Stochastic Control Theory for Optimal Investment

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Abstract

This paper illustrates the application of stochastic control methods in ruin theory. We present concepts, such as the Hamilton-Jacobi-Bellman equation, and review recent results to illustrate their use in ruin theory. In particular, given an insurance business and a fixed amount for investment in a portfolio consisting of one riskless asset and one risky asset, the optimal investment strategy that minimizes infinite time ruin probability is determined using the Hamilton-Jacobi-Bellman equation of the optimization problem.

In collective risk theory for non-life insurance, one is concerned with modelling surplus described as

$$\text{Surplus} = \text{Initial capital} + \text{Income - Outflow}.$$

A particular concern in this area of study is to describe the dynamics of the risk process often modelled as

$$R(t) = r + ct - \sum_{k=1}^{N(t)} X_k$$

where $r \geq 0$ is the initial value, $c > 0$ is the premium rate, $\{N(t), t \geq 0\}$ is a Poisson process with intensity λ used here to model the number of claims in the time interval $(0, t]$ and $\{X_k : k = 1, 2, ...N(t)\}$ is a family of independent, identically distributed positive random variables, each independent of $\{N(t), t \geq 0\}$, used to model the claim size. The aggregate claim

$$S(t) = \sum_{k=1}^{N(t)} X_k$$

represents the outflow in the risk process and the income is the premium paid by the policy holders. It is assumed that the premium income accrues linearly through time at some constant rate c, so that at a certain time t, the total premium accrued is ct.

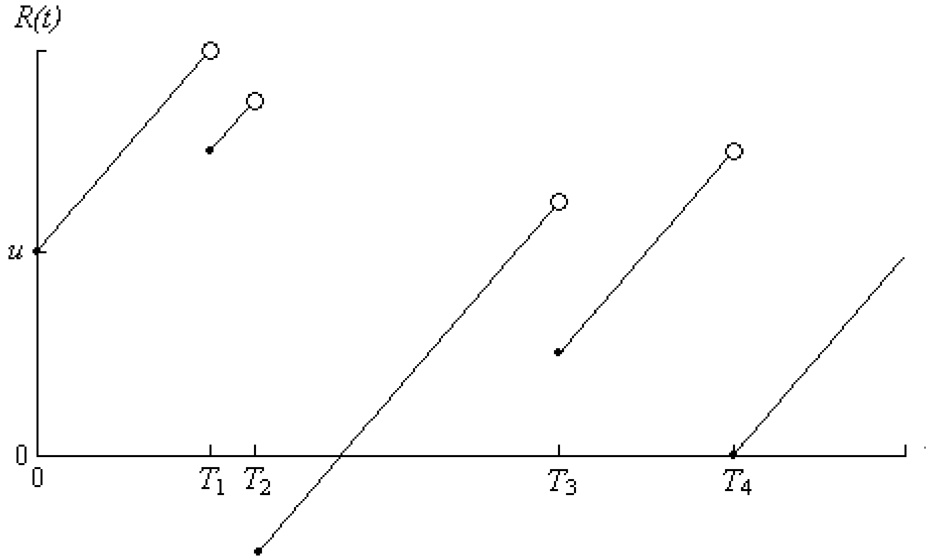

Figure 1: A typical risk process for an insurance business

Figure 1 shows a typical plot of the risk process. Note that the value of the process increases linearly with slope c, until a claim occurs. The time when the i^{th} claim occurs is designated by T_i, where $i = 1, 2, \ldots$. When a claim occurs, the process has a downward jump indicating a decrease in the company's surplus due to payment of the claim. The risk process assumes an instantaneous payment of claims that is why the wealth immediately goes down at the claim times.

This classical risk model was first considered by Filip Lundberg in 1903. His use of the standard compound Poisson model was later made mathematically rigorous by Harald Cramer in the 1930s. This model, also known as the *Cramer-Lundberg* risk model, has since then been extended in various ways: general renewal processes and Cox processes replace the Poisson process; a random environment allows for random changes in the intensity of the claims process and of the claim size distribution; dependent claims for the claims process;

interest rates are considered in the premium income side; and piecewise deterministic Markov processes provide new insight and models.

An exciting area of collective risk theory is ruin theory, where first-passage events above a high threshold are considered. In particular, the first time surplus becomes negative is investigated. Of interest in ruin theory are the probabilities associated with the random time of ruin. If $\{U(t), t \geq 0\}$ is the surplus process for the insurance business, the time of ruin, denoted by τ, is defined to be

$$\tau = \inf\{t \geq 0 : U(t) < 0\}.$$

That is, ruin is defined to be the event when surplus becomes negative and τ is the first time ruin happens. Ruin as described is just a technical term. It does not mean that the business is insolvent. Here, one can interpret u, the initial surplus, as the amount the company is willing to risk for an insurance business and ruin can be interpreted as a signal that the company has to take action in order to make the business profitable.

The quantities of interest in ruin theory are the ruin probabilities associated with the initial capital u namely

$$\psi(u) = \Pr\{U(t) < 0 \;\text{ for some }\; t \geq 0\}$$

and

$$\psi(u, s) = \Pr\{U(t) < 0 \;\text{ for some }\; t \leq s\}.$$

One would like to minimize these ruin probabilities subject to the dynamics of the surplus process. In practice, an actuary is often asked to take decisions with regards to new or existing insurance business. Strategies that result from minimizing the above ruin probabilities may be considered when business policies are determined.

Recently, a lot of interest is generated by the use of mathematical tools from stochastic control theory in addressing the problem of minimizing the infinite time ruin probability. Investment, new business, reinsurance and dividend payment are only a few of the many control variables that are adjusted dynamically in an insurance business. By means of a standard control tool such as the Hamilton-Jacobi-Bellman equation, optimal solutions can be characterized and computed, often numerically, and the smoothness of the value can be shown. The optimization of reinsurance programs in the framework of controlled diffusion was considered by Hojgaard and Taksar in [8] and [9] , Schmidli in [10] and recently, Hipp and Vogt [7]. Issuance of new business to deal with ruin was considered by Hipp and Taksar in [6]. Works involving investment strategies for controlled processes are presented in [1],[3],[4] and [5] Recently, Schmidli in [11] has obtained results on the simultaneous dynamic control of proportional reinsurance and investment.

In this paper we give an illustration of how stochastic control notions are applied in computing the investment strategy that would minimize the probability of ruin for a given block of business. For the sake of simplicity, we consider the hypothetical situation where the company has current surplus u, a fixed amount for investment a available at any time t which is independent of the business surplus, and an investment portfolio consisting of one risky asset and one riskless asset. At each time t, a proportion $b(t) \in [0,1]$ of a is invested in the risky asset and this changes the dynamics of the business. The investment strategy chosen is the proportion $b(t)$ of the fixed amount a to be invested in the risky asset. The strategy $b(t)$ is chosen predictable, i.e. it depends on all information available before time t.

To give a mathematical formulation to the optimization problem, we start with a probability space (Ω, F, P) with filtration $\{F_t, t \geq 0\}$ and a stochastic process $\{W(t)\}$, which is standard Brownian motion adapted to $\{F_t, t \geq 0\}$. The filtration represents the information available at time t and any decision is made based on this information.

We model the surplus process of an insurance business, its risk modeled by a Cramér-Lundberg process, i.e.

$$dR(t) = cdt - dS(t), \qquad R(0) = r. \tag{1}$$

The company is about to decide on an investment strategy for a block of business that is aimed to minimize the infinite time ruin probability. The investment portfolio consists of a riskless asset whose price $B(t)$ follows

$$\frac{dB(t)}{B(t)} = \rho dt, \qquad B(0) = b, \tag{2}$$

and a risky asset whose price $Z(t)$ follows a geometric Brownian motion

$$\frac{dZ(t)}{Z(t)} = \mu dt + \sigma dW(t), \qquad Z(0) = z \tag{3}$$

where ρ, μ, and σ are positive constants. The company has the following position:

a. A fixed amount a will be invested at any time t.

b. A fraction $b(t)$ of a, where $b(t) \in [0,\ 1]$, will be invested at time t in the risky asset, the remaining part in the riskless asset.

c. The fraction $b(t)$ may change through time depending on which combination of risky and riskless asset minimizes the infinite time ruin probability.

Following the position of the company, the investment return process $I(t)$ from the amount a is given by

$$\begin{aligned} dI(t) &= a[1-b(t)]\,\frac{dB(t)}{B(t)} + ab(t)\,\frac{dZ(t)}{Z(t)} \\ &= a[1-b(t)]\rho\ dt + ab(t)\mu\ dt + ab(t)\sigma\ dW(t)\ . \end{aligned} \tag{4}$$

The surplus process of the business is then given as the sum of the risk process from the insurance business and the investment return process,

$$dU(t) = dR(t) + dI(t), \qquad U(0) = u. \tag{5}$$

$U(t)$ depends on the composition of the investment portfolio in which the fixed amount a is invested. The surplus process is then influenced by the investment strategy $b = \{b(t), t \geq 0\}$ and we want to find the strategy b^* that minimizes the infinite time ruin probability subject to the dynamics of the surplus process, i.e.

$$\begin{array}{c}\text{minimize} \quad \psi(u) \\ b\end{array}$$

where $\psi(u) = Pr\{U(t) < 0 \quad \text{for some} \quad t \geq 0\}$. The strategy b^* solving the problem above will be determined through the Hamilton-Jacobi-Bellman equation of the control problem. For this task, we make use of the the survival probability $\delta(u) = 1 - \psi(u)$ and the equivalent control problem

$$\begin{array}{c}\text{maximize} \quad \delta(u). \\ b\end{array}$$

The Hamilton-Jacobi-Bellman equation

Let $\psi_b(u)$ denote the ruin probability associated with the arbitrary strategy $b = \{b(t), t \geq 0\}$ and the corresponding survival probability by $\delta_b(u) = 1 - \psi_b(u)$. If b^* is the optimal strategy, then the associated survival probability function is maximal and it follows that

$$\delta_{b^*}(u) \geq \delta_b(u)\ .$$

For simplicity, the subscript b^* will be dropped and $\delta(u)$ will denote the survival probability for b^*. Furthermore, we assume that $\delta(u)$ is twice differentiable.

Consider a time interval $[0,\ h]$ in which the fixed amount a is placed in the investment portfolio. Note that the surplus process has the following dynamics in the given time interval:

a. A claim amount X is incurred with probability $\lambda h + o(h)$ and the probability that there are no claims within this period is $1 - \lambda h + o(h)$. This follows from the assumption that the number of claims is a Poisson process with intensity λ.

b. An amount $ch + o(h)$ is received as a premium income from the business.

c. An amount $\int_0^h a[1 - b(s)]\rho \; ds + o(h)$ is received as an investment income from the riskless asset.

d. An amount $\int_0^h ab(s)\mu \; ds + \int_0^h ab(s)\sigma \; dW(s) + o(h)$ is received as an investment income from the risky asset.

From the above discussion, the dynamics of the surplus process can be described as follows:

a. A claim of amount X occurs with probability $\lambda dt + o(dt)$ and no claim occurs with probability $1 - \lambda dt + o(dt)$.

b. An amount $cdt + o(dt)$ is received as a premium income.

c. An amount $a[1 - b(t)]\rho \; dt + o(dt)$ is received as an investment income from the riskless asset.

d. An amount $ab(t)\mu \; dt + ab(t)\sigma \; dW(t) + o(dt)$ is received as an investment income from the risky asset.

Two distinct cases are considered over the time interval $[t, \; t + dt]$: either there is no claim or there is exactly one claim. If there is no claim, the surplus of the business grows to $u + cdt + dI(t)$, where $dI(t)$ is given in equation (4). The quantities cdt and $dI(t)$ can be seen as increments in the premium income and investment income, respectively. Now, if there is a claim, the surplus of the company reduces to $u + dI(t) - X$, where X is the random claim size. Here we assume that no premium is received during the period $[t, t + dt]$.

For an arbitrary strategy b, the associated survival probability $\delta_b(u)$ can now be determined by considering the described cases. Taking expectations,

$$\delta_b(u) = \lambda dt \; E[\delta_b\left(u + dI(t) - X\right)] + (1 - \lambda dt) \; E[\delta_b\left(u + cdt + dI(t)\right)] \; .$$

The first term on the right-hand side represents the expected survival probability if there is a claim. The second term gives the expected survival probability if there is no claim. Since $\delta(u) \geq \delta_b(u)$, it follows that

$$\delta(u) \geq \lambda dt \; E\left[\delta_b\left(u + dI(t) - X\right)\right] + (1 - \lambda dt) \; E[\delta_b\left(u + cdt + dI(t)\right)] \; ,$$

or equivalently,

$$\begin{aligned} 0 &\geq E[\delta\,(u+cdt+dI(t))-\delta(u)]+\lambda dt\ E[\delta\,(u+dI(t)-X)] \\ &- \lambda dt\ \delta\,(u+cdt+dI(t)) \\ 0 &\geq \frac{1}{dt}\ E[d\delta(V(t))]+\lambda\ E\,[\delta\,(u+dI(t)-X)]-\lambda\ \delta\,(u+cdt+dI(t))\ , \end{aligned} \tag{6}$$

where $V(t)=u+pt+I(t)$. and the process $V(t)$ can be interpreted as the income process. Using equation (4) and the Itô's formula, $d\delta(V(t)$ can be written as

$$\begin{aligned} d\delta(V(t)) &= \frac{1}{2}\ \sigma^2a^2\ b(t)^2\ \delta''(V(t))\ dt \\ &+ \ \{\ cdt+a[1-b(t)]\rho\ dt+ab(t)\mu\ dt+\sigma ab(t)\ dW(t)\ \}\,\delta'(V(t))\ . \end{aligned}$$

Hence,

$$\begin{aligned} E\,[d\delta(V(t))] &= \frac{1}{2}\ \sigma^2a^2\ b(t)^2\ \delta''(V(t))\ dt \\ &+ \ \{\ cdt+a[1-b(t)]\rho\ dt+ab(t)\mu\ dt+\sigma ab(t)\ E[dW(t)]\ \}\,\delta'(V(t))\ . \end{aligned}$$

Since the Brownian motion $W(t)$ is a martingale, $E[dW(t)]=0$. Applying this, the preceding equation becomes

$$E\,[d\delta(V(t))]=\frac{1}{2}\ \sigma^2a^2\ b(t)^2\ \delta''(V(t))\ dt+\{\ pdt+a[1-b(t)]\rho\ dt+ab(t)\mu\ dt\ \}\,\delta'(V(t))\ .$$

Inequality (6) can now be written as

$$\begin{aligned} 0 &\geq \frac{1}{2}\ \sigma^2a^2b(t)^2\delta''(V(t))+\{c+a[1-b(t)]\rho+ab(t)\mu\}\ \delta'(V(t)) \\ &+ \ \lambda\ E[\delta(u+dI(t)-X)]-\lambda\ \delta(u+cdt+dI(t))\ . \end{aligned}$$

As dt approaches 0, the value of $dI(t)$ also approaches zero and the preceding inequality becomes

$$0\geq\frac{1}{2}\ \sigma^2a^2b(t)^2\delta''(V(t))+\{c+a[1-b(t)]\rho+ab(0)\mu\}\ \delta'(V(t))+\lambda\ E[\delta(u-X)-\delta(u)]\ .$$

The above inequality must hold for all admissible strategies b. If b is optimal, its corresponding survival probability $\delta_b(u)$ should be close to $\delta(u)$, so intuitively equality should be derived. For an arbitrary startegy b, the HJB equation for the problem is given by

$$0=\sup_b\ \left\{\frac{1}{2}\ \sigma^2a^2b^2\delta''(u)+[c+a(1-b)\rho+ab\mu]\ \ \delta'(u)+\lambda\ E\,[\delta(u-X)-\delta(u)]\right\}, \tag{7}$$

where $\delta(u) = 0$ for $u < 0$ and $\delta(\infty) = 1$. Equation (7) is called the *Hamilton-Jacobi-Bellman equation* for the control problem.The optimal survival probability $\delta(u)$ and its underlying proportion process b gives the survival behavior upon acquiring a surplus level u. At this point the probability $\delta(u)$ can be regarded as a function of the surplus alone.

An optimal strategy is derived from a solution $(\delta(u), b(u))$ of this HJB equation for all surplus u. Letting $b(u) = b$, each solution, if any, has the following properties:

$$\frac{1}{2}\,\sigma^2 a^2 b^2 \delta''(u) + [c + a(1-b)\rho + ab\mu]\;\delta'(u) + \lambda\; E\left[\delta(u - X) - \delta(u)\right] = 0,$$

and for arbitrary strategy b we have

$$0 \geq \frac{1}{2}\,\sigma^2 a^2 b^2 \delta_b''(u) + [c + a(1-b)\rho + ab\mu]\;\delta_b'(u) + \lambda\; E\left[\delta_b(u - X) - \delta_b(u)\right].$$

Several properties of the survival probability function $\delta(u)$ are immediately drawn out. First, one can show that $\delta(u)$ is an increasing function of u. If u and v are initial capital values such that $0 \leq u < v$, then ruin cannot occur for initial capital v before ruin occurs for initial capital u. This shows that $\delta(u) \leq \delta(v)$.

As a special case, if the surplus is zero, the optimal strategy will be $b(u) = 0$. This is evident from the fact that if most of the amount a is invested in the risky asset then there will be greater chances of shortage of surplus to pay out possible early claims. Since the investment income can be negative, the premium income alone, accumulated for a short period of time, will not be sufficient to counter claims. It should be noted that the amount a is taken as a separate fund and is not considered a part of the company's surplus. It will follow from equation(7) that

$$\begin{aligned} 0 &= (c + a\rho)\;\delta'(0) + \lambda\; E\left[\delta(0 - X) - \delta(0)\right] \\ &= (c + a\rho)\;\delta'(0) - \lambda\delta(0)\;, \end{aligned}$$

and

$$\delta'(0) = \frac{\lambda\delta(0)}{c + a\rho}\;. \tag{8}$$

The quantity inside the braces in the HJB equation is maximized by b satisfying

$$\sigma^2 a^2 b\;\delta''(u) + (-a\rho + a\mu)\;\delta'(u) = 0$$

which gives

$$\tilde{b} = \frac{(\rho - \mu)\;\delta'(u)}{a\sigma^2\delta''(u)}\;. \tag{9}$$

If $\tilde{b} \in [0, 1]$ then the optimal strategy is $b^*(u) = \tilde{b}$. Notice, however, that the value of $\tilde{b}$ is not necessarily inside the indicated interval. For values of $\tilde{b}$ outside the interval, it will be necessary to consider the endpoints. Observe that if $\tilde{b}$ is less than 0, the riskless asset is more advantageous than the risky asset. In this case, $b^*(u)$ should be defined $b^*(u) = 0$. A different scenario is achieved when $\tilde{b}$ is greater than 1. This time, the risky asset is more advantageous than the riskless asset and the optimal strategy therefore should be $b^*(u) = 1$.

Notice that equation (7) is quadratic in b. The supremum value of the quantity inside the braces is therefore attained when $b = 0$, $b = 1$, or $b = \tilde{b}$. Specifically, if the supremum is attained at b^* then

$$b^*(u) = \begin{cases} 0 & \text{if} \quad \tilde{b} < 0 \\ \tilde{b} & \text{if} \quad 0 \le \tilde{b} \le 1 \\ 1 & \text{if} \quad \tilde{b} > 1 \end{cases}$$

The optimal survival probabilities

The survival probabilities $\delta(u)$ for the optimal strategy cases $b^*(u) = 0$, $b^*(u) = 1$, and $b^*(u) = \tilde{b}$, denoted by $\delta_0(u)$, $\delta_1(u)$, and $\delta_{\tilde{b}}(u)$, respectively, are next determined.

Case 1: $b^*(u) = 0$

The classical risk process for insurance business with premium rate $c + a\rho$ results if $b^*(u) = 0$. Here it follows that

$$dR(t) = (c + a\rho)\ dt - dS(t), \qquad R(0) = u\ . \tag{10}$$

In addition, with $b^*(u) = 0$, all the terms related to the risky asset on the HJB equation (7) vanishes and will simplify to the equation

$$0 = (c + a\rho)\ \delta_0'(u) + \lambda\ E\left[\delta_0(u - X) - \delta_0(u)\right]\ .$$

When solved for $\delta_0'(u)$, the preceding equation is transformed to the form

$$\delta_0'(u) = \frac{\lambda}{c + a\rho}\ E\left[\delta_0(u) - \delta_0(u - X)\right]\ . \tag{11}$$

The above equation gives the survival probability for the insurance business without any investments and solves equation (10). An explicit solution for the above stochastic differential equation has been derived for an assumed distribution of X, see [2]. Generally, however, the differential equation is solved numerically.

Case 2: $b^*(u) = 1$

A similar procedure as above was done to derive an expression for $\delta_1'(u)$. If b is replaced by 1, the HJB equation (7) will simplify to the equation

$$0 = \frac{1}{2}\ \sigma^2 a^2 \delta_1''(u) + (c + a\mu)\delta_1'(u) + \lambda\ E[\delta_1(u - X) - \delta_1(u)]\ .$$

where, obviously, the term related to the riskless asset was eliminated. It will follow that

$$\delta_1''(u) = \frac{2}{\sigma^2 a^2} \{\lambda\ E[\delta_1(u) - \delta_1(u - X)] - (c + a\mu)\delta_1'(u)\}\ . \tag{12}$$

It is necessary to simplify the above equation by expressing $\delta_1''(u)$ in terms of $\delta_1'(u)$. This is done by integrating both sides of the equation from u_1 to u. The following differential equation results.

$$\delta_1'(u) = \frac{2}{\sigma^2 a^2} \int_{u_1}^{u} \{\lambda\ E[\delta_1(t) - \delta_1(t - X)] - (c + a\mu)\delta_1'(t)\}\ dt + \delta_1'(u_1)\ . \tag{13}$$

Equation(13) defines the survival probability when the amount a is fully invested in the risky asset. Note that the value u_1 can be taken as the least capital such that $\tilde{b} = 1$.

Case 3: $b^*(u) = \widetilde{b}$

Assuming that $b = \tilde{b}$ where $\tilde{b}$ is given in equation (9), the HJB equation (7) becomes

$$\begin{aligned} 0 &= \frac{1}{2}\frac{(\rho-\mu)^2\delta_{\tilde{b}}'(u)^2}{\sigma^2\delta_{\tilde{b}}''(u)} + \left[c + a\rho - \frac{(\rho-\mu)\delta_{\tilde{b}}'(u)}{\sigma^2\delta_{\tilde{b}}''(u)}\rho + \frac{(\rho-\mu)\delta_{\tilde{b}}'(u)}{\sigma^2\delta_{\tilde{b}}''(u)}\mu\right]\delta_{\tilde{b}}'(u) \\ &+ \lambda\ E[\delta_{\tilde{b}}(u-X) - \delta_{\tilde{b}}(u)] \\ &= -\frac{1}{2}\frac{(\rho-\mu)^2\delta_{\tilde{b}}'(u)^2}{\sigma^2\delta_{\tilde{b}}''(u)} + (c + a\rho)\delta_{\tilde{b}}'(u) + \lambda\ E[\delta_{\tilde{b}}(u-X) - \delta_{\tilde{b}}(u)] \end{aligned}$$

or equivalently,

$$-\frac{\delta_{\tilde{b}}''(u)}{\delta_{\tilde{b}}'(u)^2} = \frac{(\rho-\mu)^2}{2\sigma^2\left\{\lambda\ E[\delta_{\tilde{b}}(u) - \delta_{\tilde{b}}(u-X)] - (c + a\rho)\ \delta_{\tilde{b}}'(u)\right\}}\ . \tag{14}$$

Both sides of the above equation are integrated from 0 to u to remove the second derivative $\delta_{\tilde{b}}''(u)$. Thus,

$$\frac{1}{\delta_{\tilde{b}}'(u)} = \frac{(\rho-\mu)^2}{2\sigma^2}\int_0^u \frac{1}{\lambda\ E[\delta_{\tilde{b}}(t) - \delta_{\tilde{b}}(t-X)] - (c + a\rho)\delta_{\tilde{b}}'(t)}\,dt + \frac{1}{\delta_{\tilde{b}}'(0)}\ .$$

Solving for $\delta_{\tilde{b}}'(u)$, the following differential equation results

$$\delta_{\tilde{b}}'(u) = \left\{\frac{(\rho-\mu)^2}{2\sigma^2}\int_0^u \frac{1}{\lambda\ E[\delta_{\tilde{b}}(t) - \delta_{\tilde{b}}(t-X)] - (c + a\rho)\delta_{\tilde{b}}'(t)}\,dt + \frac{1}{\delta_{\tilde{b}}'(0)}\right\}^{-1}. \tag{15}$$

If the optimal strategy is attained at $\tilde{b}$ and $\delta(u) = \delta_{\tilde{b}}(u)$, from equation (9)

$$\tilde{b} = \frac{(\rho-\mu)\ \delta_{\tilde{b}}'(u)}{a\sigma^2\delta_{\tilde{b}}''(u)}\ .$$

If $\mu > \rho$ then $\tilde{b} > 0$ and as u approaches 0, $\tilde{b}$ must approach 0. Since $\delta'_{\tilde{b}}(0) > 0$ is finite, it follows that

$$\delta''_{\tilde{b}}(0^+) = -\infty \ .$$

The left hand side of equation (14) is infinite and it will follow that the denominator of the right hand of the equation should be zero. Thus

$$\lambda \ E[\delta_{\tilde{b}}(0) - \delta_{\tilde{b}}(-X)] = (c + a\rho) \ \delta'_{\tilde{b}}(0^+)$$

Note that $\delta_{\tilde{b}}(-X) = 0$ thus

$$\delta'_{\tilde{b}}(0^+) = \frac{\lambda \ \delta_{\tilde{b}}(0)}{c + a\rho}$$

which is similar to equation (8). Equation(7) can be written as

$$\delta'_{\tilde{b}}(u) \ = \left\{ \frac{(\rho-\mu)^2}{2\sigma^2} \int_0^u \frac{1}{\lambda \ E[\delta_{\tilde{b}}(t) - \delta_{\tilde{b}}(t-X)] - (c+a\rho)\delta'_{\tilde{b}}(t)} \ dt + \frac{c+a\rho}{\lambda \delta_{\tilde{b}}(0)} \right\}^{-1} . \tag{16}$$

Equation (16) will be the basis in the determination of the optimal strategy $b^*(t)$. Once the solution to this equation is characterized, $\tilde{b}$ will also be solved which in turn will determine the value of $b^*(u)$.

Existence of a Solution to the HJB Equation

Notice that equation (7) determines solutions up to a multiplicative constant. It will therefore follow that $g(u) = \omega\delta(u)$, where $\omega > 0$ solves (7) with boundary condition $g(\infty) = \omega$. The computations below will consider a solution using $g(0) = \delta_0(0)$. A similar approach was used by C. Hipp and M. Taksar [6] and H. Schmidli [10] in their work. Using the function $g(u)$ instead of $\delta(u)$, equation (16) can be transformed to

$$g'_{\tilde{b}}(u) \ = \left\{ \frac{(\rho-\mu)^2}{2\sigma^2} \int_0^u \frac{1}{\lambda \ E[g_{\tilde{b}}(t) - g_{\tilde{b}}(t-X)] - (c+a\rho)g'_{\tilde{b}}(t)} \ dt + \frac{c+a\rho}{\lambda g_{\tilde{b}}(0)} \right\}^{-1} . \tag{17}$$

The initial task is to ensure that the integral in equation (17) is finite for a surplus u. The procedure starts by showing that $g_{\tilde{b}}(u)$ exists on an interval close to zero. The quantity $E[g(t) - g(t-X)]$ is expressed first in terms of $g'(t)$. By the definition of the expectation,

$$E[g(t) - g(t-X)] = g(t) - \int_0^t g(t-x) f(x) \ dx$$

where $f(x) = F'(x)$ is the probability distribution function of X. Using integration by parts, it can be shown that

$$E[g(t) - g(t-X)] = g(t) - g(0)F(t) + g(t)F(0) - \int_0^t F(x) \ g'(t-x) \ dx \ .$$

Recall that the cumulative distribution function of X satisfy the property $F(0) = 0$. Therefore

$$E[g(t) - g(t-X)] = g(t) - g(0)F(t) - \int_0^t F(t-z)\; g'(z)\; dz\;.$$

Further manipulations to the preceding equation give

$$E[g(t) - g(t-X)] = g(0)[1-F(t)] + \int_0^t [1-F(t-z)]\, g'(z)\; dz\;. \tag{18}$$

If the expression $E[g_{\tilde{b}} - g_{\tilde{b}}(t-X)]$ in equation (17) is replaced by a corresponding expression based on the formula above, the equation becomes

$$g'_{\tilde{b}}(u) = \left[\frac{(\rho-\mu)^2}{2\sigma^2}\int_0^u \frac{dt}{\lambda\left\{g_{\tilde{b}}(0)\,[1-F(t)] + \int_0^t [1-F(t-z)]\,g'_{\tilde{b}}(z)\;dz\right\} - (c+a\rho)g'_{\tilde{b}}(t)} + \frac{c+a\rho}{\lambda g_{\tilde{b}}(0)}\right]^{-1}$$

enabling the right-hand side of the equation to be expressed in terms of $g'_{\tilde{b}}(u)$ alone. Define a function $k(u)$ by

$$k(u) = \frac{\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - g'_{\tilde{b}}(u^2)}{u}\;. \tag{19}$$

Then the following equations are derived.

$$\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - uk(u)$$

$$= \left[\frac{(\rho-\mu)^2}{2\sigma^2}\int_0^{u^2} \frac{dt}{(c+a\rho)\sqrt{t}\;k(\sqrt{t}) - \lambda g_{\tilde{b}}(0)F(t) + \lambda\int_0^t [1-F(t-z)]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - \sqrt{z}\;k(\sqrt{z})\right]\;dz} + \frac{c+a\rho}{\lambda g_{\tilde{b}}(0)}\right]^{-1}$$

$$= \left[\frac{(\rho-\mu)^2}{\sigma^2}\int_0^{u} \frac{dt}{(c+a\rho)\;k(t) - \lambda g_{\tilde{b}}(0)\frac{F(t^2)}{t} + \lambda\int_0^1 [1-F(t^2-t^2z)]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - t\sqrt{z}\;k(t\sqrt{z})\right]\;t\;dz} + \frac{c+a\rho}{\lambda g_{\tilde{b}}(0)}\right]^{-1}$$

Solving for the value of $k(u)$,

$$k(u) = \frac{l(u)}{u}\;\frac{\lambda^2 g_{\tilde{b}}(0)^2(\rho-\mu)^2}{\lambda g_{\tilde{b}}(0)(c+a\rho)(\rho-\mu)^2 l(u) + \sigma^2(c+a\rho)^2} \tag{20}$$

where the function $l(u)$ is defined by

$$l(u) = \int_0^u \frac{dt}{(c+a\rho)\;k(t) - \lambda g_{\tilde{b}}(0)\;\frac{F(t^2)}{t} + \lambda t\int_0^1 [1-F(t^2-t^2z)]\left[\frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - t\sqrt{z}\;k(t\sqrt{z})\right]\;dz}\;.$$

Note that $\lim_{t\to 0}\frac{F(t^2)}{t} = \lim_{t\to 0}\; 2tf(t^2) = 0$. Furthermore, the functions $F(x)$ and $k(u)$ present in the integrand defining the function $l(u)$ are bounded, therefore, the inner integral and the integrand itself are bounded. Thus, the following statements hold.

a. $\lim_{u\to 0} l(u) = 0$

b. $\lim_{u\to 0} \frac{l(u)}{u} = \frac{1}{(c+a\rho)\lim_{u\to 0} k(u)}$

Taking the limit of both sides of equation (20) as $u \to 0$ and applying the preceding properties, the equation below follows

$$\left[\lim_{u\to 0} k(u)\right]^2 = \frac{\lambda^2 g_{\tilde{b}}(0)^2(\rho-\mu)^2}{\sigma^2(c+a\rho)^3} \, .$$

Equivalently,

$$\lim_{u\to 0} k(u) = -\frac{\lambda g_{\tilde{b}}(0)(\rho-\mu)}{\sigma(c+a\rho)^{\frac{3}{2}}} \, .$$

Knowing the behavior of $k(u)$ for small values of u, it is possible to generate a corresponding behavior for $g'_{\tilde{b}}(u)$. Using equation (19),

$$g'_{\tilde{b}}(u) = \frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} - k(\sqrt{u})\sqrt{u} \, .$$

As $u \to 0$ it follows that

$$g'_{\tilde{b}}(u) = \frac{\lambda g_{\tilde{b}}(0)}{c+a\rho} + \frac{\lambda g_{\tilde{b}}(0)(\rho-\mu)}{\sigma(c+a\rho)^{\frac{3}{2}}}\sqrt{u} \, . \tag{21}$$

Equation (21) gives the derivative of $g_{\tilde{b}}(u)$ for small values of u. Since $g_{\tilde{b}}(0)$ is known, the above equation can be integrated to get $g_{\tilde{b}}(u)$. Therefore, a solution to (17) exists.

A Numerical Algorithm

The optimal strategy is solved numerically as follows. Given a discretization step size Δu where

$$\Delta u = \frac{u}{n}$$

consider an initial iterate $g'_{\tilde{b}}(i\Delta u)_0$ for $i = 1, 2, \ldots, n$. For values of u near zero, $g'_{\tilde{b}}(i\Delta u)_0 = \delta'_{\tilde{b}}(i\Delta u)$ where $\delta'_{\tilde{b}}(i\Delta u)$ is obtained from equation (21). Next, define a sequence $\{g'_{\tilde{b}}(i\Delta u)_j\}$ by the following recursion.

$$g'_{\tilde{b}}(i\Delta u)_{j+1} = \left[\frac{(\rho-\mu)^2}{2\sigma^2}\int_0^{i\Delta u} \frac{dt}{\lambda\left\{\delta_0(0)\left[1-F(t)\right] + \int_0^t \left[1-F(t-z)\right] g'_{\tilde{b}}(z)_j \, dz\right\} - (c+a\rho)g'_{\tilde{b}}(t)_j} + \frac{c+a\rho}{\lambda\delta_0(0)}\right]^{-1}$$

The preceding equation is locally a contraction and therefore the scheme converges to $g'_{\tilde{b}}(i\Delta u)$. The initial solution $g'_{\tilde{b}}(i\Delta u)_0$ after the recursive equation is obtained by Euler scheme from equation (14), where $\delta_{\tilde{b}}(u)$ is replaced by $g_{\tilde{b}}(u)$. Thus,

$$g'_{\tilde{b}}(i\Delta u + \Delta u)_0 = g'_{\tilde{b}}(i\Delta u) + g''_{\tilde{b}}(i\Delta u)\Delta u \ .$$

The value of $\tilde{b}(i\Delta u)$ is computed using equations (14) and (9), where $\delta_{\tilde{b}}(u)$ is replaced by $g_{\tilde{b}}(u)$. If $\tilde{b}(i\Delta u)$ is less than 0 then the optimal proportion $b^*(i\Delta u)$ is given by $b^*(i\Delta u) = 0$. If $\tilde{b}(i\Delta u)$ is greater than 1 then $b^*(i\Delta u) = 1$, that is, $b^*(i\Delta u)$ is defined as follows.

$$b^*(i\Delta u) = \begin{cases} 0 & \text{if} \quad \tilde{b}(i\Delta u) < 0 \\ \tilde{b}(i\Delta u) & \text{if} \quad 0 \le \tilde{b}(i\Delta u) \le 1 \\ 1 & \text{if} \quad \tilde{b}(i\Delta u) > 1 \end{cases}$$

The preceding criteria determines the proportion of the amount a to be invested in the risky asset to maximize the survival probability. The optimal survival probability, in turn, is dependent on the above proportions. Consider the following formulas derived from equations (11) and (13)

$$\begin{aligned} g'_0(u) &= \frac{\lambda}{c + a\rho} \left\{ \delta_0(0)[1 - F(u)] + \int_0^u [1 - F(u - z)]\, g'_0(z)\ dz \right\} . \\ g'_1(u) &= \frac{2}{\sigma^2 a^2} \int_0^u \left(\lambda \left\{ \delta_0(0)[1 - F(t)] + \int_0^t [1 - F(t - z)]\, g'_1(z)\ dz \right\} - (p + a\mu) g'_1(t) \right) dt + \delta'_0(0) \ . \end{aligned}$$

Next, define the functions $g'(u)$ and $h(u)$, respectively, by

$$g'(u) = \begin{cases} g'_0(u) & \text{if} \quad \tilde{b}(u) < 0 \\ g'_{\tilde{b}}(u) & \text{if} \quad 0 \le \tilde{b}(u) \le 1 \\ g'_1(u) & \text{if} \quad \tilde{b}(u) > 1 \end{cases}$$

and

$$h(u) = \delta_0(0) + \int_0^u g'(t)\ dt \ .$$

The optimal survival probability $\delta(u)$ at the optimal proportion $b^*(u)$ is determined by the norming

$$\delta(u) = \frac{h(u)}{h(\infty)} \ .$$

Numerical Examples

In the following examples, an exponential distribution with mean γ is assumed. The optimal investment portfolio will be determined on a given insurance business for different investment scenarios. The examples consider two different amounts a allocated for investment and two different values of drift coefficients μ. The results obtained are very much dependent on the author's numerical implementation and tolerance level. The numerical implementation used a discretization size of $\Delta u = 0.001$ and a tolerance level of 1×10^{-12}. The derived optimal proportions are assumed to be near, if not identical to, the correct values.

Example 1

Consider an insurance business with premium $c = 1$ and whose aggregate claim is such that $\lambda = 1$ and $\gamma = 1$. An amount $a = 2$ will be invested in an investment portfolio where $\rho = 4\%$, $\mu = 6\%$, and $\sigma = 0.4$. Figure 2 shows the graph of the optimal proportion $b^*(u)$ as a function of the surplus level u.

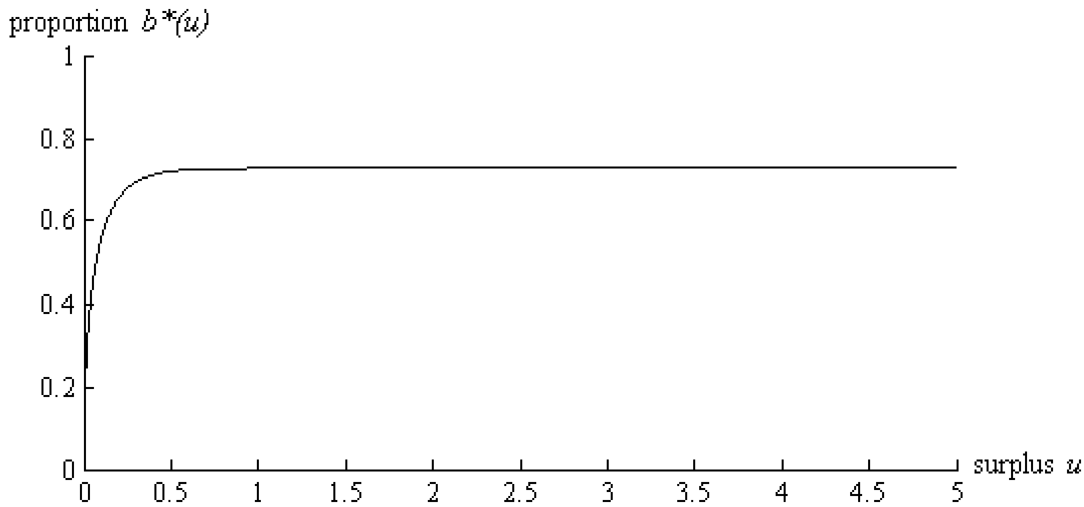

Figure 2: Optimal proportion $b^*(u)$ at $a = 2$ and $\mu = 6\%$.

The graph has an asymptote at $b^*(u) = 0.7271$. This implies that for sufficiently large values of u, the optimal strategy is to invest a constant proportion 0.2729 on the riskless asset and the remaining proportion 0.7271 on the risky asset. Notice that the value of $b^*(u)$ have not reached 1, thus, both assets comprise the optimal investment portfolio for all positive surplus levels u. Table 1 provides a summary of the optimal strategy for some small values of u. This will show how a company should invest when its surplus process is near ruin.

Surplus level u	Risky asset $b^*(u)$	Riskless asset $1-b^*(u)$
0.0	0.0000	1.0000
0.1	0.5566	0.4434
0.2	0.6545	0.3455
0.3	0.6939	0.3061
0.4	0.7115	0.2885
0.5	0.7197	0.2803
0.6	0.7236	0.2764
0.7	0.7254	0.2746
0.8	0.7263	0.2737
0.9	0.7267	0.2733
1.0	0.7269	0.2731

Table 1: Optimal strategy for small values of u at $a=2$ and $\mu=6\%$.

It can be seen that as the surplus decreases the fraction invested on the risky asset decreases. As a company moves to a higher surplus level, the fraction of the risky asset in the investment portfolio increases. This is expected since higher surplus level implies greater capability in handling claims in the insurance business and losses incurred brought by an investment in the risky asset.

Example 2

For the same insurance business, the same amount $a=2$ will be invested in another investment portfolio where μ is considerably higher than ρ. Here, $\rho=4\%$, $\mu=8\%$, and $\sigma=0.4$. The result is different with that of Example 1. Figure 3 shows the graph of $b^*(u)$ for this investment portfolio.

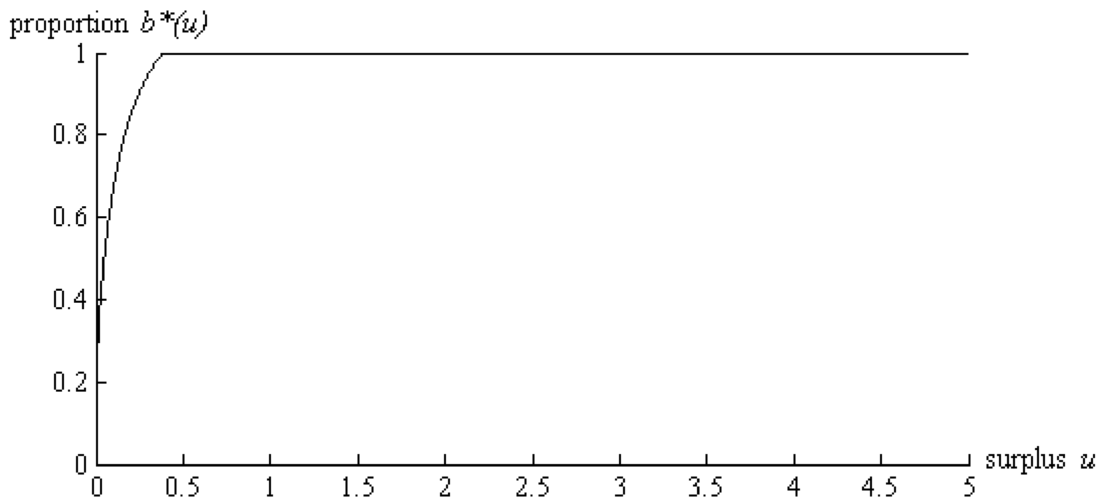

Figure 3: Optimal proportion $b^*(u)$ at $a = 2$ and $\mu = 8\%$.

It is interesting to note that $b^*(u)$ is less than 1 on the interval from 0 to 0.388. Thereafter, the optimal strategy is $b^*(u) = 1$. At $u = 0.388$, a company has to invest fully on the risky asset to achieve the optimal survival probability. At this level of surplus, the risk brought by the diffusion coefficient of the risky asset is upset by the surplus itself and the expected returns on investment. This creates a remarkable advantage of the risky asset over the riskless asset. Table 2 gives a summary of the optimal strategy for some small values of u. Notice that the risky asset with greater drift coefficient constitutes a higher proportion on the optimal investment portfolio than one with a smaller drift coefficient. This result is not surprising. Assuming that the diffusion coefficients are the same, the expected investment return will be bigger for the asset with the larger drift coefficient.

Example 3

For the same insurance business, a bigger amount $a = 5$ will be invested in the same investment scenario as in Example 1. Here, the characteristics of the investment portfolio are also $\rho = 4\%$, $\mu = 6\%$, and $\sigma = 0.4$. Observe that the result is also different with that of Example 1. Figure 4 shows the graph of $b^*(u)$.

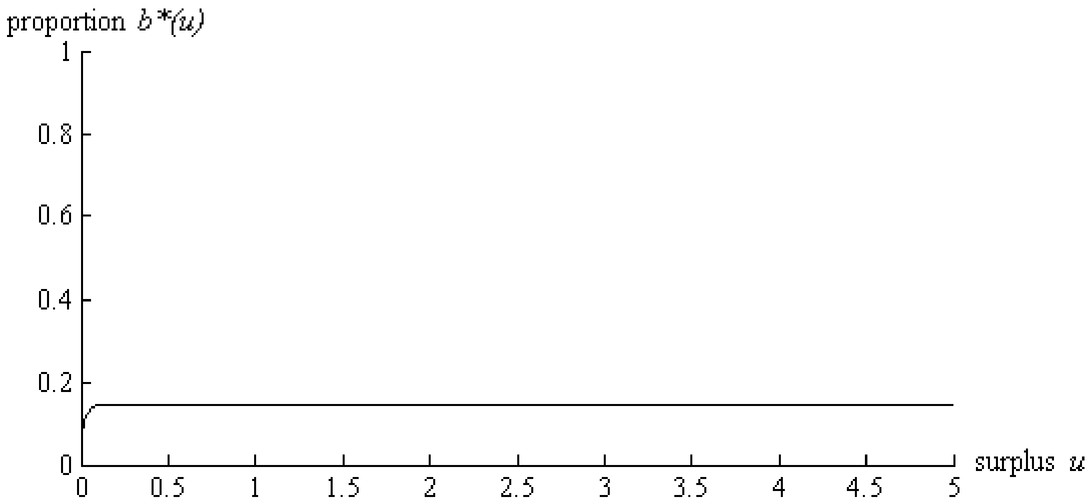

Figure 4: Optimal proportion $b^*(u)$ at $a = 5$ and $\mu = 6\%$.

Both types of asset constitute the optimal investment portfolio for all positive surplus levels since the value of $b^*(u)$ has not reached 1. The graph has an asymptote at $b(u) = 0.1459$ which implies that for sufficiently large values of u, the optimal strategy is to invest a constant proportion 0.1459 on the risky asset. This is less than the corresponding proportion in Example 1. It is possible therefore to have a completely different optimal strategy with similar scenarios but having different investment allotments. The optimal strategy for small values of u are given in Table 3.

It is important to note here that based on the numerical results, the optimal combinations of the riskless and the risky assets varies with the value of a allotted for investment. No single optimal combination can optimize every investment amount.

Conclusions and Recommendations

The results obtained in the study show that stochastic control is a very helpful tool in insurance risk management. In particular, an optimal combination of an investment portfolio with a riskless and a risky asset can stochastically be determined to minimize the ruin probability associated with an insurance business. The results show that risky assets with greater drift coefficients tend to be more advantageous, thereby, making their proportions on the optimal portfolio greater than those with smaller drift coefficients. Different amounts allotted for investment require different asset combinations. This implies that an optimal

Surplus level u	Risky asset $b^*(u)$	Riskless asset $1-b^*(u)$
0.0	0.0000	1.0000
0.1	0.6759	0.3241
0.2	0.8503	0.1497
0.3	0.9475	0.0525
0.4	1.0000	0.0000
0.5	1.0000	0.0000
0.6	1.0000	0.0000
0.7	1.0000	0.0000
0.8	1.0000	0.0000
0.9	1.0000	0.0000
1.0	1.0000	0.0000

Table 2: Optimal strategy for small values of u at $a = 2$ and $\mu = 8\%$.

Surplus level u	Risky asset $b^*(u)$	Riskless asset $1-b^*(u)$
0.0	0.0000	1.0000
0.1	0.1427	0.8573
0.2	0.1457	0.8543
0.3	0.1459	0.8541
0.4	0.1459	0.8541
0.5	0.1459	0.8541
0.6	0.1459	0.8541
0.7	0.1459	0.8541
0.8	0.1459	0.8541
0.9	0.1459	0.8541
1.0	0.1459	0.8541

Table 3: Optimal strategy for small values of u at $a = 5$ and $\mu = 6\%$.

portfolio for a given investment allotment does not necessarily give optimality on a different investment allotment.

The optimal proportion is always a function of the business surplus. It is assumed in the problem that for business with zero surplus, the amount should be invested fully on the riskless asset. This is to insure that the company will be able to pay out possible early claims. Note that this may not be possible if some or all of the amount a is invested in the risky asset. The diffusion coefficient brought by the risky asset may bring so much risk that the surplus of the business will be negative. With zero surplus, full investment on the risky asset could lead to the ruin of the insurance business.

If the insurance business has gained a sufficient surplus, an ample amount of the risky asset may be considered to come up with greater investment gains. This surplus, together with the returns on the investment, is expected to upset the risk brought by the diffusion coefficient of the risky asset. The survival probability function is an increasing function of the business surplus so a greater surplus implies a greater survival probability.

A very good extension of this work is to consider the case where the amount to be invested is not fixed but rather an increasing function of the surplus. The amount could also be bounded, e.g., less than or equal to the present surplus. The numerical computation of the optimal survival probability may also be included since this was not done in this work. Another study, which is as interesting as the one mentioned, is to find the optimal amount to be invested in the risky asset simultaneously with finding the optimal combination of the riskless and the risky asset in the investment portfolio. This may be done by differentiating first the preceding HJB equation with respect to the amount a giving

$$0 = a\sigma^2 b^2 \delta''(u) + [(1-b)\rho + b\mu)\delta'(u)$$

and solving for the value of a,

$$a = -\frac{[(1-b)\rho + b\mu]\ \delta'(u)}{\sigma^2 b^2 \delta''(u)} \ .$$

The preceding expression for a can be substituted back to the HJB equation reducing the problem with just the determination of the optimal proportion $b^*(u)$.

References

[1] Browne, S. (1994) *Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin.* Mathematics of Operations Research, 20, 4, 937-958.

[2] Fausett, L. (1999), *Applied Numerical Analysis Using MATLAB*, Prentice-Hall, Inc., New Jersey, 1999.

[3] Hipp, C. (2000) *Optimal investment with state dependent income, and for insurers.* Download available at http://www.ecom.unimelb.edu.au/actwww/no82.pdf.

[4] Hipp, C. (2002) *Optimal investment for insurers: the coxian case.* Preprint. Download available under wwwrz.rz.uni-karlsruhe.de/ ~Im09/workshop.

[5] Hipp, C. and Plum, M. (2000) *Optimal investment for insurers.* Insurance: Mathematics and Economics 27, 251-262.

[6] Hipp, C. and Taksar, M. (2000) *Stochastic control for optimal new business.* Insurance: Mathematics and Economics 26, 185-192.

[7] Hipp, C. and Vogt, M. (2002) *Optimal dynamic XL-reinsurance for a compound Poisson risk process.* As preprint under www. mathpreprints.com, MPS, Applied Mathematics/0106069

[8] Hojgaard, B. and Taksar, M. (1998) *Optimal proportional reinsurance policies for diffusion models.* Scandinavian Actuarial Journal, 81, 166-180.

[9] Hojgaard, B. and Taksar, M. (1999) *Optimal proportional reinsurance policies for diffusion models with transaction costs.* Insurance: Mathematics and Economics 22, 41-51.

[10] Schmidli, H. (1999) *Optimal proportional reinsurance policies in a dynamic setting.* Scandinavian Actuarial Journal, 77, 1-25.

[11] Schmidli, H. (2002) *Minimizing ruin probability by control.* Preprint. Download available under wwwrz.rz.uni-karlsruhe.de/ ~Im09/workshop.