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## Lifetime Financial Planning with Regime-switching: the Case of Persistence and Mean Reversion

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## Lifetime Financial Planning with Regime-switching: the Case of Persistence and Mean Reversion

### Abstract

This paper carries forward investigations into optimal solutions for lifetime financial planning, comprising consumption, asset allocation and life insurance and annuity purchase decisions. Closed form solutions to this problem exist only when there is a single stochastic state variable, investment returns, with a simple geometric Brownian motion returns process. Stochastic dynamic programming must be used when these restrictions are relaxed. Evidence from financial markets suggests that some degree of long-run predictability is a feature of stock markets – a feature which can be captured by a particular characterisation of regime-switching. The present paper proposes and implements computational procedures for generating approximations to optimal solutions to the lifetime financial planning problem when stock returns exhibit persistence and mean reversion. This is an example of a stochastic dynamic control problem with two state variables. Illustrative results are presented which show that while expected consumption continues to increase smoothly increasing under these assumptions, for the cases we study, the path of expected wealth accumulations can be affected if changes in expected volatility are introduced into the regimes considered. It is expected that the same set of algorithms will provide the basis for numerically solving other generalisations of the core problem.

## Lifetime Financial Planning with Regime-switching: the Case of Persistence and Mean Reversion

Optimal solutions to stylised representations of intractable problems often provide a benchmark with which actual outcomes can be compared. The major motivation for the present research program is to develop such a benchmark for lifetime financial planning. The project takes as its point of departure the lifetime asset allocation model of Merton (1969, 1971). It casts this analytically tractable model in an operational framework by employing explicit behavioural functions and specific parameter values, and operationalises Richard's (1975) extensions to capture the role of life insurance and annuity markets in lifetime planning.

Numerical simulation of lifetime financial planning using this structure has already generated a number of insights into the role of government policy and private financial institutional behaviour on individual choice. The impact of social security on aggregate private accumulations, and on asset allocation, was assessed, along with age phased investment strategies and the role of bequests and financial institution loadings on annuity demand. These results are reported in Purcal and Piggott (2001).

That analysis, and the findings reported there, rely on a number of crucial simplifying assumptions. The analytic solutions provided by Merton and Richard were exploited to allow the quantitative analysis to proceed. Numerical simulation took these closed form solutions as a starting point. They in turn relied on the assumption of simple geometric Brownian motion as the basis of stochastic innovation in the model –

essentially a random walk mechanism, in which returns on stocks and bonds are unpredictable, even in the long run.

This paper reports an algorithm for the numerical analysis of optimal lifetime financial planning which does not rely on any closed form solution of which we are aware. Rather than use the conditions for a closed form solution as the basis for numerical optimisation, it builds on a stochastic dynamic programming representation of the problem, and generates approximate solutions using Markov chain methods. While far more complex computationally, this provides much greater flexibility in introducing “real-world” considerations into the modeling. It allows for the introduction of more than one state variable in the financial planning problem. For example, a lifetime planning model with stochastic labour returns can be solved. In addition, phenomena such as bankruptcy and liquidity constraint may be introduced into the analysis.

The present paper applies this new solution algorithm to modeling persistence and mean reversion in stock returns. Essentially, this relaxes the assumption of geometric Brownian motion as the sole governor of stock returns. While not altering its essential economic structure, this extension converts the problem from a one-state-variable problem into one in which two state variables operate: the return on investment, and the level of wealth.

Through the 80s and 90s, a series of important papers lent credence to the view that, at least in the long term, some predictability was present in financial asset returns. Persistence and mean reversion were seen as phenomena which were encountered in careful analysis of data, and which needed to be accommodated in numerical analysis of financial markets, particularly where long time horizons were the focus of attention.

Brownian motion, embracing as it does the random walk hypothesis, is thus called into question as the only sensible and tractable basis for numerical analysis of life cycle financial planning, since if the hypothesis were true, persistence and mean reversion would not be observed.

Cochrane (2001) offers a useful summary of the change in received wisdom about financial market behaviour that has occurred over the last 20 years. At the start of the 80s, a student of finance would have been expected to take from his training that stock and bond returns are unpredictable, and that stock market volatility does not change much through time. They would also have been told that professional fund managers do not reliably outperform passive portfolios, corrected for risk.

These beliefs are now all seriously challenged by a large volume of research. While explanations for these findings are not always settled, and the findings of particular pieces of work are criticised, it seems fair to say that what might be termed the “simple random walk” view of financial markets is no longer uncritically accepted.

For example, recent research suggests strongly that observable variables such as the dividend/price ratio can predict substantial amounts of stock return variation, over the business cycle and longer horizons. Stock market volatility changes over time. And bond and foreign exchange returns are predictable at least some of the time.<sup>1</sup>

Although a simple descriptive account of past history provides no evidence either for or against the random walk hypothesis, for contextual relevance we do report (in Table 1) the decade-by decade Japanese experience with stock market returns over the

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<sup>1</sup> Recent literature focuses on how an investor might manage his investments better by exploiting predictability in market returns (Cochrane 1999), or by following derived rules for market timing (Cochrane 2001). We make no attempt to represent this kind of sophisticated financial strategy.

last 30 years. The data certainly appear consistent with the idea that stocks do not follow a random walk, and that stock market volatility varies through time. Interestingly, Kim et al (1988) find that in the US, the mean-reversion phenomenon is a feature of the 1926-1946 period, while the period from 1946 exhibits persistence of returns.

**Table 1: Mean returns and volatility in the Nikkei index, 1970-2000**

	Mean stock market return	Mean volatility
1970-1980	11.0%	18.9%
1980-1990	17.6%	13.7%
1990-2000	- 5.4%	24.5%
1970-2000	6.4%	19.9%

Preliminary findings from our model suggest that allowing for persistence and mean reversion will affect expected consumption and wealth accumulation trajectories, and asset allocations, especially if regime switching in volatilities is present.

Section I recapitulates the basic optimising model with which we are working. In section II we describe our solution methodology and outline the regime-switching process, while section III reports and discusses our preliminary results. Section IV concludes.

## **I. The economic model**

Merton (1971) introduced mortality to optimal consumption and investment models, incorporating a parametric survival model of mortality into his formulation.

While much of the literature that has followed focuses only on optimal consumption and portfolio selection, Richard (1975) extended this model to consider additionally the optimal amount of life insurance. In doing so, he substituted the parametric survival model of Merton (1971) with a more realistic tabular, or nonparametric, survival model along the lines of Yaari (1965).

The Richard model contains all the elements of personal financial planning. The model tries to capture the central uncertainties confronting an individual trying to plan his finances over a life cycle – investment returns on a risky asset, and his own mortality. The major simplifying assumption is that we assume deterministic labour income, and fixed labour supply.

The investor's objective in this model is to maximize his expected lifetime discounted utility (including the utility from bequest). This naturally leads to a stochastic control problem with a risky state variable, wealth, and several controls—the investor will attempt to achieve his objective through optimal choice of consumption, investment in risky assets and bequest.

In our model, we assume the investor will start at age 30 with initial wealth equal to one year of salary. The investor's stock of wealth will then be reduced by consumption and expenditures on life insurance; it will be augmented by the receipt of income and annuity payments, as well as returns from his investments in risky and safe assets.

The investor's purchase of life insurance is determined by both his optimal desired legacy and the current level of his financial wealth. Should the investor's desired legacy exceed this wealth on hand, he necessarily buys life insurance to provide for this shortfall on his death. For this, he pays a premium each period, which can be specified as

either actuarially fair, or comprising some loading. Should the investor have no bequest motive, or should his financial wealth grow to exceed his desired bequest, then this excess will be annuitized. In this respect an annuity is the negative of life insurance.<sup>2</sup>

In the model we assume markets are complete, and the investor receives a deterministic wage. With respect to investment decisions, then, the investor considers his total wealth—consisting both his wealth on hand and the capitalized value of future labour income.

For isoelastic utility, and bequest and if geometric Brownian motion governs the stochastic process associated with risky asset returns, Richard has determined the values of the optimal controls. His optimal consumption decision is made such that he considers his total wealth as having to provide both for his lifetime consumption and an insurance benefit on his death. Richard's model also shows that investors place a constant fraction of total (human plus financial) wealth in the risky asset.

The formal details of the model, including the specification of the utility and bequest functions, are laid out in Purcal and Piggott (2001). Model parameters are also discussed there. Here we provide a brief recapitulation of the formal model, and reproduce the parameters selected for our numerical specification.

Richard models a multi-period utility maximizing investor with objective

$$\max E \left[ \int_{\tau}^T U(C(t), t) dt + B(Z(T), T) \right], \tag{1}$$

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<sup>2</sup> This symmetry has been observed by earlier authors. See, for example, Fischer (1973). In the case of life insurance, the insured makes a payment, in return for which the insurance company will pay a benefit contingent on death. In the case of a life annuity, the annuitant receives a payment, in return for which the insurance company will receive a benefit if the annuitant dies.

where  $T$  is the investor's uncertain time of death, and  $U$ ,  $C$ ,  $Z$  and  $B$  are the investor's utility, consumption, legacy at death and utility from bequest. The investor is able to choose between two securities, one risky and one risk-free, with the price of the risky asset,  $Q$ , following geometric Brownian motion

$$\frac{dQ(t)}{Q(t)} = \alpha dt + \sigma dq(t), \quad (2)$$

where  $dq(t)$  is a Wiener increment.

The investor's change in wealth is given by the stochastic differential equation

$$dW(t) = -C(t) - P(t)dt + Y(t)dt + rW(t)dt + (\alpha - r)\pi(t)W(t)dt + \sigma\pi(t)W(t)dq(t), \quad (3)$$

where  $P(t)$ ,  $Y(t)$ ,  $W(t)$  are, respectively, the investor's life insurance premium paid, income (assumed to be non-stochastic), and wealth at time  $t$ . From equation (2), the mean return on risky investment is  $\alpha$ , with standard deviation  $\sigma$ , while the risk-free investment returns  $r$ ; the investor places a proportion  $\pi$  of wealth in the risky asset.

The investor buys instantaneous term life insurance to the amount of  $Z(t) - W(t)$ . For this, a premium of  $P(t)$  is paid. If we denote the force of mortality by  $\mu(t)$ , then the amount of premium paid for actuarially fair insurance will be

$$P(t) = \mu(t)(Z(t) - W(t)). \quad (4)$$

The investor's problem is to solve equation (1), subject to budget constraint (3) and initial wealth condition  $W(0) = W_0$ , by optimal choice of controls  $C$ ,  $\pi$  and  $Z$ . The function  $U$  is assumed to be strictly concave in  $C$  and  $B$  is assumed strictly concave in  $Z$ .

Following the method outlined in Purcal and Piggott, the Hamilton-Jacobi-Bellman equation is therefore

$$0 = \max_{C, Z, \pi} \{ \mu(t)B(Z(t), t) + U(C(t), t) - \mu(t)J + J_t + [\pi\alpha W + (1 - \pi)rW + Y - C - P]J_w + \frac{1}{2}\sigma^2\pi^2W^2J_{ww} \} \quad (5)$$

As  $Y(t)$  in equation (5) is non-stochastic, Richard demonstrates that (5) is equivalent to an equation involving capitalized  $Y(t)$ . That is, *adjusted wealth* is defined as

$$\tilde{W}(t) \equiv W(t) + b(t), \quad (6)$$

where  $b(t)$  is defined as the capitalized value of future income:

$$b(t) = \int_t^\omega Y(\theta) \frac{S(\theta)}{S(t)} e^{-r(\theta-t)} d\theta. \quad (7)$$

The standard approach (Richard 1975, Bodie, Merton & Samuelson 1992) is to remove  $Y(t)$  from (5) and substitute  $\tilde{W}(t)$  for  $W(t)$ . Income is thus treated as a traded asset.

Finally, we note that discounted values of the functions of interest can produce very small values, so we may facilitate the numerical solution of the model by converting the discounted Hamilton-Jacobi-Bellman equation (5) to current values. If we set

$h(t) = e^{-\rho t}$ , so  $U(C(t), t) = e^{-\rho t}\tilde{U}(C(t))$ , and  $m(t) = e^{-\rho t}\phi(t)$ , so

$B(Z(t), t) = e^{-\rho t}\phi(t)\tilde{B}(Z(t))$ , where  $\rho$  represents an individual's rate of time preference, equation (5) becomes

$$0 = \max_{C, Z, \tilde{\pi}} \{ \mu(t) \tilde{B}(Z(t), t) + \tilde{U}(C(t), t) - \mu(t) \tilde{J} + \tilde{J}_t + [ \tilde{\pi} \alpha \tilde{W} + (1 - \tilde{\pi}) r \tilde{W} - C - P ] \tilde{J}_w + \frac{1}{2} \sigma^2 \tilde{\pi}^2 \tilde{W}^2 \tilde{J}_{ww} \} . \quad (8)$$

These conclusions are similar to the simpler results of Merton (1969), where, in a model with certain lifetime and no bequest or insurance considerations, the optimal solutions for a model with isoelastic utility and geometric Brownian motion underlying the risky asset, optimal consumption at a point in time was determined by annuitizing wealth at that moment over the consumer's remaining certain life. The optimal investment in risky was also a constant fraction of wealth.

The values of parameters used in numerically specifying the model are given in Table 2.

**Table 2. Parameters used in numerical simulation of the model.**

$\alpha$	0.025
$r$	0.005
$\rho$	0.005
$\sigma$	0.2
Mortality	JLT18 (male)
$\gamma$	- 0.5
$\omega$	110
$Y (=W(0))$	¥ 4 375 686

The analytic solutions to these models, obtained by Merton and Richard, rely on the assumption of standard geometric Brownian motion. While closed form solutions have been found for certain special cases where this assumption is relaxed (e.g., Wachter 1999), no general analytic solution is available. For this reason, numerical approximations must be generated. In what follows, we document our approach to a

solution algorithm which potentially allows considerable generalisation of the Richard model.

## II. Solution procedures

In this section we document the Markov chain approximation method to the solution of the Richard model, together with a discussion of its computer implementation and its application. The section concludes with a description of the regime-switching model we have added to that of Richard.

### II.1 Markov Chain Solution Method

The numerical solution of finite horizon stochastic optimal control problems is well described in Kushner (1977, Chapter 7) and Kushner and Dupuis (2001, Chapter 12). The ‘‘explicit’’ solution approach involves a finite difference approximation to the HJB equation (8), a second-order linear parabolic partial differential equation, as described below.

Firstly, consider the coefficient of  $\tilde{J}_w$  in (8). Partition the terms that make up this coefficient into a positive group,  $d^+ = (\alpha - r)\tilde{\pi}\tilde{W} + r\tilde{W} + \mu\tilde{W}$ , and a negative group  $d^- = C + \mu Z$ , where  $d = d^+ + d^-$ ,  $d$  being the coefficient of  $\tilde{J}_w$ . Let us approximate the partial derivatives in equation (8) as follows:

$$\begin{aligned}
f_t(x,t) &\rightarrow \frac{f(x,t+\delta) - f(x,t)}{\delta} \\
f_x(x,t) &\rightarrow \frac{f(x+h,t+\delta) - f(x,t+\delta)}{h} \text{ for } d^+ \\
f_x(x,t) &\rightarrow \frac{f(x,t+\delta) - f(x-h,t+\delta)}{h} \text{ for } d^- \\
f_{xx}(x,t) &\rightarrow \frac{f(x+h,t+\delta) + f(x-h,t+\delta) - 2f(x,t+\delta)}{h^2}
\end{aligned}$$

and write (8) as follows, where  $V(\cdot, \cdot)$  represents the solution to the finite difference equation:

$$\begin{aligned}
0 = \max_{C, \tilde{\pi}, Z} & \left\{ \mu\phi\tilde{B}(Z(t)) + \tilde{U}(C(t)) - (\mu + \rho)V(\tilde{W}, t) + \right. \\
& \frac{V(\tilde{W}, t+\delta) - V(\tilde{W}, t)}{\delta} + \frac{V(\tilde{W}+h, t+\delta) - V(\tilde{W}, t+\delta)}{h} \delta^+ \\
& \left. - \frac{V(\tilde{W}, t+\delta) - V(\tilde{W}-h, t+\delta)}{h} d^- + \right. \\
& \left. \frac{1}{2} (\sigma\tilde{\pi}\tilde{W}^2) \frac{V(\tilde{W}+h, t+\delta) + V(\tilde{W}-h, t+\delta) - 2V(\tilde{W}, t+\delta)}{h^2} \right\}. \tag{9}
\end{aligned}$$

Equation (9) can be written as

$$\begin{aligned}
V(\tilde{W}, t) = \max_{C, \tilde{\pi}, Z} & \frac{1}{1 + \mu\delta + \rho\delta} \left\{ \tilde{U}(C(t)) + \mu\phi(t)\tilde{B}(Z(t)) + \right. \\
& V(\tilde{W}, t+\delta) \left[ 1 - \frac{\delta}{h}d - \frac{\delta}{h^2}\sigma^2 \right] + \\
& \left. V(\tilde{W}+h, t+\delta) \left[ \frac{\delta}{h}d^+ + \frac{\delta}{h^2}\frac{\sigma^2}{2} \right] + V(\tilde{W}-h, t+\delta) \left[ \frac{\delta}{h}d^- + \frac{\delta}{h^2}\frac{\sigma^2}{2} \right] \right\}
\end{aligned}$$

or, more conveniently,

$$\begin{aligned}
V(\tilde{W}, t) = \max_{C, \tilde{\pi}, Z} & \frac{1}{1 + \mu\delta + \rho\delta} \left\{ \left[ \sum_{\theta=-1}^1 p(\tilde{W}, \tilde{W} + \theta h) V(\tilde{W} + \theta h, t + \delta) \right] \right. \\
& \left. \tilde{U}(C(t)) + \mu\phi(t)\tilde{B}(Z(t)) \right\} \tag{10}
\end{aligned}$$

where the  $p(\cdot, \cdot)$  may be interpreted as transition probabilities of a Markov chain. These transition probabilities are locally consistent with equation (3)—when we approximate

the diffusion of the state variable,  $\tilde{W}$ , by the above Markov process, the characteristics of the approximation follow those of the underlying diffusion of equation (3). The transition probabilities are given by:

$$\begin{aligned} p(\tilde{W}, \tilde{W} + h) &= \frac{\delta}{h^2} \left\{ \frac{1}{2} (\sigma \tilde{\pi} \tilde{W})^2 + h [(\alpha - r) \tilde{\pi} \tilde{W} + r \tilde{W} + \mu \tilde{W}] \right\} \\ p(\tilde{W}, \tilde{W} - h) &= \frac{\delta}{h^2} \left\{ \frac{1}{2} (\sigma \tilde{\pi} \tilde{W})^2 + h [C + \mu Z] \right\} \\ p(\tilde{W}, \tilde{W}) &= 1 - p(\tilde{W}, \tilde{W} + h) - p(\tilde{W}, \tilde{W} - h) \end{aligned} \quad (11)$$

The boundary condition is  $V(\tilde{W}, \omega) = \phi(\omega) \tilde{B}(Z(\omega))$ . Thus, the solution to the investor's stochastic control problem (equation (1)) is approximated by the solution to equation (10) as  $h \rightarrow 0$  and  $\delta \rightarrow 0$  together.

## II.2 Implementing the Method

Equation (10) was solved on a grid of the state variable ( $\tilde{W}$ ) by backwards iteration, using a computer. The  $N$  point grid  $S_h^N = \{ih \mid 0 \leq i \leq N\}$ , for a positive integer  $N$ , represented the state space for the Markov chain. Following Fitzpatrick and Fleming (1991), let us give the set of admissible controls as

$$\Gamma_{i,h}^N = \{(C, \tilde{\pi}, Z) \mid 0 \leq C, \tilde{\pi}, Z \leq KNh\}$$

where the constant  $K$  gives an artificial bound on the controls.<sup>3</sup>

Our control dependent transition probabilities for the grid are, following equation (11),

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<sup>3</sup> This constraint is relaxed in the limit as  $h \rightarrow 0$  and  $Nh \rightarrow \infty$ .

$$\begin{aligned}
P_{ii+1}^{C,\tilde{\pi},Z} &= \frac{1}{Q} \left\{ \frac{1}{2} (\sigma \tilde{\pi} \tilde{W})^2 + h [(\alpha - r) \tilde{\pi} \tilde{W} + r \tilde{W} + \mu \tilde{W}] \right\} \\
P_{ii-1}^{C,\tilde{\pi},Z} &= \frac{1}{Q} \left\{ \frac{1}{2} (\sigma \tilde{\pi} \tilde{W})^2 + h [C + \mu Z] \right\} \\
P_{ii}^{C,\tilde{\pi},Z} &= 1 - P_{ii+1}^{C,\tilde{\pi},Z} - P_{ii-1}^{C,\tilde{\pi},Z}
\end{aligned} \tag{12}$$

for  $1 \leq i \leq N-1$ , and  $Q$  is a normalizing constant, given by  $h^2/\delta$ . At the boundaries, the

probabilities are set to

$$\begin{aligned}
P_{00}^{C,\tilde{\pi},Z} &= 1 \\
P_{NN-1}^{C,\tilde{\pi},Z} &= \frac{1}{Q} \left\{ \frac{1}{2} (\sigma \tilde{\pi} \tilde{W})^2 + h [C + \mu Z] \right\} \\
P_{NN}^{C,\tilde{\pi},Z} &= 1 - P_{NN-1}^{C,\tilde{\pi},Z}
\end{aligned}$$

In order to ensure we have a valid Markov chain,  $Q$  must be chosen such that all  $P$  lie between 0 and 1. We can ensure this by requiring

$$P_{ii+1}^{C,\tilde{\pi},Z} + P_{ii-1}^{C,\tilde{\pi},Z} \leq 1$$

and so set  $Q$  as follows:

$$\begin{aligned}
Q &= \max_{0 \leq i \leq N} P_{ii+1}^{C,\tilde{\pi},Z} + P_{ii-1}^{C,\tilde{\pi},Z} \\
&= \max_{0 \leq i \leq N} \left[ \sigma^2 \tilde{\pi}^2 + h [(\alpha - r) \tilde{\pi} \tilde{W} + r \tilde{W} + \mu \tilde{W} + C + \mu Z] \right] \\
&= \sigma^2 (KNh)^2 + Nh^2 [(\alpha - r)K + r + K] + [(K+1)Nh^2]
\end{aligned} \tag{13}$$

Convergence theories for these approximations exist—see Fitzpatrick and Fleming (1991) for details. These assume that  $\tilde{J}$  is twice continuously differentiable and bounded and the solutions are compact. Convergence occurs as  $h \rightarrow 0$  and  $\delta \rightarrow 0$ , then  $V(\cdot, \cdot) \rightarrow \tilde{J}(\cdot, \cdot)$ .

### II.3 Applying the Method to the Richard Model

The model was solved on a computer by backward iteration from age 110 to age 30 over the grid of state variables. As we implement the above theory for a computer, let the function  $V(\tilde{W}, t)$  from section II.1 be represented by  $V_n^t$ , where  $t$  represents the iteration counter and  $n$ ,  $0 \leq n \leq N$ , represents the position on the grid of state variables corresponding to the value  $\tilde{W}$ . Our optimal controls,  $C_n^{t*}$ ,  $\tilde{\pi}_n^{t*}$  and  $Z_n^{t*}$  are found by solving equation (10) as follows:

$$V_n^t = \max_{C, \tilde{\pi}, Z} \frac{1}{1 + \mu\delta + \rho\delta} \left\{ P_{ii+1}^{C, \tilde{\pi}, Z} V_{n+1}^{t+1} + P_{ii-1}^{C, \tilde{\pi}, Z} V_{n-1}^{t+1} + P_{ii}^{C, \tilde{\pi}, Z} V_n^{t+1} + \delta \left[ \frac{(C_n^t)^\gamma}{\gamma} + \mu^t \phi^t \frac{(Z_n^t)^\gamma}{\gamma} \right] \right\}$$

Performing this maximisation we find:

$$C_n^{t*} = \left[ \frac{h}{Q\delta} [V_n^{t+1} - V_{n-1}^{t+1}] \right]^{\frac{1}{\gamma-1}}$$

$$\tilde{\pi}_n^{t*} = \frac{h(\alpha - r)W [V_n^{t+1} - V_{n+1}^{t+1}]}{(\sigma W)^2 [V_{n+1}^{t+1} + V_{n-1}^{t+1} - 2V_n^{t+1}]}$$

$$Z_n^{t*} = \left[ \frac{h}{Q\phi\delta} [V_n^{t+1} - V_{n-1}^{t+1}] \right]^{\frac{1}{\gamma-1}}$$

It is interesting to note that as the  $V_n^t$  represent the investor's indirect utility function, and  $(V_{n+1}^{t+1} - V_n^{t+1})/h$  and  $(V_{n+1}^{t+1} + V_{n-1}^{t+1} - 2V_n^{t+1})/h^2$  represent finite difference approximations to the first and second derivatives of  $V_n^t$  respectively, then  $\tilde{\pi}_n^{t*}$  is given by the product of  $(\alpha - r)/\sigma^2$  and the reciprocal of the investor's coefficient of relative risk aversion. This agrees with established theoretical results.

Using the parameters set out in Table 2, the Richard model was solved using the above methodology.

One modification was made to the theory described above. In order to speed up solution time  $Q$  was not fixed as a constant according to equation (13) above, but allowed to vary according to an algorithm (generating values of  $Q_i$ ) given in Fitzpatrick and Fleming (1991, pp.836-837), and described below.

After each iteration, calculate  $Q_i$  as follows:

$$Q_i = \max_{1 \leq i \leq N} \sigma^2 (\tilde{\pi}_i')^2 + h [(\alpha - r) \tilde{\pi}_i' \tilde{W} + r \tilde{W} + \mu \tilde{W} + C_i' + \mu Z_i']$$

Use this value instead of  $Q$  in the next iteration. This adaptive adjustment of  $Q$  accelerates the solution of the model by allowing lower values of  $Q$ , and consequently higher values of  $\delta$ , where  $\delta$  gives the amount of time the solution method steps back at each iteration.<sup>4</sup>

The solution method was coded in C, and run on a 1GHz PC with 256M of memory operating under Windows 2000 Professional. Solving the model (going from terminal age of 110 back to 30) took under 20 minutes.

Finally, we note that in addition to providing the values of the control variables for a variety of wealth and age values, the solution method should also provide us with the expected paths of these variables. From an economic point of view, we are interested in our expectation at the outset of the path consumption and the other controls will take over one's lifetime. These expected paths can be readily determined using the Markov chain solution technique, as it provides us with transition probabilities. Using these

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<sup>4</sup> Recall that  $\delta = h^2/Q$

transition probabilities, we can calculate the expected future value of a state or control variable,  $\xi_t$ . Specifically, at any wealth node-time pair  $(n, t)$ ,

$$E[\xi_{n,t+1} | \mathcal{F}_{n,t}] = p_{n,n-1}\xi_{n-1,t+1} + p_{n,n}\xi_{n,t+1} + p_{n,n+1}\xi_{n+1,t+1}.$$

Thus, by the tower property of conditional expectation we may obtain  $E[\xi_t | \mathcal{F}_{n,0}]$  by backward recursion on the value of  $\xi_t$  using the transition probabilities.

**Table 3**  
**Approximate and exact solutions to the standard Richard model**  
**( $\alpha = 0.025$ ; Expected values; ¥ million, annual)**

Age	Wealth		Consumption		Proportion in risky		Desired bequest	
	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.
30	138.6	138.6	2.51	2.48	.33	.32	106.0	104.6
50	117.7	116.3	2.84	2.76	.33	.32	95.1	92.3
70	86.3	85.3	3.17	3.06	.33	.33	75.0	72.3
90	48.1	47.9	3.52	3.46	.33	.33	44.3	43.4

In Table 3 we report the results of the Markov chain solution method, using the same parameterisation of the Richard model as described in Purcal and Piggott (2001), and compare them to the values reported in that earlier paper. Their close agreement indicates the value of the approximation method.

#### II.4 Regime-Switching in the Richard Model

As described in the introduction, we are interested in considering the issue of regime-switching because it adds realism to our model. Below we describe how we implemented a simplified version of a regime-switching model which we find is able to provide a rich set of results.

We are interested in examining persistence in both the mean return on the risky asset and the volatility of this asset. Specifically, we consider the following two versions of equation (1).

1. Persistence in mean return

$$\frac{dQ(t)}{Q(t)} = \alpha(t)dt + \sigma dq(t)$$

where

$$\alpha(t) = \begin{cases} 0.02 & \text{where } i \leq t \leq i + 20 \text{ with probability } \frac{1}{3} \\ 0.025 & \quad \quad \quad \text{"} \quad \text{"} \\ 0.03 & \quad \quad \quad \text{"} \quad \text{"} \end{cases}$$

and  $i=0, 20$  and  $40$  (i.e., breakpoints at ages 50, 70 and 90).

2. Persistence in mean return and volatility

$$\frac{dQ(t)}{Q(t)} = \alpha(t)dt + \sigma(\alpha(t))dq(t)$$

where

$$\alpha(t) = \begin{cases} 0.02 & \text{where } i \leq t \leq i + 20 \text{ with probability } \frac{1}{3} \\ 0.025 & \quad \quad \quad \text{"} \quad \text{"} \\ 0.03 & \quad \quad \quad \text{"} \quad \text{"} \end{cases}$$

and

$$\sigma(t) = \begin{cases} 0.25 & \text{when } \alpha(t) = 0.02 \\ 0.20 & \text{when } \alpha(t) = 0.025 \\ 0.15 & \text{when } \alpha(t) = 0.03 \end{cases}$$

and  $i=0, 20$  and  $40$  (i.e., breakpoints at ages 50, 70 and 90).

The solution method we have described can readily be adapted to solve this stochastic control problem. In essence, we have a two state variable problem, our first

state being wealth and our second being the mean return on the risky asset. We assume transition between the  $\alpha$ -state values occurs only at ages 50, 70 and 90 for simplicity—the mean level of risky asset returns persists for twenty year periods. At the end of any one of these periods, the mean return can jump with equal probability to any of the three given levels. Persistence in both mean return and volatility is implicit in the above characterisation of regime-switching, since these values persist for 20 year periods by construction. Mean reversion is introduced through the balance of probabilities on the direction of future regime shifts when either of the two extremes of the set of three available mean values is selected. At these points, there is a 0.67 probability of a reverting movement in the mean return at the next breakpoint.

The level of risky asset returns was set at a mean level of 0.025 to reflect the real rate of return on the Nikkei over the period from January 1970 to June 2000. The values above it and below it were arbitrarily set.

In the introduction we noted that a considerable body of evidence exists regarding changes in stock market volatility over time. We have chosen to implement this observation in a second experiment, combining both persistence in risky asset returns and volatility over time. Here, asset returns and volatility persist in one of three pairs (0.02, 0.25), (0.025, 0.20) or (0.03, 0.15) over twenty year periods. Although somewhat arbitrarily chosen, these pairs have some grounding in observed Japanese experience. In Table 1 above we displayed evidence of an association between mean asset returns and volatility — that higher periods of market return were associated with lower volatility, and lower periods of market return were associated with higher volatility. We make no claim that this is an established empirical regularity. Instead, we use this observation to

parameterise our simple model of persistence in mean return and volatility in furtherance of our overall goal of determining the impact on consumption, investment patterns and life insurance and annuity purchases if such realistic characteristics exist in an economy.

### **III. Preliminary results**

The introduction of regime switching was undertaken in two parts. The first set of experiments simulated regime shifts related to the mean risky rate of return. In the second, these shifts in mean return were linked with inverse movements in volatility, a pattern detected in the Nikkei over the last 30 years. Overall, the impact of shifts from the first set of experiments in the mean risky rate of return on expected wealth and consumption were not significant. This finding probably relates to the narrow span of mean risky returns – a limitation imposed by computational difficulty. But the exposure to investment risk did respond to the expected rate of return under each regime switch. Further, when volatility shifts were introduced in the second set of experiments, significant variations in the patterns of expected wealth accumulation were generated.

In our characterisation of regime switching, the individual knows what the expected rate of return is in each period (of 20 years), and he knows the actual mean rate of return he will be confronting during the first period of his life. This latter piece of information conditions his overall expectation of the mean rate of return over his life span.

Table 4 reports computed values for expected wealth and consumption, and exposure to risk, for the three possible cases of regime switching in mean returns, with no regime switch in volatility. Once the minor adjustments for expected rates of return resulting

from the initial regime selection are taken into account, no impact of regime switching can be discerned on expected wealth and consumption, an impression reinforced by Table 5, in which constant regime and regime-switching variants are compared for the same overall expected rates of return.

The proportion of full (including human) wealth held in risky, however, is sensitive to the value of the expected risky rate of return. The proportion in risky,  $\pi$ , is

**Table 4**  
**Expected consumption and expected wealth under regime-switching:  
alternative initial regimes (¥ million)**

Initial regime	Age	30	50	70	90
$\alpha = 0.02$	Expected consumption	2.46	2.61	2.93	3.34
	Expected wealth	138.6	110.5	81.8	46.2
	Proportion in risky ( $\pi$ )	0.25	0.25	0.33	0.33
$\alpha = 0.025$	Expected consumption	2.48	2.76	3.07	3.49
	Expected wealth	138.6	116.2	85.6	48.3
	Proportion in risky ( $\pi$ )	0.32	0.32	0.33	0.33
$\alpha = 0.03$	Expected consumption	2.52	2.93	3.23	3.66
	Expected wealth	138.6	122.6	89.8	50.6
	Proportion in risky ( $\pi$ )	0.39	0.39	0.33	0.33

close to the value predicted by both Merton and Richard:  $(\alpha - r)/[(1 - \gamma)\sigma^2]$ . For example, for  $\alpha$  set at 0.025, and other parameters set at values given by Table 2, the predicted value of  $\pi$  is 0.33, very close to the values generated by the model. This also holds on a regime-by-regime basis for other initial values. The value of  $\pi$  is close to 0.33 for second

and later regime periods in all cases, because at the start of the first period, the expected risky rate of return for these periods is 0.025.

This is surprising, because the global volatility confronted on an expected basis by the individual under regime switching significantly exceeds that for the constant regime case. This issue is discussed further below.

**Table 5**  
**Comparing constant expected returns with regime switching**  
**( $\alpha = 0.025$ ; ¥ million)**

Age	Constant regime:		Regime switching:	
	Expected Wealth	Expected Consumption	Expected Wealth	Expected Consumption
30	138.6	2.48	138.6	2.48
40	128.6	2.62	128.6	2.62
50	116.3	2.76	116.2	2.76
60	101.8	2.90	101.9	2.91
70	85.5	3.06	85.6	3.07
80	67.1	3.24	67.4	3.26
90	47.9	3.35	48.3	3.49
100	27.5	3.81	27.8	3.85
110	4.3	5.46	4.4	5.52

Table 6 provides the most interesting of this preliminary set of results. As we pointed out in the introduction, one important change in perceptions about the behaviour of financial markets to take place over the last two decades is the realisation that the volatility of stocks changes over time. We have tried to capture this effect by coupling the change in mean returns with changes in stock volatility in our regime specifications.

Expected consumption retains the smooth upward path exhibited in constant regime scenarios, but the path of expected wealth changes significantly, depending on the initial regime. In the case of a high return-low volatility draw, full wealth is largely

preserved throughout the initial regime sequence, even though human capital is depreciating over this period. Financial wealth adjusts dramatically to the differential opportunities presented in the initial draw, and these adjustments take place in the first decade of the 20 year regime period. For example, in the first decade, financial wealth increases by a factor of six in the low return-high volatility draw, but by a factor of 10 in the high return-low volatility case. In the second decade, financial wealth increases uniformly by 80% across all three scenarios. More minor differences between scenarios, perhaps reflecting differences in accumulations, can be found in financial wealth changes in the decade following the first regime shift, with no difference being found in the second decade of that regime. Thereafter, proportional differences in financial wealth adjustments are insignificant.

**Table 6**  
**The role of volatility change under regime switching (¥ million)**

<b>Initial Regime</b>	<b>Age</b>	<b>30</b>	<b>40</b>	<b>50</b>	<b>60</b>	<b>70</b>	<b>80</b>	<b>90</b>	
$\alpha = 0.02$	Expected consumption	2.46	2.51	2.56	2.77	2.98	3.22	3.52	
	$\sigma = 0.25$	Expected: Full wealth	138.6	123.7	107.3	96.1	82.37	66.1	48.5
		Financial wealth	4.3	26.4	47.9	75.4	82.4	66.1	48.5
		Proportion in risky ( $\pi$ )	0.16	0.16	0.16	0.40	0.40	0.40	0.40
$\alpha = 0.025$	Expected consumption	2.50	2.64	2.78	2.98	3.18	3.42	3.73	
	$\sigma = 0.20$	Expected: Full wealth	138.6	128.4	115.6	102.3	87.8	70.3	51.1
		Financial wealth	4.3	31.1	56.2	81.6	87.8	70.3	51.1
		Proportion in risky ( $\pi$ )	0.32	0.32	0.32	0.39	0.40	0.40	0.40
$\alpha = 0.03$	Expected consumption	2.66	3.04	3.27	3.40	3.58	3.84	4.17	
	$\sigma = 0.15$	Expected: Full wealth	138.6	138.1	132.9	116.5	98.7	78.8	57.5
		Financial wealth	4.3	40.8	73.5	95.8	98.7	78.8	57.5
		Proportion in risky ( $\pi$ )	0.63	0.60	0.63	0.37	0.39	0.40	0.40

The proportion in risky again follows the Merton rule:  $(\alpha - r)/[(1 - \gamma)\sigma^2]$ . For the first (known) regime, the proportion in risky is as predicted for the combinations of the expected risky rate of return and volatility stipulated. Thereafter, from the termination of the first regime period, the estimated proportion in risky is close to the predicted average of the proportion in risky that would be found under each of the regimes separately: 0.41.

These results point to a limitation in our modeling of regime switching. There appears to be a disjunction between the periodicity of the value function, and that implicit in our representation of regime switching. Once the individual is sure of the initial regime, asset allocation decisions are made as if no future regime shift is contemplated. The exposure to risk which is chosen is consistent with the Merton prescription for the risk and return specification associated with the initial regime, even though the individual has knowledge of possible future shifts in regime. In expectation, the proportion allocated to risky assets under future regimes is simply the average of the Merton prescriptions for the admissible regimes. Yet a rational consumer with a lifetime horizon would in general take account of possible regime shifts in deciding current asset allocations.

#### **IV. Concluding remarks**

This paper has presented and demonstrated the use of a stochastic dynamic programming algorithm suitable for generating numerical solutions to complex life-cycle financial planning problems. The computational procedure allows expected values of relevant control variables to be determined throughout the lifetime, so that expected paths of these variables may be traced under alternative conditions. It is also feasible to report

other points on the distributions of these control variables at different points in time: for example, the probability that the value of a variable lies below 75 per cent of the expected value. The algorithm reproduces, to a high degree of accuracy, values of model variants for which analytic results can be computed.

To illustrate the use of the procedure, we have applied a simple representation of regime switching which incorporates persistence and mean reversion into the investment returns innovations process. Results are very preliminary, but nevertheless indicate that wealth accumulation decisions, and investment allocation choices, are likely to be systematically conditioned by the introduction of such phenomena. Since persistence and mean reversion have been reported in careful analyses of financial market behaviour, this is a potentially significant consideration that has hitherto been addressed only for special analytically tractable cases (e.g., Wachter 1999). At the same time, however, our results have limitations of their own, and these we readily acknowledge.

The major strength of the methodology is that it allows computational research into lifetime financial planning to be liberated from the confines of formulations admitting closed form solution. It therefore has the potential to address not only more complex and complete representations of persistence and mean reversion, but also the introduction of stochastic labour market innovations, and other “real world” considerations such as liquidity constraints. In future research, we plan to apply the procedure developed here to introduce a model-consistent representation of labour market uncertainties.

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