The Pricing of Tranched Longevity Bonds

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Outline

1. Longevity and Mortality Risk
2. Risk Management Strategies
3. Longevity Risk Securitization
4. Models for Mortality
5. The Proposed Mortality Model
6. The Longevity Bond
7. The Pricing Model
8. Data and Assumptions
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1. Longevity and Mortality Risk

An Increasing Exposure:

- Longevity is improving with greater variability.
- OECD Male 60-64 Labour Participation:
  - 60-90% (1970s) to 20-50% (today).
- Shift to DC Superannuation.
- Australian Super Industry:
  - $912b assets (June 2006),
  - 2/3 DC or Hybrids.
- Australian Life Annuities:
  - $4.3b assets (June 2006).
- Supply/demand constraints (Purcal, 2006).
- Reinsurance:
  - Longevity is “toxic” (Wadsworth, 2005).
2. Risk Management Strategies

1. Avoidance
   - Participating Annuities
   - Reverse Mortgages

2. Retention
   - Capital Reserves
   - Contingent Capital

3. Transfer
   - Reinsurance
   - Bulk Purchase Annuities
   - Securitization

4. Hedging
   - Natural Hedges
   - Survivor Bonds
   - Mortality Swaps
   - Longevity Options and Futures
3. Longevity Risk Securitization

Securitization Background

- Vehicle for risk transfer: CDOs in the late 1980s.
- Insurance-Linked Securitization – USD 5.6b issued in 2006 (Lane and Beckwith, 2007).
  - Insurance-Linked Bonds
  - Industry Loss Warranties
  - Sidecars
- Survivor Bond Issues (BNP Paribas/EIB, 2004).
- Benefits:
  - Improved capacity for risk transfer. Tranching broadens appeal to investors.
  - Issuer can tailor issue to manage basis risk vs. moral hazard / info. asymmetry.
  - Diversification benefits for investors.
4. Models for Mortality


\[
\ln[m(x, t)] = a_x + b_x k_t + \varepsilon_{x,t}
\]

where

\[
\sum_{x} b_x = 1 \quad \text{and} \quad \sum_{t} k_t = 0
\]


\[
d \mu(t, x) = \alpha(t, x, \mu(t, x)) dt + \sigma(t, x, \mu(t, x)) dB_t
\]

- With a specific form based on Cox et al (1985):

\[
d \mu(t, x + t) = \left( \beta(t, x) - \gamma(t, x) \mu(t, x + t) \right) dt + \rho(t, x) \sqrt{\mu(t, x + t)} dB_t
\]

c. Forward Rate Models:

- Model the dynamics of the forward mortality surface.

- Based on work by Heath, Jarrow and Morton (1992).
4. Models for Mortality


- Pricing employs the Wang (1996, 2000, 2002) transform that shifts the survival curve using a fixed ‘price of risk’, $\lambda$:

$$F^*(t) = \Phi[\Phi^{-1}(F(t)) - \lambda]$$


- This method has been subject to criticism (Cairns et al, 2006; and Bauer and Russ, 2006) as it does not incorporate varying $\lambda$ over age and time.
5. The Proposed Model

i) A Multivariate Mortality Process

- For lives at time $t$, initially aged $x$, the mortality rate $\mu(x,t)$ is given by:

$$d\mu(x,t) = \left(a(x+t) + b\right)\mu(x,t)dt + \sigma \mu(x,t)dW(x,t) \text{ for all } x.$$  

- This falls within the Dahl (2004) family of models.

- To incorporate dependence, we introduce a M.V. random vector $dW(t)$, length $N$:

$$dW(t) = \Delta dZ(t),$$

with each element

$$dW(x,t) = \sum_{i=1}^{N} \delta_{x,i}dZ_i(t) \text{ for all } x.$$  

- Where $dZ(t)$ is a random vector of independent B.M. of length $N$; and $\Delta$ is a $N \times N$ matrix of constants, such that:

$$\begin{bmatrix} dW(x_1,t) \\ \vdots \\ dW(x_N,t) \end{bmatrix} = \begin{bmatrix} \delta_{11} & \ldots & \delta_{1N} \\ \vdots & \ddots & \vdots \\ \delta_{N1} & \ldots & \delta_{NN} \end{bmatrix} \begin{bmatrix} dZ_1(t) \\ \vdots \\ dZ_N(t) \end{bmatrix}$$

Note: the dimension of $dZ(t)$ can be reduced using PCA.
5. The Proposed Model

i) A Multivariate Mortality Process

- The covariance matrix of \( dW(t) \), \( \Sigma \), has each element:

\[
\text{Cov}(\{dW(x_n, t), dW(x_m, t)\}) = \sum_{i=1}^{N} \delta_{ni} \delta_{im} \text{Var}(dZ_i(t))
\]

\[
= \sum_{i=1}^{N} \delta_{ni} \delta_{im} dt.
\]

- This gives the Cholesky decomposition of \( \Sigma \).

\[
\Sigma = (\Delta \sqrt{dt}) (\Delta \sqrt{dt})^	op
\]

ii) Incorporating Age-Dependence

- Using PCA, decompose \( \Sigma \) into its eigenvectors \( V \), and eigenvalues (diagonal matrix \( T \)):

\[
\Sigma = VTV' \\
V \sqrt{T} = \Delta \sqrt{dt}
\]

- Simulations of \( dW(t) \) can be generated with the same dependence properties:

\[
d\hat{W}(t) = V \sqrt{T} \eta
\]

<table>
<thead>
<tr>
<th># of Eigenvectors</th>
<th>% of Observed Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29.3%</td>
</tr>
<tr>
<td>5</td>
<td>69.8%</td>
</tr>
<tr>
<td>10</td>
<td>85.1%</td>
</tr>
<tr>
<td>15</td>
<td>92.4%</td>
</tr>
<tr>
<td>20</td>
<td>96.5%</td>
</tr>
<tr>
<td>25</td>
<td>97.1%</td>
</tr>
<tr>
<td>30</td>
<td>99.1%</td>
</tr>
<tr>
<td>31</td>
<td>99.5%</td>
</tr>
<tr>
<td>32</td>
<td>100.0%</td>
</tr>
</tbody>
</table>
6. The Longevity Bond

The Proposed Longevity Bond Structure

- Both the PL and the LL are based on the percentage cumulative losses incurred on an underlying annuity portfolio:

\[ CL(t) = \frac{\sum_{s=1}^{t} L(s)}{FV} \]

- Where the loss on the portfolio in each period is:

\[ L(t) = \left( A \sum_{\text{all } x} l(x, t) - E \left[ A \sum_{\text{all } x} l(x, t) \right] \right)^+ \]

\[ \approx \left( A \sum_{\text{all } x} l(x, 0) tP_x - A \sum_{\text{all } x} l(x, 0) t\bar{P}_x \right)^+ \]
6. The Longevity Bond

- The total variance of the number of lives alive at time $t$, initially aged $x$ is given by:

$$Var[l(x,t)] = E[Var[l(x,t)|tp_x]] + Var[E[l(x,t)|tp_x]].$$

- The first term gives the binomial variability in the portfolio given a fixed $tp_x$ (the focus of Lin and Cox, 2005).

- The second is the variability due to changes in the mortality rate, which accounts for almost all of the portfolio variance:

Variability in $tp_x$ accounts for almost all the variability in $l(x,t)$. 
6. The Longevity Bond

Tranching

- Tranche losses are allocated by the cumulative loss on the portfolio. From this we can find the cumulative tranche loss:

\[
CL_j(t) = \begin{cases} 
0 & \text{if } L(t) < K_{A,j}; \\
CL(t) - K_{A,j} & \text{if } K_{A,j} \leq L(t) < K_{D,j}; \\
K_{D,j} - K_{A,j} & \text{if } L(t) \geq K_{D,j},
\end{cases}
\]

where

\[
CL(t) = \sum_{j=1}^{J} CL_j(t),
\]

Portfolio cumulative loss simulations.

- The tranche loss as a percentage of its prescribed principal is given by:

\[
TCL_j(t) = \frac{E[CL_j(t)]}{K_{D,j} - K_{A,j}}.
\]

- The assumed tranche thresholds are:

<table>
<thead>
<tr>
<th>Tranche</th>
<th>(K_{A,j})</th>
<th>(K_{D,j})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0%</td>
<td>15%</td>
</tr>
<tr>
<td>2</td>
<td>15%</td>
<td>30%</td>
</tr>
<tr>
<td>3</td>
<td>30%</td>
<td>100%</td>
</tr>
</tbody>
</table>
7. The Pricing Model

- The premium on tranche \(j\), \(P^*_j\), is set to equate the cashflows on the premium leg \((PL_j)\), and the loss leg \((LL_j)\):

\[
PL_j = \sum_{t=1}^{T} P_j B(0, t-1)[1 - TCL_j(t-1)]
\]

\[
LL_j = \sum_{t=1}^{T} B(0, t)[TCL_j(t) - TCL_j(t-1)]
\]

such that \(PL_j(P^*_j) - LL_j(P^*_j) = 0\).

where:
- \(B(0,t)\) is the price of a ZCB.
- \(TCL_j(t)\) is the tranche % cum. loss at time \(t\).

- Premiums need to be set under a risk-adjusted \(Q\) mortality measure. Using the Cameron-Martin-Girsanov Theorem:

\[
dW^Q(x, t) = \sum_{i=1}^{N} \delta_{xi} (dZ_i(t) + \lambda_i(t)dt)
\]

\[
= dW(x, t) + \sum_{i=1}^{N} \delta_{xi} \lambda_i(t)dt.
\]

and for all ages: \(dW^Q(t) = dW(t) + \Delta\lambda(t)dt\)

where \(\Delta\lambda(t)\) is a ‘risk adjustment’ that can differ for each age and time.

- and the risk adjusted mortality process is:

\[
d\mu^Q(x, t) = \left[\mu(x + t) + b + \sum_{i=1}^{N} \delta_{xi} \lambda_i(t)\right] \mu^Q(x, t)dt + \sigma \mu^Q(x, t)dW(x, t)
\]
7. The Pricing Model

- However, the choice of \( Q \), and thus \( \Delta \lambda(t) \) is not unique (like IR derivatives). It thus needs to be calibrated to market prices.

- These are approximated using an empirical model proposed by Lane (2000), fit to the price of 2007 mortality bond issues using non-linear least squares:

\[
\hat{P}_j^L = EL_j + EER_j
\]

\[
EER_j = \gamma(PFL_j)^\alpha(CEL_j)^\beta
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>2006-07 Mortality Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>0.9980</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.8965</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.5034</td>
</tr>
<tr>
<td>( X^2 )</td>
<td>0.04</td>
</tr>
<tr>
<td>( \chi_5^2 ) at 99%</td>
<td>2.09</td>
</tr>
</tbody>
</table>

- To facilitate calibration with limited data, simplifying assumptions are made on the risk adjustment:

\[ \Delta \lambda(t) = \lambda^* \text{ where } \lambda^* = [\lambda^*, \ldots, \lambda^*]' \]

So that for each \( x \) and \( t \):

\[ d\mu^Q(x, t) = \left( a(x + t) + b - \sigma \lambda^* \right) \mu^Q(x, t)dt + \sigma \mu^Q(x, t)dW(x, t) \]

- \( \lambda^* \) is chosen so that: \( P_j^{\lambda^*} = P_j^L \)  
- As a result, \( \lambda_j^* = f(PFL_j, CEL_j, \gamma, \alpha, \beta) \)
8. Data and Assumptions

Data

Assumptions

<table>
<thead>
<tr>
<th>Parameter</th>
<th>MLE: Male</th>
<th>MLE: Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>-9.4398E-04</td>
<td>2.6993E-04</td>
</tr>
<tr>
<td>$b$</td>
<td>0.1347</td>
<td>0.0608</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.0906</td>
<td>0.0873</td>
</tr>
</tbody>
</table>

Mortality process parameter estimates.

- $dW(t)$ is modeled under 3 assumptions of age dependence:
  1. Perfect age independence.
  2. Observed age dependence using PCA.
  3. Perfect age dependence.

### Proposed Longevity Bond Assumptions

- **Bond Face Value:** $FV = \$750,000,000$.
- **Term to Maturity:** $T = 20$ years.
- **Payment Frequency:** Annually, for both premium and loss payments.
- **Number of Tranches:** $J = 3$.
- **Initial Age of Annuities:** $x = 50, \ldots, 79$.
- **Initial No. of Annuities:** $n(x, 0) = 60,000$. We assume this is evenly distributed between the 30 ages, with $l(x, 0) = 2,000/yx$.
- **Annuity Payments:** $A = \$50,000$ paid at the end of each year to each living annuitant.
9. Results

The Mortality Model

A 20 year projection of male expected mortality (linear and log scales).

95% confidence intervals for projected male mortality, under 3 age-dependence assumptions.
9. Results
The Mortality Model

- Analysis of Fit:

Fitted residuals (left) and descriptive statistics (above).

Asymptotic variance/covariance matrix for MLE estimates.

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>5.53E-13</td>
<td>5.13E-13</td>
</tr>
<tr>
<td>b</td>
<td>4.24E-11</td>
<td>3.94E-11</td>
</tr>
<tr>
<td>σ</td>
<td>5.01E-11</td>
<td>-4.48E-11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Error</th>
<th>Standard Deviation</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Confidence Level (95.0%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>-3.12E-03</td>
<td>0.024</td>
<td>0.972</td>
<td>-6.320</td>
<td>6.277</td>
<td>0.047</td>
</tr>
<tr>
<td>Female</td>
<td>4.84E-08</td>
<td>0.025</td>
<td>1.000</td>
<td>-4.396</td>
<td>14.459</td>
<td>0.049</td>
</tr>
</tbody>
</table>

Pearson’s chi-square statistic.

\[ X^2 \text{ Male} = 71.08, \quad X^2 \text{ Female} = 23.16, \quad \chi^2_{388} \text{ at 99%} = 326.15 \]
9. Results
The Longevity Bond

Portfolio expected cumulative loss and 95% bounds.

Tranche expected cumulative loss and 95% bounds under 3 age-dependence assumptions.
9. Results
The Longevity Bond

Tranche cumulative losses, disaggregated by age.
9. Results
The Pricing Model

- Calibrated tranche premiums and associated ‘prices of risk’ $\lambda$. Consistent with risk averse investors.

- In the absence of a closed form, sensitivities of $\lambda$ to the inputs into the Lane (2000) model are provided based on mortality rates under observed age dependence.

$$\lambda_j^* = f(P\hat{F}L_j, C\hat{E}L_j, \gamma, \alpha, \beta)$$

<table>
<thead>
<tr>
<th>Tranche</th>
<th>Premium</th>
<th>$\lambda_j^*$</th>
<th>$P\hat{F}L_j$</th>
<th>$C\hat{E}L_j$</th>
<th>$\gamma$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2058</td>
<td>2.52</td>
<td>-</td>
<td>1.39</td>
<td>2.04</td>
<td>-</td>
<td>-3.52</td>
</tr>
<tr>
<td>2</td>
<td>371</td>
<td>0.31</td>
<td>1.24</td>
<td>0.76</td>
<td>1.21</td>
<td>-1.76</td>
<td>-2.33</td>
</tr>
<tr>
<td>3</td>
<td>31</td>
<td>0.25</td>
<td>0.29</td>
<td>0.17</td>
<td>0.31</td>
<td>-0.95</td>
<td>-0.79</td>
</tr>
</tbody>
</table>
9. Results

Implications of Results

- Mortality can effectively be modelled as a dynamic, multi-age process.
- Tranched longevity bonds provide an effective vehicle for managing longevity risk.
- Dynamic mortality models are well suited to pricing longevity-linked securities.

Further Research

- Calibration of the risk-adjusted mortality process.
- Application of the proposed mortality model to a broader range of ages
- Alternative definitions for portfolio loss, eg. changes in future obligations on the annuity portfolio (Sherris and Wills, 2007).
10. Conclusion

The Mortality Model

- The first time that the Dahl (2004) framework has successfully fit changes in mortality by age and time simultaneously. Importance of age-dependence. Implications for modelling mortality-linked securities on multi-age portfolios.

- First time that the Dahl family has been considered in an Australian context.

The Longevity Bond

- The first consideration of a longevity-linked security on multiple ages.

- The first detailed analysis of the impact of tranching, under a range of age dependence assumptions.

The Pricing Model

- Mortality model is sufficiently flexible to allow the ‘price of risk’ to vary by age and time. Incorporate range of investor sentiments.

- Calibrated price of risk consistent with risk averse investor with non-linear risk/return tradeoff.
References


- Lane, M.N., and Beckwith, R., 2007, That was the Year that was! The 2007 Review of the Insurance Securitization Market, Lane Financial L.L.C. (Available at http://lanefinancialllc.com/)


Questions and Comments