. (See figure 1.) This is a consequence of the zero off-diagonal values of the variance-covariance matrix in (8); non-zero values would lead to more possible local movements.

4.3 Local consistency

The above probabilities define a locally consistent Markov chain approximation, in the sense of Kushner & Dupuis (2001), to the model described in equations (1)–(8). This can be demonstrated by calculating $\mathbf{E} \left[\begin{pmatrix} W(t + \delta) - W(t) \\ Y(t + \delta) - Y(t) \end{pmatrix} \right]$ and $\mathbf{var} \left[\begin{pmatrix} W(t + \delta) - W(t) \\ Y(t + \delta) - Y(t) \end{pmatrix} \right]$ and showing these values are close to the expectations and variances of the dynamic processes underlying the original model, given

by the stochastic differential equations in (8) above.

Firstly, let us examine $\mathbf{E} \left[\begin{pmatrix} W(t + \delta) - W(t) \\ Y(t + \delta) - Y(t) \end{pmatrix} \right]$:

$$\begin{aligned} \mathbf{E} \left[\begin{pmatrix} W(t + \delta) - W(t) \\ Y(t + \delta) - Y(t) \end{pmatrix} \right] &= h \frac{\delta}{h^2} \begin{pmatrix} hd^+ + \frac{1}{2}(\sigma\pi W)^2 \\ h\hat{\alpha}Y + \frac{1}{2}(\hat{\sigma}Y)^2 \end{pmatrix} \\ &\quad - h \frac{\delta}{h^2} \begin{pmatrix} hd^- + \frac{1}{2}(\sigma\pi W)^2 \\ \frac{1}{2}(\hat{\sigma}Y)^2 \end{pmatrix} \\ &= \frac{\delta}{h} \begin{pmatrix} hd^+ - hd^- \\ h\hat{\alpha}Y \end{pmatrix} \\ &= \begin{pmatrix} d \\ \hat{\alpha}Y \end{pmatrix} \delta \end{aligned}$$

which approximates $\mathbf{E} \left[\begin{pmatrix} dW(t) \\ dY(t) \end{pmatrix} \right]$ well. For the case of the variance consider first the second moments of the above differences:

$$\begin{aligned} \mathbf{E} \left[\begin{pmatrix} (W(t + \delta) - W(t))^2 \\ (Y(t + \delta) - Y(t))^2 \end{pmatrix} \right] &= h^2 \frac{\delta}{h^2} \begin{pmatrix} hd^+ + hd^- + (\sigma\pi W)^2 \\ h\hat{\alpha}Y + (\hat{\sigma}Y)^2 \end{pmatrix} \\ &= \begin{pmatrix} (\sigma\pi W)^2 \\ (\hat{\sigma}Y)^2 \end{pmatrix} \delta + \begin{pmatrix} hd^+ + hd^- \\ h\hat{\alpha}Y \end{pmatrix} \delta. \end{aligned}$$

Thus

$$\mathbf{var} \left[\begin{pmatrix} W(t + \delta) - W(t) \\ Y(t + \delta) - Y(t) \end{pmatrix} \right] = \begin{pmatrix} (\sigma\pi W)^2 \\ (\hat{\sigma}Y)^2 \end{pmatrix} \delta + o(\delta)$$

which approximates $\mathbf{var} \left[\begin{pmatrix} dW(t) \\ dY(t) \end{pmatrix} \right]$ well as both $\delta \rightarrow 0$ and $h \rightarrow 0$. Hence the Markov chain approximation (15) has the ‘local properties’ of the diffusion processes (8).

We can then rely on Kushner that our finite difference scheme approximation converges to the solution of (1) as both $\delta \rightarrow 0$ and $h \rightarrow 0$. However, as mentioned in Purcal & Piggott (2001*b*), this solution approach does not have any known solution criteria that apply to the values of δ and h in a scheme to guarantee convergence. One must examine the results corresponding to a variety of values of δ and h to determine whether the results are stable.

5 Computational approach

In this section we discuss how the theory treated in the last section, the Markov chain method, was implemented on a computer. The method was implemented in

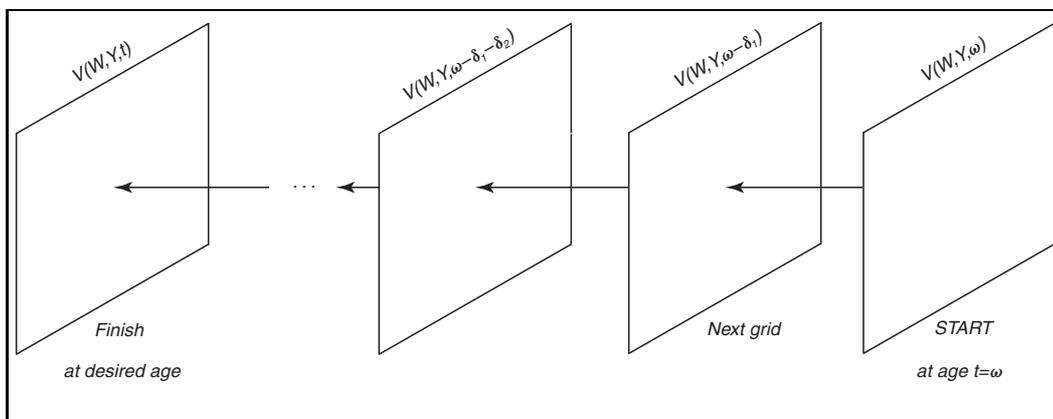


Figure 2: Recursive solution approach.

a recursive fashion on a two-dimensional grid. The details of how the probabilities were modified at particular positions on the grid are given, as is the method used for determining the backward time step at each point.

To run the code in a reasonable time-frame required the use of high performance computing. The code was parallelised by the High Performance Computing Support Unit of the University of New South Wales. Several lines of C code were added to the existing program. The program was then compiled and run on the Compaq AlphaServer SC system of the Australian Partnership for Advanced Computing (APAC) National Facility in Canberra. This system is made up of 125 Compaq AlphaServer ES45's (500 processors) with 1GHz chips, each with at least four gigabytes of memory and seventy-two gigabytes of disk space. Our work made use of only one ES45, or four processors.

5.1 Markov chain implementation

5.1.1 Recursive approach

Equation (15) was solved recursively over a two-dimensional grid of state variables ($W \times Y$) using a computer. The recursion started at the optimising individual's terminal age, $t = \omega$, and moved backwards to a chosen initial age. At the terminal age all individuals must die, and so we can make use of the boundary condition $V(W, Y, \omega) = \phi(\omega)B(Z(\omega))$ to initialise the recursion.

Generally speaking, earlier values of V are then determined by application of equation (15), using the equations (11)–(13) to give the values of the optimal controls, and using equations (16)–(24) to give the transition probabilities. The process is illustrated in figure 2. Note that given the matrix of values of V at a point in time t we then know the optimal value of the three controls at that point in time by virtue of equations (11)–(13). Thus recursing through equation (15)

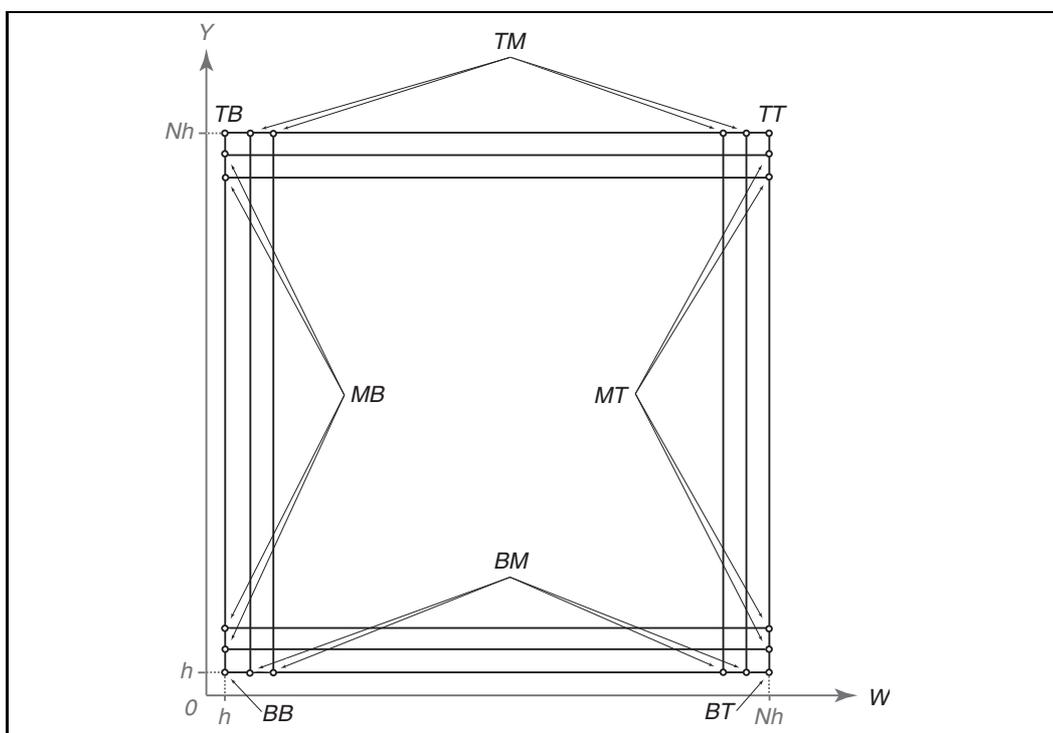


Figure 3: Troublesome transition areas.

and storing the resulting V matrices enables us to know optimal consumption, investment and insurance at *any* point in time, at *any* required wealth/salary combination on the grid. This is a very rich and very useful set of information. In particular, it provides a mapping from an investor's current wealth and income position to his optimal consumption, portfolio and insurance decisions—one of the most important objectives of financial planning.

More specifically, at certain points of the two-dimensional grid both the controls and the probabilities mentioned above need to be modified because we are working on a finite state-space. We turn to these modifications next.

5.1.2 Adjusting controls and probabilities

For the majority of backward transitions from a grid at time $t + \delta$ to a grid at time t , the controls and probabilities given by equations (11)–(13) and (16)–(24) will suffice for our Markov chain approach of equation (15). However, there are some transitions on the grid, notably at the tops and bottoms of the grids, that must be specially dealt with. These troublesome transitions are illustrated in figures 3 and 4. We adopt the notation of these figures for the following discussion. Thus when we speak of transition TB we mean transitions from the top of the salary grid (T) and the bottom of the wealth grid (B).

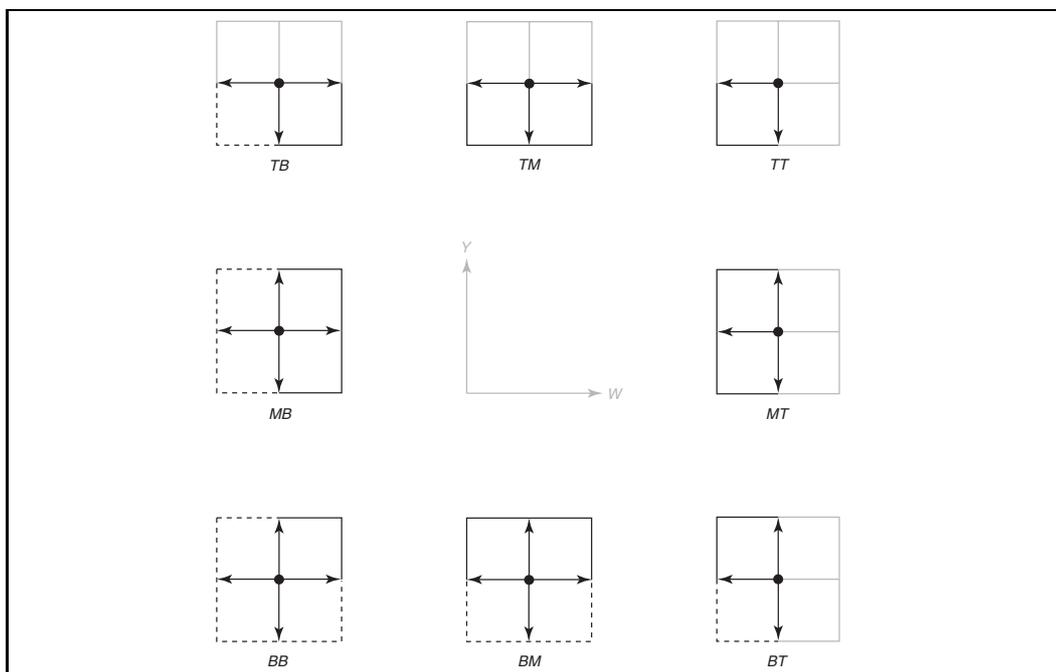


Figure 4: Permissible transitions.

Controls Equations (11) and (12) hold for transitions TT, TM, MT, MM, BT and BM. Transitions TB, MB and BB involve transitions from the bottom of the wealth grid, including the possibility of moving from the lowest wealth value, h , to 0. Although our two-dimensional grid doesn't include zero, we follow our earlier methodology (Purcal & Piggott 2001*b*) and set the value function $V(0, \cdot, \cdot) \equiv V(\cdot, 0, \cdot) \equiv V_0$ to a constant negative large number. In this way (11) and (12) can be calculated for transitions TB, MB and BB.

Calculations of (13) for transitions MB and BB also involve V_0 .

Following our earlier one-dimensional approach the optimal value of π is set to zero whenever we are at the top of a grid. Thus (13) is set to zero for transitions TT, TM, TB, MT and BT.

Probabilities For transitions MM, MB, BM and BB we use probabilities (17), (19), (20), (21) and (23). Transitions TM, TB, MT and BT are more limited, and as result the probabilities we use here are (17), (19), (21) and $1 - (17) - (19) - (21)$. Transition TT is even more limited, and the probabilities used for this transition are (17), (19) and $1 - (17) - (19)$.

5.1.3 Backward time step for the recursion

The backwards time step, δ , for the implementation of the recursive Markov chain equation (15) is determined in a manner similar to our earlier approach in one-dimension (Purcal & Piggott 2001*b*). For the first recursion we use a fixed step size while for all later recursions we calculate a varying step size with the aim of speeding up the solution of the model.

First recursion Here we follow the approach of Fitzpatrick & Fleming (1991), and described in Purcal & Piggott (2001*b*). Recall that this approach involved calculating the largest value of Q , given by the sum of our probabilities (16)–(19) and (21)–(24) multiplied by h^2/δ . In this approach we fix the values of the control variables used in these probabilities to lie within a certain range. By maximising Q we maximise the probability of mixing and this has good convergence properties. The value of δ is then determined by setting h^2/δ equal to Q ; given h and Q we can find δ .

To determine the largest value of Q we consider the following:

$$h(d^+d^- + \hat{\alpha}Y) + (\sigma\pi W)^2 + (\hat{\sigma}Y)^2,$$

which is just the sum of the probabilities referred to above, multiplied by h^2/δ . As we did earlier, we bound the controls C and Z to lie in the set $[0, KNh]$ and the control π to lie in the set $[0, K]$. In that case the maximum value of Q on our two dimensional grid is given by

$$Nh[(\alpha - r)K + r + 1 + \hat{\alpha}] + (\sigma KNh)^2 + (\hat{\sigma}Nh)^2,$$

and we use this to determine the first backwards time step.

Subsequent recursions Here we again follow the approach of Fitzpatrick & Fleming (1991, pp. 836–7), also described in Purcal & Piggott (2001*b*). Recall that this approach involved calculating a value of Q for each grid, then using the resulting value of Q in the *next* recursion.

The adaptive value of Q on a particular two-dimensional grid is determined by selecting the largest value on the grid of the sum of our probabilities (16)–(19) and (21)–(24) multiplied by h^2/δ . In this approach the values of the control variables used in these probabilities are not fixed to lie within a certain range. The value of δ is determined by dividing h^2 by the value of Q we have calculated.

6 Results

The development of the computer program to solve this extension of the Richard model proceeded in a number of steps. In order to check the code for reasonable-

ness it was initially run with zero volatility for the salary process ($\hat{\sigma} = 0$) and zero growth rate ($\hat{\alpha} = 0$). The results were compared with those from the simulation runs of Purcal & Piggott (2001a). They were persuasive—the code approximates the closed-form solution for deterministic income quite well. These results are discussed in subsection 6.1.

Buoyed with this success we then moved on to work with stochastic income. Results for the optimal controls were calculated with a stochastic salary process calibrated to Japanese data ($\hat{\alpha}=1.92\%$, $\hat{\sigma}=2.2\%$). These are sensible. The incorporation of a risky income process produced less consumption, investment in the risky asset and life insurance demand over the life cycle when compared to the case of safe income ($\hat{\alpha}=1.92\%$, $\hat{\sigma}=0.0\%$). These results are discussed in subsection 6.2.1 below.

Lastly we calculated the expected paths of the state and control variables over the pre-retirement period to get a sense of optimal life cycle behaviour with risky income. We find that with risky income less wealth is accumulated, on average. The lifetime expected consumption profile is now hump-shaped, in accordance with empirical evidence. Age-phased investment occurs, with less investment in risky assets when young. More life insurance tends to be bought later in working life. Annuities are smaller on average, and the date of annuitisation is later. We report these findings below in subsection 6.2.2.

6.1 Deterministic income check

As a check the two-dimensional code developed to solve the Richard model was run with zero salary drift and volatility. With such parameter values for equation (7) the stochastic income model collapses to one with deterministic income. The results from such a model can be compared to our earlier work in Purcal & Piggott (2001a) to check its performance.

The results of the comparison between the simulation code of Purcal & Piggott (2001a) and the extended model code with $\hat{\alpha} = \hat{\sigma} = 0$ were tabulated. The three controls are in general accord for ages 35 to 65. At age 30 the controls, most notably the proportion invested in the risky asset, appear to differ between the simulation results and the Markov chain results. This is not altogether surprising, however.

In earlier discussion of Markov chain approximation results (Purcal & Piggott 2001b) we found that the values of controls at both the bottom and top of the grid differed from the correct theoretical value. This is a common problem with numerical methods as we approach boundaries. In this instance, at a wealth and salary level of 4.4 million yen we are very close to the bottom of *both* the wealth grid and salary grid (which represent wealth and salary levels from two hundred thousand yen to two hundred million yen), and we should expect some disagree-

ment. In fact the numerical method is approximating the infinite negative value of the value function by a fixed large negative integer—this will lead to inaccuracies. Also the vast amount of change that occurs in the value function between wealth levels of two hundred thousand yen and zero is condensed in the numerical method to one discrete step.

We can draw comfort from the observation that although the control values do not match at low wealth and salary levels the values are on the conservative side. The numerical results are not pushing the investor to consume and invest more than the theoretical values, but rather the opposite.

Overall the agreement between the theoretical values with deterministic income and the Markov chain approximation method set to deterministic income are quite good and we can proceed to examine the case of stochastic income with some degree of confidence.

6.2 Stochastic income

Before we ran the computer code with the stochastic income parameterisation appropriate for Japan ($\hat{\alpha}=1.92\%$, $\hat{\sigma}=2.2\%$), the simulation code developed earlier (Purcal & Piggott 2001a) was modified to allow for salary growth. In this way we could produce theoretical results corresponding to an environment of deterministic exponential salary growth. This served as another check to the numerical results. Intuitively, and indeed from the existing literature treating the optimal consumption and investment model with stochastic income, we would expect the optimal consumption in an environment of risky income to be smaller than in a safe income environment. We would also expect investment in risky assets to be less.

In addition, we modified our initial wealth parameter from 4 375 656 yen to 10 939 140 yen (two and a half times yearly earnings at age thirty) to mitigate the problem of being at both the bottom of the wealth grid and the bottom of the salary grid.

6.2.1 Optimal controls: comparative statics

In figures 5 through 10 we display cross sections of the age 30 grid output from the solution of our extended Richard model; same cross sections from the age 60 grid are also shown for comparison. These results provide the comparative statics at age 30 for this model: what happens to the optimal controls as we vary wealth, holding income fixed; what happens as we vary income, holding wealth fixed?

We find that salary is very important to an individual's optimal decision-making process. Indeed with respect to life insurance demand, although we have

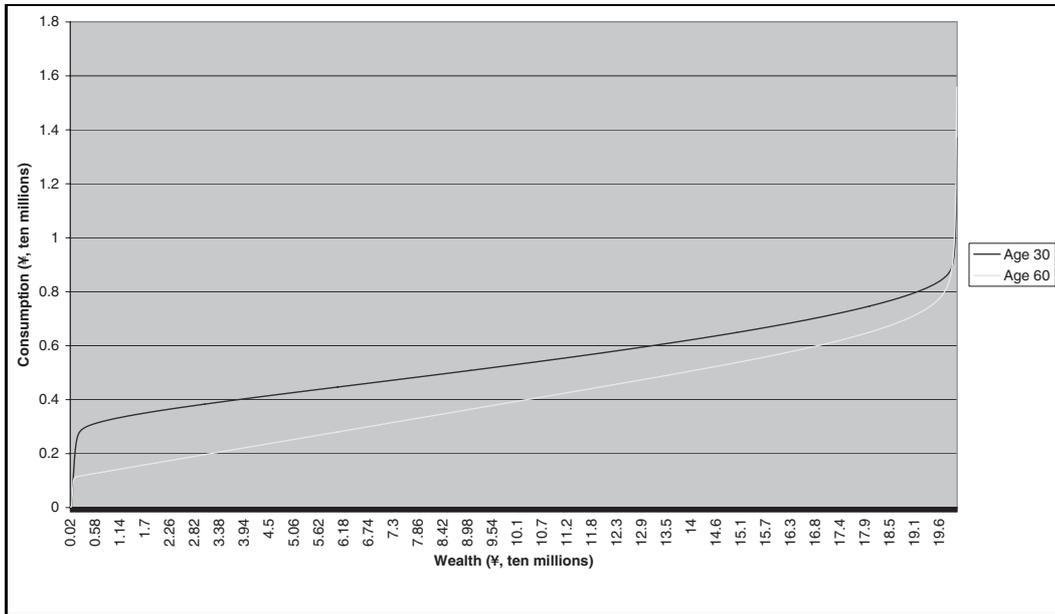


Figure 5: Consumption and wealth relationship, salary fixed.

seen that the human life value concept yields flawed levels of optimal life insurance demand (Purcal & Piggott 2001a), its emphasis on salary as a determinant of life insurance is not misplaced.

Varying wealth, holding income fixed One would expect these results to be similar to the ones in Purcal & Piggott (2001a), which discussed the one dimensional Richard model—with wealth as the only state variable. The results are in fact similar. As before, apart from the behaviour at small wealth values and large wealth values on the grid, consumption in figure 5 appears close to affine in wealth at ages 30 and 60, holding income fixed. The slope of the function is quite low, approximately 0.02, showing relatively little sensitivity of consumption to wealth.

The behaviour of the optimal bequest is similar, but with larger values (roughly from 150 to 350 million yen). The slope is also greater. It has a value of approximately one.

The graph of the optimal proportion in risky, figure 6, displays neither the linearity of the optimal consumption function above nor the constant value of the proportion in risky one observes in the one dimensional Richard case. In the one dimensional case (Purcal & Piggott 2001a) *total* wealth was the state variable; here it is financial wealth. In figure 6 we see the a high proportion of financial wealth invested in risky assets by the poor, which falls in a convex fashion with rising wealth. The wealthier one is, the less they invest in the risky asset.

Note also the rapid rise in the proportion invested in risky at extremely low

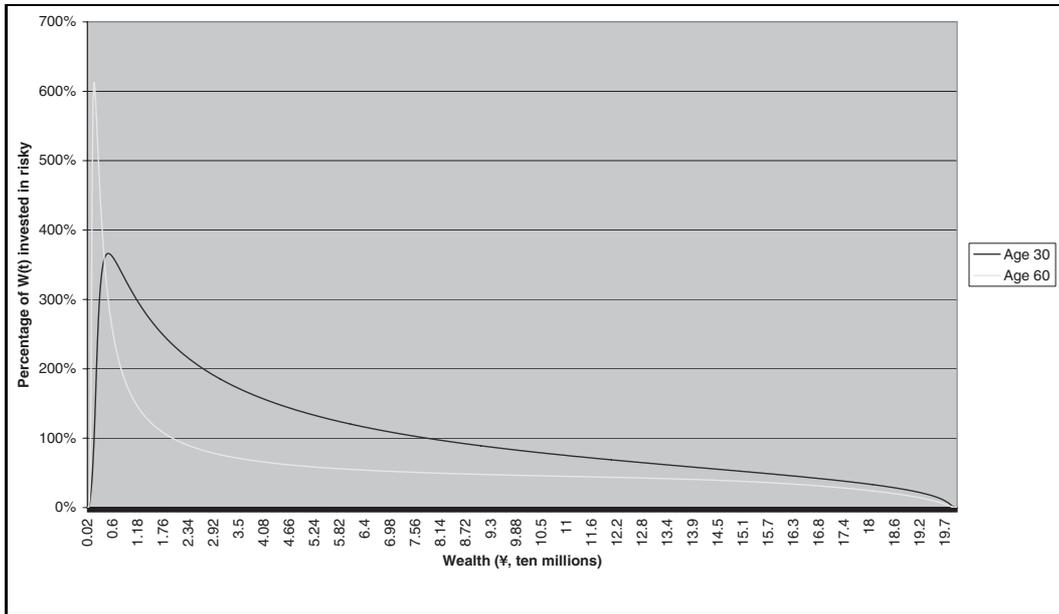


Figure 6: Investment and wealth relationship, salary fixed.

wealth levels. And its concave shape at high values of wealth. These are the results of the numerical approximation. To get accurate values at the extreme values of the state variable one has to work very hard. A much finer grid is required.

In figure 7, we see how the optimal life insurance premium amount varies at age 30 over a wealth cross section. For most of the wealth range the function appears mildly convex. At extremely low wealth values it is concave, rising steeply to a value close to 0.1. At higher wealth levels the function is convex, and rises very sharply at the extreme wealth levels of the grid. It would appear that wealth has largely no effect on life insurance demand at this particular age and salary level.

The above result stems from our choice of bequest function. As wealth rises so does the optimal bequest. As we noted above the slope of the bequest function is roughly one. Thus it is not surprising that the optimum life insurance premium graph is largely constant. It is interesting to note that for two individuals with the same salary their optimal amount of life insurance is roughly the same, regardless of whether they are poor or rich. That is, life insurance demand is wealth inelastic.

This pattern doesn't hold in general. At age 60 there is a clear negative relationship between the optimal insurance premium and wealth. The relationship is almost affine apart from, again, very low values of wealth and for higher values of wealth. The slope of the function is however quite low, roughly around -0.002 , indicating that optimal life insurance is not particularly sensitive to changes in wealth.

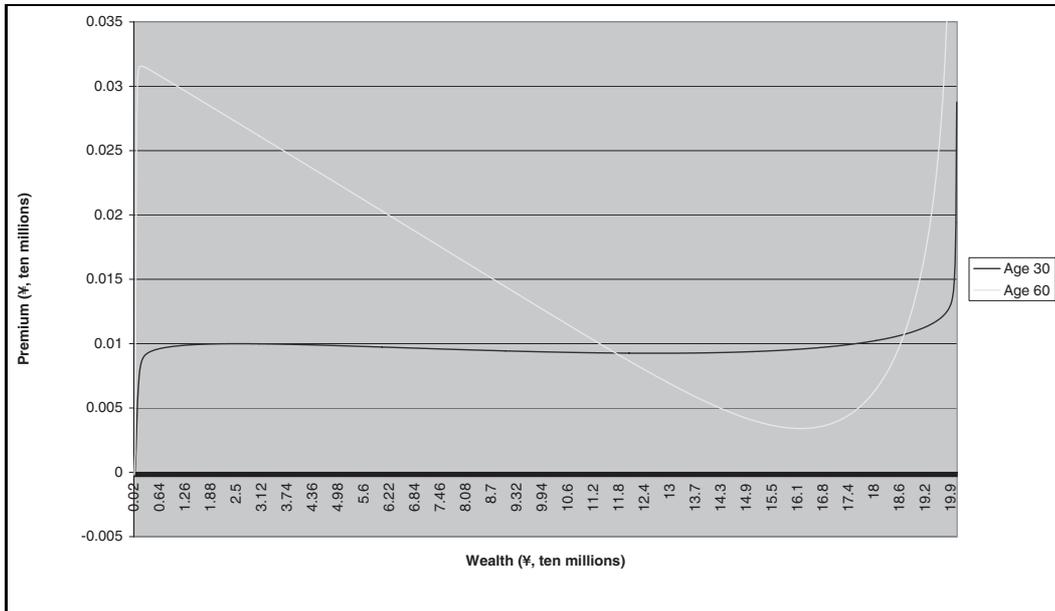


Figure 7: Premium and wealth relationship, salary fixed.

Varying income, holding wealth fixed Salary plays a far more important role in decision making than wealth at younger ages. In figure 8 we see salary and optimal consumption have a largely linear relationship. The slope of the curve is close to one. Compare this with the slope of figure 5 above which illustrated the relationship between optimal consumption and the financial wealth and had a value of about 0.02.

Figure 9 illustrates the relationship between the proportion of financial wealth invested in risky assets as salary increases. Unlike the case of increasing wealth, where the relationship was negative, here increasing salary results in increasing willingness to bear risk. This, no doubt, arises in part from our assumption of zero correlation between the salary and risky asset process.

The relationship between salary and the proportion invested in risky is largely close to affine, apart from low salary levels and high salary levels, where the proportion invested in risky is very sensitive to salary changes. At very low salary levels it is concave, while at very high salary levels it is convex.

The optimal bequest level has a largely linear relationship with salary, as in the case of consumption above. Again this is not surprising as our parameterisation of the model has a bequest function which depends on consumption levels.

In the case of optimal life insurance premium payments the results of varying salary indicate a clear difference with the case of changing wealth. Here, in figure 10 we see a largely linear relationship between the optimal insurance premium paid and salary. The relationship is positive. Young individuals on large salaries

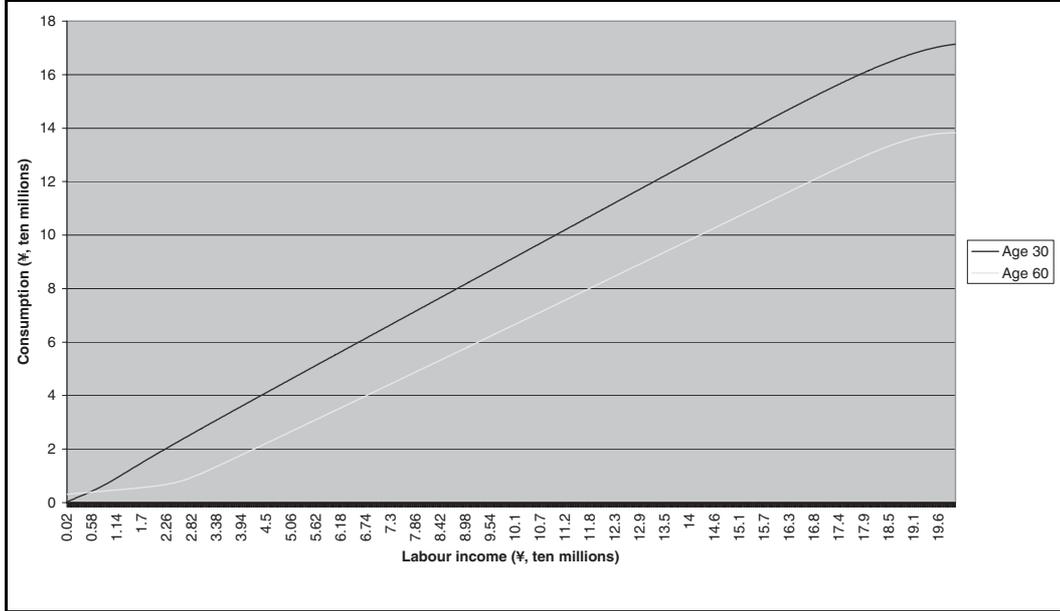


Figure 8: Consumption and salary relationship, wealth fixed.

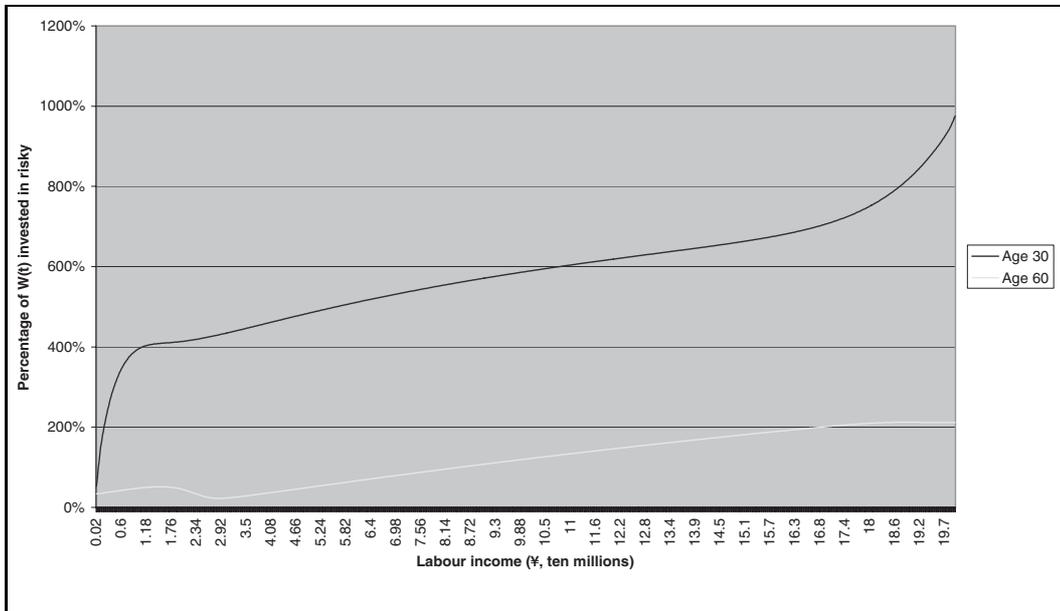


Figure 9: Investment and salary relationship, wealth fixed.

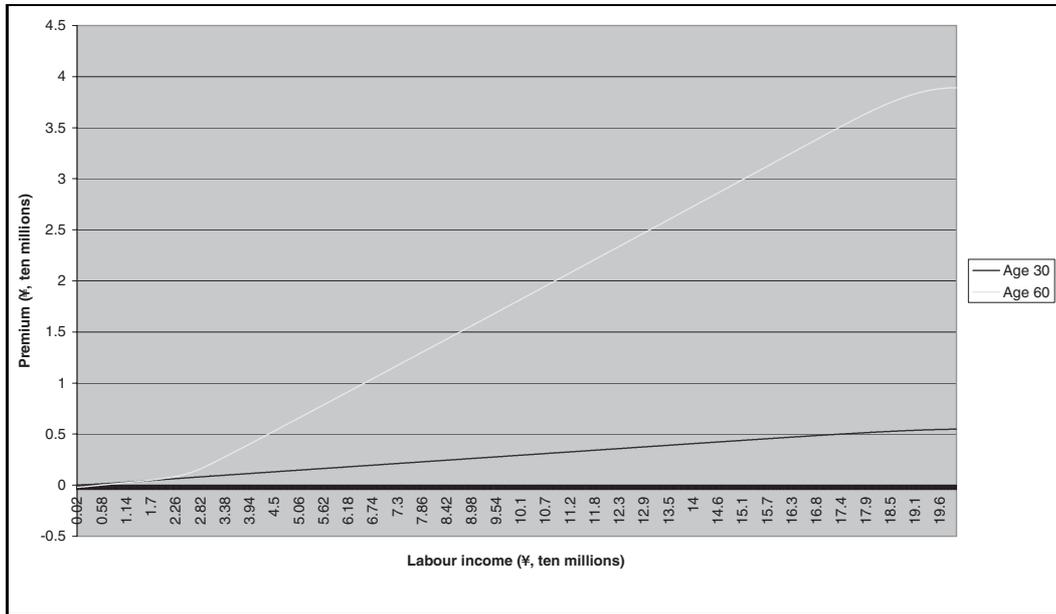


Figure 10: Premium and salary relationship, wealth fixed.

will buy more life insurance than those with low salaries, given the same level of financial wealth.

The human life value concept (Huebner 1964) also implies an increasing linear relationship between salary and premium paid at a fixed age. However, as we have seen earlier (Purcal & Piggott 2001a) the levels differ between the optimal amounts given by the human life value concept and the Richard model. It would appear however that the emphasis of the human life value concept on salary as the determinant of the optimal amount of life insurance was not misplaced.

While there is general similarity between the optimal control results at ages 30 and age 60 there is a startling contrast in the optimal life insurance results between these two age groups. As can be seen from figure 10, the level of optimal life insurance premium is considerably larger at age 60 than at age 30. Two reasons would appear to produce this result.

Firstly the life insurance product we are dealing with here is term insurance, which gets more expensive with age. At age 60 the force of mortality is almost fifteen times greater than that at age 30. Such an increase will result in larger premium costs. The second factor contributing to the large life insurance premiums is the chosen parameterisation of the bequest function. Recall our discussion in section 3 above concerning the implications of the bequest function selected. We are effectively assuming an annuity certain is provided to the insured's spouse on his death. The difference between the costs of an annuity certain and a life annuity at higher ages is considerable. A less generous modelling of the bequest function

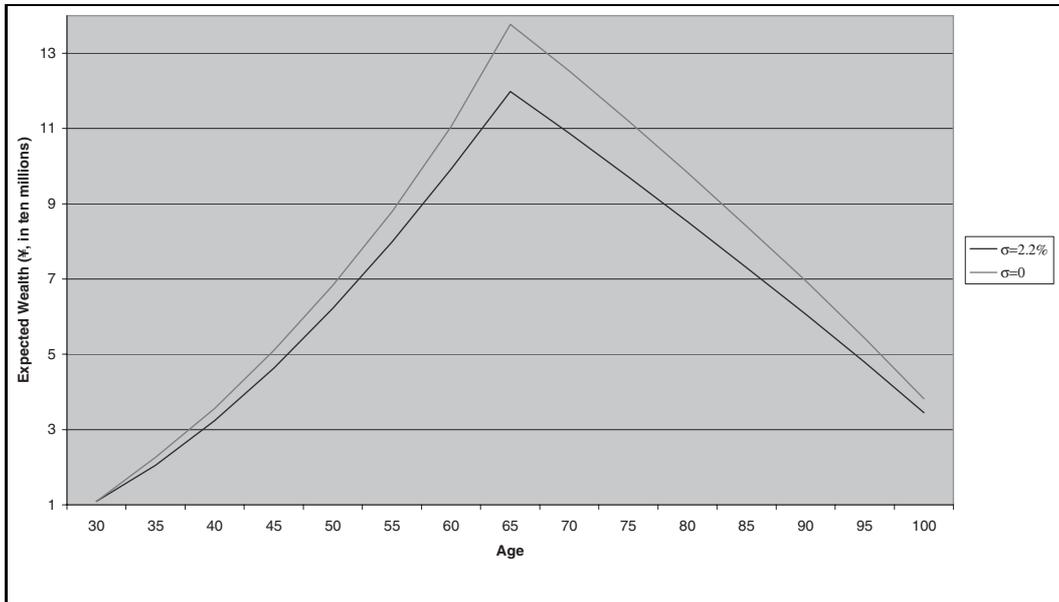


Figure 11: Expected wealth path.

would result in less expenditure on life insurance at age 60.

6.2.2 Expected paths

Using the methodology of Purcal & Piggott (2001b) the expected paths for the state and control variables were calculated in an environment of stochastic income ($\hat{\alpha}=1.92\%$, $\hat{\sigma}=2.2\%$).³ The results, together with the corresponding results for the deterministic income case ($\hat{\sigma}=0$), are presented in figures 11–14. In each instance the expected paths are determined from the perspective of a thirty year old with current wealth of eleven million yen.

In figure 11 we see the effect of including risky labour income in the Richard model. In an environment of risky labour income our thirty year old can expect to accumulate less wealth over his working life than another agent with safe income. In the safe income environment he accumulates 137.7 million yen of financial wealth. In the risky income environment he accumulates only 87% of this—120 million yen.

It is instructive to compare this reduced wealth accumulation in the stochastic labour income model with a similarly reduced wealth accumulation in the deter-

³Using the Markov chain approach described above we have the transition probabilities at every grid point. By simple recursion we can then use these transition probabilities to determine expected values. We do this by using these transition probabilities to move back from a grid at age x ($x > 30$) to the original grid at age 30. This gives us the expected values of a state or control variables at age x conditional on wealth and salary at age 30.

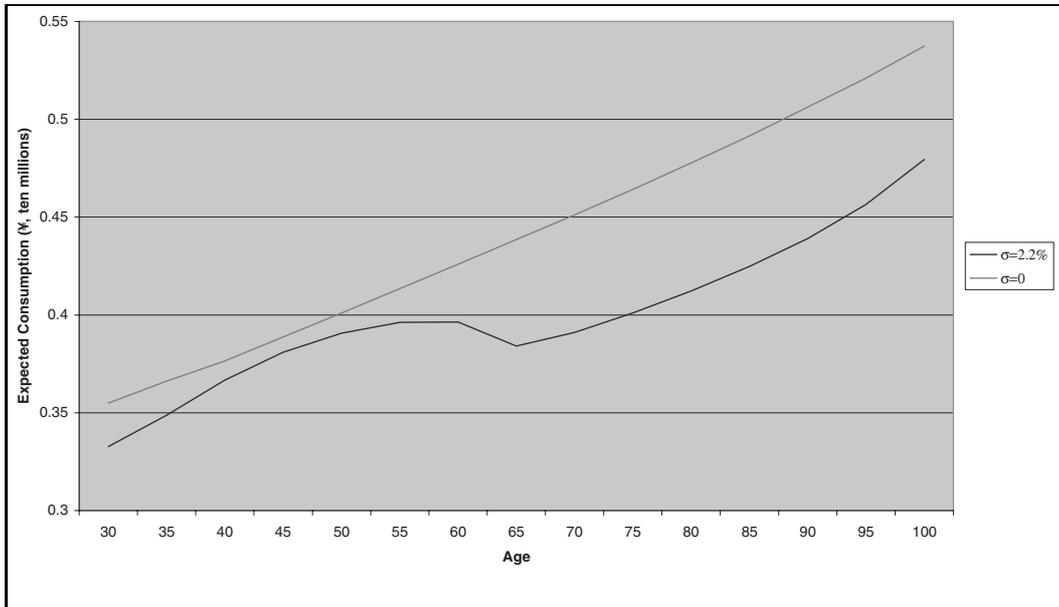


Figure 12: Expected consumption path.

ministic income model—but where the reduction is driven by increased volatility in the investment market, and not by volatility in labour income. Hence we ask: what volatility in investment market earnings would result in a retirement accumulation of 120 million yen, all other parameters being equal? The answer, generated by our simulation code (Purcal & Piggott 2001a), is a volatility of 37%. Thus the introduction of random income into the model with a volatility of only 2.2% produces the same reduction in retirement wealth as an almost *doubling* of the investment volatility in our model with deterministic income. Seen in this context, labour income volatility has strong behavioural repercussions.

Risky income also results in less expected consumption in the pre-retirement period (figure 12). The results accord with theory (e.g. Büttler 2001). In the case of deterministic income, the expected consumption path appears to grow at a constant rate. In the case of stochastic income, the incomplete markets case, the expected consumption path does not. In fact the hump shaped path for consumption accords with empirical evidence (e.g. Gourinchas & Parker 2002) in the pre-retirement period. Figure 12, however, indicates that post retirement consumption becomes convex and increasing—the situation we expect to be in when there is no labour income uncertainty.

Gourinchas & Parker explain the humped shaped consumption they found by arguing consumers behave in different ways at different stages in their lives. Early in their lives precautionary motives dominate, and consumers behave like ‘buffer-stock’ consumers (Carroll 1997). Were it not for income uncertainty they would

borrow against future labour income. That they cannot means they have to endure lower consumption levels. Later, between ages 40 and 45, Gourinchas & Parker believe households change to traditional life cycle consumers, saving for retirement and bequests.

To answer why exactly we get hump shaped consumption is not straightforward. The hump shape appears to be produced by a changing relationship between consumption and wealth over time. In the case of deterministic income the optimal consumption/wealth graph shifts more or less uniformly downwards as an individual age and his salary increases. In the case of stochastic income as young investors age and their salaries increase the optimal consumption/wealth graph shifts down by very small amounts at first. These downward shifts then increase by much larger amounts as they approach retirement. Such a non-uniform pattern in the changing relationship between optimal consumption and wealth leads to the humped pattern of consumption as wealth is accumulating in the pre-retirement period.

But why does this dynamic relationship between optimal consumption and wealth change between the safe income and risky income cases? That this has something to do with the period of remaining working life seems a reasonable hypothesis. When ones remaining working life is long one needn't be so cautious—on average income will grow. Workers hold back on consumption only to protect themselves against income uncertainty, and as the income horizon is long they needn't hold back too much (buffer-stock consuming). As the earning horizon decreases, however, the chance of stochastic earnings going awry and not having enough time to grow out of a bad patch is more palpable. Thus one saves more and is more cautious (life cycle saving).

Without the benefit of further investigation this is, however, only speculation. Despite our best efforts one cannot dismiss the possibility of a coding error.

In figure 13 we see the impact of a stochastic income environment on the proportion invested in the risky asset over the pre-retirement period. No longer is the graph convex, it is now largely concave. In addition for the first few years the proportion invested in risky is less than that of the safe income case. This changes to be largely greater than the safe income case over the pre-retirement period. This is not surprising—in the risky income environment less wealth is accumulated. In our cross-section analysis of the solution grid at ages 30 and 60 (figure 6) we showed that, for a fixed income level, reducing wealth produced a more than proportional increase in the percentage of wealth invested in the risky asset. Thus, the combination of a lower wealth path and a higher percentage invested in risky assets would not be unusual.

Figure 13 also indicates that age-phasing is appropriate over the life cycle. Age-phasing in the stochastic income environment appears much more gradual than the safe income case; it seems almost affine between ages forty and fifty-five.

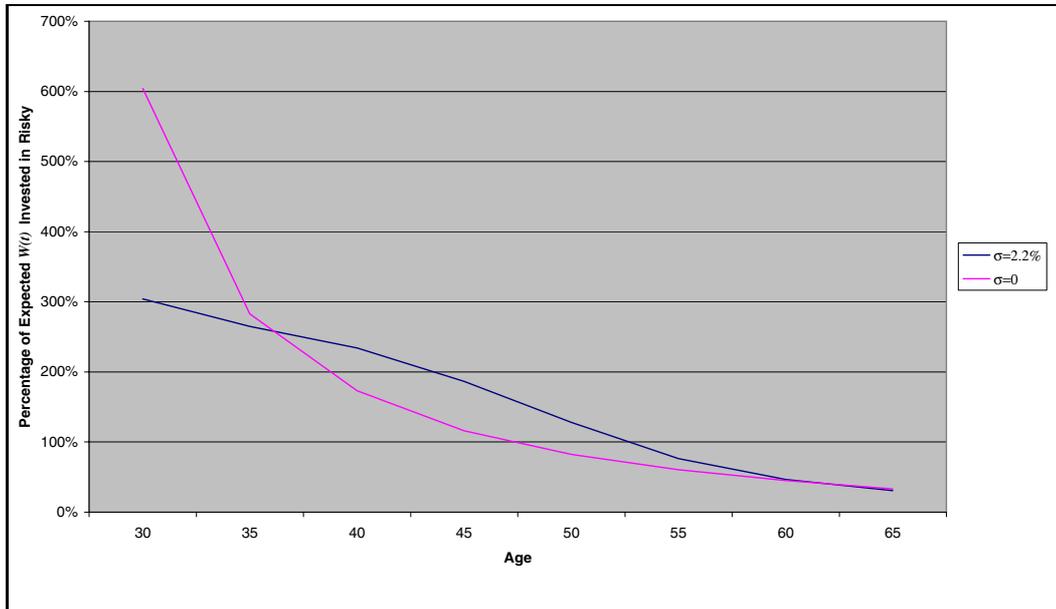


Figure 13: Expected investment path.

Also, the proportion invested in risky at younger ages is a lot less than the safe income case. This is also more in accord with casual real world empiricism.

Campbell & Viciera (pp. 218-19) report a number of cases in their model that the profile of financial wealth invested in the risky asset can hump over the early period of life. One can imagine that for an appropriate parameterisation the investment in risky profile in the present model could be hump shaped, given its concave nature.

The impact of risky income on life cycle life insurance and annuity demand can be seen in figure 14 for our chosen parameterisation. The largest differences appear to be a higher demand for life insurance in the ten or so years prior to retirement in the stochastic income case and a lower demand for annuities. Also the shift to annuity purchase occurs somewhat later in the stochastic income case. This can be explained. As we have seen in figure 11 less wealth is accumulated when labour income is stochastic. Consumption, on the other hand, has not fallen by as much. As our desired bequest is related to consumption we find insurance demand will rise. Similarly, near retirement, the smaller wealth accumulation will lead to later (and lower) annuity demand. It is interesting to note that stochastic labour income can also contribute to the thinness of annuity markets.

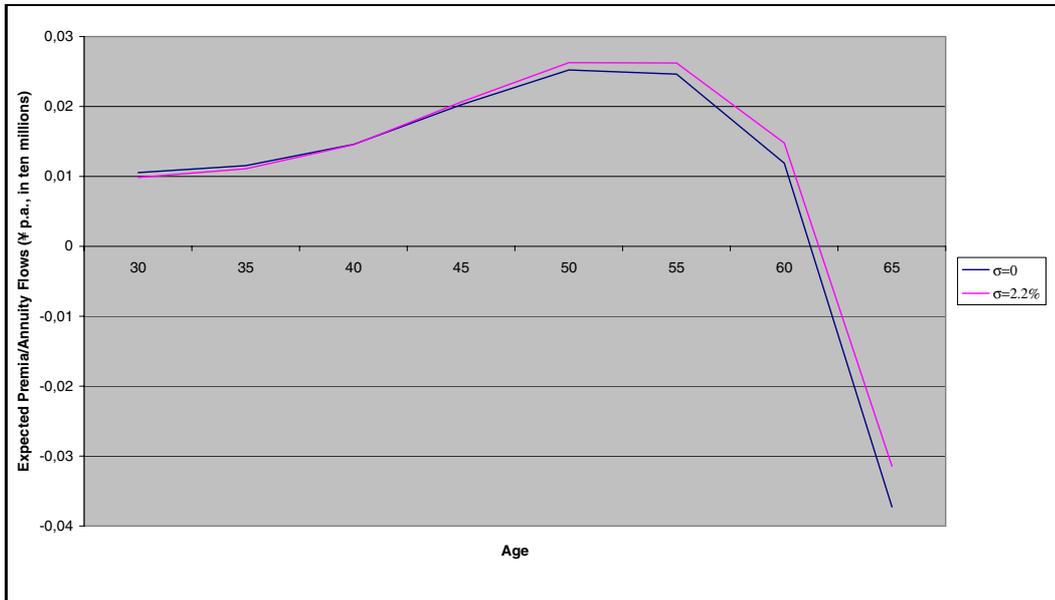


Figure 14: Expected premium path.

6.3 Remarks

The above subsections have detailed our findings from our extended Richard model. We close the discussion of our results by making a number of remarks on the solution methodology adopted.

The Markov chain approximation technique is a powerful approach to solving continuous time stochastic control problems. It is flexible. Interesting questions for future research include using utility functions more complicated than isoelastic utility, using a more complicated stochastic labour process (with time-varying coefficients or a jump process to model unemployment) and using a more complicated stochastic process for the risky returns process (with stochastic volatility or jump processes). All of these extensions can be handled by this technique.

The solution technique produces a rich set of results. Essentially it gives us the optimal behaviour of an agent at any time period in his life—given his financial wealth and salary. This information is ideal for questions of financial planning. In addition the fine time scale used, in contrast to other approximate discrete time approaches (where the model is solved in larger intervals, every year or so), could be of benefit following abrupt changes in wealth or salary. An investor could immediately set himself on his optimal path, rather than on some interpolated path of questionable accuracy.

Such accuracy comes at a cost. To solve the pre-retirement problem in a reasonable time-frame required a high performance computer program. (Post-retirement is a one-dimensional problem which can be solved in a few hours.)

Thus it seems using the above implementation of the solution approach our modelling remains limited to two state variables. An interesting avenue for future research would be to alter the implementation of the solution approach to use an implicit finite difference scheme rather than the explicit scheme adopted here. This approach is more involved and can be more accurate (Kushner & Dupuis 2001). It often involves in fewer backward time steps and thus arrives at a solution in a shorter period of time. Finite element methods, again more involved, may also be promising.

7 Conclusion

This paper has extended Richard by introducing stochastic income into the model. The result now includes the market incompleteness presented by labour income. It is not, however, able to be solved analytically and must be solved by numerical methods. We have used the Markov chain techniques and high performance computing to solve the model. We find, for our chosen parameterisation, that the expected consumption path is humped shaped up to retirement, then increases in a convex fashion. We speculate that this is an example of consumers acting as precautionary savers early in their lives and later as life cycle savers, saving for retirement and bequest motives. Investors age phase their investments in the risky asset over their lifetimes, with initial investment levels lower than if they earned a deterministic income flow. Expected life insurance demand is higher later in working life, while annuity demand is lower.

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