Synthetic CDO Pricing Using the Student t Factor Model with Random Recovery

UNSW Actuarial Studies Research Symposium 2006
University of New South Wales
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Overview

1. Introduction — Motivation
2. Setting and Notations
3. Modelling the Joint Default Time Distribution with Copulas
4. Random Recovery and Portfolio Distribution
5. Large Portfolio Approximation
6. Numerical Illustration
7. Conclusion and Possible Extensions
Single-Name CDS Cash Flows

Credit risk transfer (notional)

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<th>Protection Buyer</th>
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<td>Buys CDS</td>
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<td>“Short credit risk”</td>
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<td>Sells CDS</td>
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Periodic fee/premium

Contingent payment upon a credit event
A CDO (collateralized debt obligation) is a portfolio of debt securities that is split into tranches of debt subordination. Each tranche protects the tranches senior to it from losses on the underlying portfolio.

Typical tranches are:
- equity tranche (no subordination)
- mezzanine tranche
- senior tranche
- super senior tranche

Individual tranches can be rated by a rating agency.
CDO: Protection Buyers and Sellers

**Protection Buyers**

- **Banks**: 51%
- **Corporates**: 3%
- **Securities firms**: 16%
- **Mutual funds**: 3%
- **Hedge funds**: 16%
- **Insurance companies**: 7%
- **Government/export credit agencies**: 1%

**Protection Sellers**

- **Banks**: 38%
- **Corporates**: 2%
- **Securities firms**: 16%
- **Insurance companies**: 20%
- **Hedge funds**: 15%
- **Pension funds**: 4%
- **Mutual funds**: 4%
- **Government/export credit agencies**: 1%
Literature

- Benchmark model: one factor Gaussian model
  - Li (2000)
  - Used by all major investments banks to communicate quotes

- The opponents
  - Within the “copula family”
    - Student $t$ (O’Kane & Schloegl, Lindskog & McNeil), Clayton (Schönbucher), double $t$ (Hull & White), multifactor Gaussian, Marshall-Olkin (Giesecke, Lindskog & McNeil), random factor loadings (Andersen & Sidenius)
  - Intensity models
    - Affine Jump Diffusion (Hutt), Gamma (Joshi), Stochastic Networks (Davis, Backhaus & Frey, Giesecke),
  - Structural models
Setting and Notations

- **Notation**
  - $i = 1, \ldots, n$ credits
  - $T_1, \ldots, T_n$ default times
  - $N_i$ nominal of credit $i$
  - $R_i$ recovery rate
  - $I_{[0,T]}(T_i)$ default indicator
  - $L_i = N_i(1 - R_i)$ loss given default

- **Semi-explicit pricing for CDO tranches**
  - Default payments are based on the accumulated losses on the pool of credits:
    \[
    L(T) = \sum_{i=1}^{n} L_i I_{[0,T]}(T_i)
    \]
  - Tranche premiums only involves call options on the accumulated losses:
    \[
    E[(L(T) - K)_+]\]
Setting and Notations

- Default occurs whenever a stochastic variable \( X_i \) (or a stochastic process in dynamic models) lies below a critical threshold \( C_i \) at the end of time period \([0, T]\).
- In the Merton model default occurs when the value of the assets of a firm falls below the value of the firms’ liabilities.
- In order to apply these models at portfolio level we require a multivariate version of a firm-value model.
- In the factor copula model the critical variable \( X_i \) is interpreted as the latent default time of company \( i \).
- In a CDO the cash flows are functions of the whole random vector \( \mathbf{X} = (X_1, \ldots, X_n) \). To evaluate a CDO, all we need is today’s (risk neutral) joint distribution of the \( X_i \)’s:

\[
P(X_1 < C_1, \ldots, X_n < C_n)
\]
Copulas

- A copula captures the **nonparametric, scale-invariant, distribution-free** nature of the association between random variables.

- **Theorem 1 Sklar (1959)** Let $X' = (X_1, \ldots, X_n)$ be a random vector with joint distribution function $F_X$ and marginal distribution functions $F_{X_i}$, $i = 1, \ldots, n$. Then there exists a copula $C$ such that for all $x \in \mathbb{R}^n$

\[
F_X(x) = C(F_{X_1}(x_1), \ldots, F_{X_n}(x_n)).
\]

- If $F_{X_1}, \ldots, F_{X_n}$ are all continuous then $C$ is unique.
- Given a copula $C$ and marginal distribution functions $F_{X_1}, \ldots, F_{X_n}$, the function $F_X$ as defined by (1) is a joint distribution function with margins $F_{X_1}, \ldots, F_{X_n}$.

- The critical variables $X_i$ adopt a Gaussian/student $t$ copula:

\[
C_{\Phi_R}(u_1, u_2, \ldots, u_n; R) = \Phi_R(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \ldots, \Phi^{-1}(u_n))
\]

\[
C_{t(v,R)}(u_1, u_2, \ldots, u_n; R) = t_{v,R}(t^{-1}_v(u_1), t^{-1}_v(u_2), \ldots, t^{-1}_v(u_n))
\]
Gaussian vs. Student $t$ Copula

1000 samples of Gaussian and student $t$ copula with Kendall’s $\tau = 0.5$
Default Time Distribution

- To reduce the high dimensionality of the modelling problem a set of common factors is chosen (assumed to drive the default dependency between firms).

- Assume that the variable $Y_i$ depends on a single common factor $Z$:

$$Y_i = a_i Z + \sqrt{1 - \|a_i\|^2} \varepsilon_i,$$

where $Z, \varepsilon_1, \ldots, \varepsilon_n$, are independent $N(0,1)$ random variables.

- The critical variable

$$X_i \overset{d}{=} \begin{cases} Y_i & \text{for Gaussian factor model} \\ \sqrt{\frac{\nu}{W}} Y_i & \text{for student } t \text{ factor model} \end{cases}$$

with $W \sim \chi^2_\nu$ independent of $Y_1, \ldots, Y_n$.

- If we didn’t choose a factor structure for the $X_i$, we would have to estimate all the $\frac{1}{2}n(n-1)$ elements of the covariance matrix of the critical variables.
The latent default times are related to the original default times in the following way:

\[ T_i \leq t \iff F_{T_i}(T_i) \leq F_{T_i}(t) \iff U_i \leq F_{T_i}(t) \]

\[ \iff X_i \leq \begin{cases} 
\Phi^{-1}(F_{T_i}(t)) & \text{for Gaussian factor model} \\
\nu^{-1}(F_{T_i}(t)) & \text{for student } t \text{ factor model} 
\end{cases} \]

The model is calibrated to observable market prices of credit default swaps, i.e. the default thresholds are chosen so that they produce risk neutral default probabilities implied by quoted credit default swap spreads:

\[ C_i = \begin{cases} 
\Phi^{-1}(F_{T_i}(t)) & \text{for Gaussian factor model} \\
\nu^{-1}(F_{T_i}(t)) & \text{for student } t \text{ factor model} 
\end{cases} \]
Factor Models

- The $i^{th}$ issuer defaults if $X_i \leq C_i$ or

$$\varepsilon_i \leq \begin{cases} \frac{C_i - a_i Z}{\sqrt{1 - \|a_i\|^2}} & \text{for Gaussian factor model} \\ \frac{\sqrt{W} C_i - a_i Z}{\sqrt{1 - \|a_i\|^2}} & \text{for student } t \text{ factor model} \end{cases}$$

- The probability that the $i^{th}$ issuer defaults

$$P(X_i \leq C_i | Z = z) = \Phi \left( \frac{C_i - a_i z}{\sqrt{1 - \|a_i\|^2}} \right)$$  

$$P(X_i \leq C_i | Z = z, W = w) = \Phi \left( \frac{\sqrt{W} C_i - a_i z}{\sqrt{1 - \|a_i\|^2}} \right)$$  

for Gaussian factor model

for student $t$ factor model
Random Recovery

- The joint default time distribution describes the joint default behavior of the debtors underlying the CDO structure and hence completely determines the CDO cash flows.
- At default only a fraction of the nominal can be recovered. The recovery rates are assumed to be random and follow the cumulative Gaussian recovery model proposed by Andersen and Sidenius (2004):

$$R_i = \Phi(\mu_i + b_i Z + \xi_i),$$

where $\xi_i \sim N(0, \sigma_{\xi_i}^2)$, $i = 1, \ldots, n$. The error terms $\xi_1, \ldots, \xi_n$ are assumed to be independent from each other and also independent from $Z, W$ and $\epsilon_1, \ldots, \epsilon_n$.
- The interdependence between default of the $i^{th}$ obligor and recovery on the $j^{th}$ obligor is controlled by $a_i \cdot b_j$. 
Recovery Rate Density Functions with $\sigma^2 = 0.25$

(a)

(b)

(c)
Portfolio Loss Distribution $L(T)$

- The conditional portfolio loss distribution:

$$P(L(T) \leq \ell | Z = z, W = w) = \left( F_{\tilde{L}_1|z,w} \cdots F_{\tilde{L}_n|z,w} \right)(\ell), \quad 0 \leq \ell \leq N,$$

where $N = \sum_{i=1}^{n} N_i,$

$$F_{\tilde{L}_i|z,w}(\ell) = P(\tilde{L}_i \leq \ell | Z = z, W = w)$$
$$= 1 - P(T_i \leq T | Z = z, W = w)(1 - P(L_i \leq \ell | Z = z, W = w)),$$

and

$$P(T_i \leq T | Z = z, W = w) = \Phi \left( \frac{\sqrt{\nu} t^{-1}_\nu(F_{T_i}(T)) - a_i z}{\sqrt{1 - \|a_i\|^2}} \right)$$
$$P(L_i \leq \ell | Z = z, W = w) = \Phi \left( \frac{\mu_i + b_i Z - \Phi^{-1} \left( 1 - \frac{\ell}{N_i} \right)}{\sigma_{\xi_i}} \right).$$
Portfolio Loss Distribution $L(T)$: recursion

- Let $K_i$ denote the loss given default of obligor $i$ expressed in loss units $u$
- For $\ell = 1, \ldots, \ell_{i}^{\text{max}} = \lfloor N_i/u \rfloor$:
  \[
P(K_i = 0|Z = z, W = w) = P(L_i \leq 0|Z = z, W = w) = 0
  \]
  \[
P(K_i = \ell|Z = z, W = w) = P(L_i \leq \ell u|Z = z, W = w)
  \]
  \[
  - P(L_i \leq (\ell - 1)u|Z = z, W = w)
  \]
- Let $K(T)$ denote the portfolio loss over $[0, T]$ expressed in loss units $u$ and let $P^{(i)}$ denote the distribution of $K(T)$ for the first $i$ obligors. Then
  \[
P^{(i)}(K(T) = \ell|Z = z, W = w)
  \]
  \[
  = \min(\ell_{i}^{\text{max}}, \ell)
  \]
  \[
  = \sum_{k=0}^{\min(\ell_{i}^{\text{max}}, \ell)} P^{(i-1)}(K(T) = \ell - k|Z = z, W = w) P(\tilde{K}_i = k|Z = z, W = w)
  \]
- The recursion starts from the boundary case of an empty portfolio for which
  \[
P^{(0)}(K(T) = \ell|Z = z, W = w) = \delta_{\ell,0}
  \]
Portfolio Loss Distribution $L(T)$: FFT

- Characteristic function of $K(T)$:

$$E \left( e^{itK(T)} \right) = E \left( \prod_{i=1}^{n} e^{itK_i I_{[0,T]}(T_i)} \right) = E \left[ \prod_{i=1}^{n} E \left( e^{itK_i I_{[0,T]}(T_i)} \big| Z, W \right) \right]$$

with

$$* = \sum_{l=0}^{\ell_i^{\text{max}}} \left( e^{it\ell} P(T_i \leq T \mid Z = z, W = w) + P(T_i > T \mid Z = z, W = w) \right) \times P(K_i = \ell \mid Z = z, W = w)$$

$$= 1 - P(T_i \leq T \mid Z = z, W = w) \left( 1 - \sum_{l=0}^{\ell_i^{\text{max}}} e^{it\ell} P(K_i = \ell \mid Z = z, W = w) \right)$$

- Apply an inverse Fourier transform to the sequence $E \left( e^{i2\pi k K(T)/(\ell_i^{\text{max}} + 1)} \right)$

$(k = 0, \ldots, \ell^{\text{max}} = \sum_{i=1}^{n} \ell_i^{\text{max}})$
LHP Approximation

- Approximate the real reference credit portfolio with a portfolio consisting of a large number of equally weighted identical instruments (having the same term structure of default probabilities, recovery rates, and correlations to the common factor).

- Homogeneous portfolio: $a_i = a$ and $C_i = C$ for all $i$ and the notional amounts and recovery rates $R$ are the same for all issuers.

- Basel II agreement: “infinitely granular” portfolios

- LHP using a one factor Gaussian copula has become a standard model in practice.
  - Closed-form analytic synthetic CDO pricing formula

- This model fails to fit the prices of different CDO tranches simultaneously: lack of tail dependence of the Gaussian copula
  - Use student $t$ copula
Stochastic Order Bounds

Theorem: (Kaas et al., 2000)
For any \((X_1, \ldots, X_n)\) and any \(Z\), we have that

\[
S^l := \sum_{i=1}^{n} E[X_i \mid Z] \leq_{cx} \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F^{-1}_{X_i}(U) =: S^c
\]

Assume that all \(E[X_i \mid Z]\) are \(\nearrow\) functions of \(Z\)
\[\Rightarrow S^l \text{ is a comonotonic sum.}\]

\[E[(S^l - d)_+] \leq E[(S - d)_+] \leq E[(S^c - d)_+]\]

The stop-loss premiums of a sum of comonotonic random variables can easily be obtained from the stop-loss premiums of the terms:

\[
E[(S^c - d)_+] = \sum_{i=1}^{n} E \left[ (X_i - F^{-1}_{X_i}(F_{S^c}(d)))_+ \right], \quad (F^{-1}_{S^c}(0) < d < F^{-1}_{S^c}(1))
\]
Stop-loss Premium

\[ E [(X - d)_+] = \int_d^\infty F_X (x) \, dx, \quad -\infty < d < \infty \]

\( \rightarrow \) = surface above the distribution function, from \( d \) on.
Stop-loss Premium

\[ E[(X - d)_+] = \int_d^{\infty} F_X(x) \, dx, \quad -\infty < d < \infty \]

\( \rightarrow \) = surface above the distribution function, from \( d \) on.

\( F_X(x) \)

\( E[(X-d)_+] \)
LHP Approximation

- \( X_i := N_i(1 - R_i)I_{[0,T]}(T_i) \): loss at time \( T \) associated with name \( i \)
- Assume \( X_i \)'s are independent conditionally upon \( Z \) and identically distributed:

\[
\frac{1}{n} \sum_{i=1}^{n} X_i \longrightarrow E[X_i|Z]_{LPA}
\]

- \( L(T) = X_1 + \cdots + X_n \): aggregated loss at time \( T \)

Convex order bounds:

\[
\sum_{i=1}^{n} E[X_i|Z] \leq_{cx} L(T) \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U),
\]

with \( E[X_i|Z] = N_i(1 - R_i)P(T_i \leq T|Z) \).
LHP Approximation

- Loss on the portfolio as a percentage of the total portfolio notional:

\[
L = (1 - R) \frac{1}{n} \sum_{i=1}^{n} I\{X_i \leq C\}
\]

- The probability that the \(i^{th}\) issuer defaults

\[
P(X_i \leq C_i | \eta) = \Phi \left( \frac{\eta}{\sqrt{1 - a^2}} \right),
\]

with

\[
\eta = \begin{cases} 
C - aZ & \text{for Gaussian factor model} \\
C \sqrt{\frac{W}{\nu}} - aZ & \text{for student } t \text{ factor model}
\end{cases}
\]
LHP Approximation

- Law of large numbers:

\[ \frac{1}{n} \sum_{i=1}^{n} I_{\{X_i \leq C\}} \longrightarrow \Phi \left( \frac{\eta}{\sqrt{1 - a^2}} \right) \]

- LPH: Assume that exactly this fraction of issuers defaults for each realization of \( \eta \), so that \( L \approx h(\eta) \), with \( h : \mathbb{R} \to [0, 1] \) and

\[ h(x) = (1 - R) \Phi \left( \frac{x}{\sqrt{1 - a^2}} \right) \]

- Distribution of \( L \):

\[ P[L \leq \theta] = F_\eta(h^{-1}(\theta)), \quad \theta \in [0, 1] \]

- CDO premiums can be easily computed!
LPH Approximation: student $t$ factor model

Distribution of the mixing variable $\eta$

- $P[W \geq x] = \Gamma \left( \frac{\nu}{2}, \frac{x}{2} \right)$
- $P[\eta \leq t | Z] = I\{Z \geq -\frac{t}{a}\} + I\{Z \leq -\frac{t}{a}\} \Gamma \left( \frac{\nu}{2}, \frac{\nu(t + aZ)^2}{2C^2} \right)$

Cumulative distribution function $F_{\eta}(t)$:

$$P[\eta \leq t] = \Phi \left( \frac{t}{a} \right) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{-t}{a}} \Gamma \left( \frac{\nu}{2}, \frac{\nu(t + au)^2}{2C^2} \right) e^{-\frac{u^2}{2}} du$$

Density $f_{\eta}(t)$:

$$f_{\eta}(t) = \frac{1}{a\sqrt{\pi}2^{\frac{\nu+1}{2}} \Gamma(\frac{\nu}{2})} \int_{0}^{+\infty} e^{-\frac{1}{2a^2}(t-C\sqrt{\frac{w}{\nu}})^2} w^{\frac{\nu}{2} - 1} e^{-\frac{w}{2}} dw$$
Conclusion and Possible Extensions

- **Synthetic CDO Pricing**
  - Dependence between default times is modelled through Student $t$ copulas.
  - Factor approach leading to semi-analytic pricing expressions that ease model risk assessment.
  - The loss amounts — or equivalently, the recovery rates — associated with defaults are random.
  - Closed-form solutions for the loss distribution can be derived under the LHP approximation.

- **Possible extensions:**
  - other copulas: skewed $t$ copula, grouped $t$ copula, ... 
  - stable distributions 
  - moment matching techniques 
  - ...
References


