

Synthetic CDO Pricing Using the Student t Factor Model with Random Recovery

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Abstract

A synthetic collateralized debt obligation, or synthetic CDO, is a transaction that transfers the credit risk on a reference portfolio of assets. The reference portfolio in a synthetic CDO is made up of credit default swaps. Much of the risk transfer that occurs in the credit derivatives market is in the form of synthetic CDOs.

While the Gaussian copula model, introduced to the credit field by Li (2000), has become an industry standard, its theoretical foundations, such as credit spread dynamics may be questioned. Various authors have considered tail dependence amongst default times or default events. This would lead to fat tails in the credit loss distributions.

In this paper dependence between default times is modelled through Student t copulas. We use a factor approach leading to semi-analytic pricing expressions that ease model risk assessment. We present an extension to the student t factor model in which the loss amounts — or equivalently, the recovery rates — associated with defaults are random. We detail the model properties and compare the semi-analytic pricing approach with large portfolio approximation techniques.

Keywords: CDO, copula, student t distribution.

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1 Introduction

A synthetic collateralized debt obligation, or synthetic CDO, is a transaction that transfers the credit risk on a reference portfolio of assets. The reference portfolio in a synthetic CDO is made up of credit default swaps. Much of the risk transfer that occurs in the credit derivatives market is in the form of synthetic CDOs.

The credit risk on the CDO is tranching, so that a party that buys insurance against the defaults of a given tranche receives a payoff consisting of all losses that are greater than a certain percentage, and less than another certain percentage, of the notional of the reference portfolio. In return for this insurance, the protection buyer pays a premium, typically quarterly in arrears, proportional to the remaining tranche notional at the time of payment; there is also an accrued amount in the event that default occurs between two payment dates.

To price or measure the risk of a synthetic CDO tranche, the probability distribution of default losses on the reference portfolio is a key input. Due to its simplicity, the Gaussian copula model has become market standard. However, this model has a number of obvious shortcomings as a model of the real world. For instance, standard implementations of the model make the assumption that recovery rates on default are known firm-specific constants. Moreover, it fails to fit the prices of different CDO tranches simultaneously which leads to the well known implied correlation smile. The main explanation of this phenomenon is the lack of tail dependence of the Gaussian copula. Various authors have proposed different ways to bring more tail dependence into the model. One approach is to use copulas such as the Student t , Clayton, double t , or Marshall-Olkin copula. This would lead to fat tails in the credit loss distributions. Incorporating the effect of tail dependence into the one factor portfolio credit model yields significant pricing improvement. Another approach is the introduction of additional stochastic factors into the model. Andersen and Sidenius (2005) extended the Gaussian factor copula model to random recovery and random factor loadings.

In this paper we combine both approaches through the introduction of random recovery rates in the student t copula model. We show how to compute the loss distribution in an analytically tractable way using the well-known factor approach. We assume that the correlation of defaults on the reference portfolio is driven by common factors. Therefore, conditional on these common factors defaults are independent. An explicit form of the number-of-default distribution, or loss distribution, can be computed and used to value synthetic CDO tranches. The sensitivity measures can be produced in a similar way. This approach allows to use semi-analytic computation techniques avoiding time consuming Monte Carlo simulations. Similar approaches have been followed by Li (2000), Laurent and Gregory (2003) and Andersen et al. (2003).

We provide some interesting comparison between the semi-analytic approach and large portfolio approximations. These convenient approximations were first proposed by Vasicek (1987a,b). The real reference credit portfolio is approximated with a portfolio consisting of a large number of equally weighted identical instruments (having the same term structure of default probabilities, recovery rates, and correlations to the common factor). Large portfolio limit distributions are

often remarkably accurate approximations for finite-size portfolios especially in the upper tail. Given the uncertainty about the correct value for the asset correlation the small error generated by the large portfolio assumption is negligible.

The structure of this paper is as follows. In section 2 we introduce the student t copula model with a latent factor structure. Section 3 presents the random recovery rates and explains how to compute the portfolio loss distribution using a recursion algorithm and Fourier techniques. Section 4 provides the large portfolio results. Finally, in Section 5 we insert some concluding remarks.

2 Joint default time distribution: Student t copula

Consider a portfolio consisting of n credit default swaps. The default time of the name underlying the i th CDS, denoted T_i , is a random variable with distribution function F_{T_i} . We assume that the marginal default time distributions are continuous and strictly increasing. For portfolio valuation we need, next to the marginal default behavior, information on default dependence. At this point copula functions enter the picture.

Definition 1 *A n -copula is a function $C : [0, 1]^n \rightarrow [0, 1]$ with the following properties*

1. for every $\mathbf{u} \in [0, 1]^n$ with at least one coordinate equal to 0, $C(\mathbf{u}) = 0$,
2. if all coordinates of \mathbf{u} are 1 except u_k then $C(\mathbf{u}) = u_k$,
3. for all $\mathbf{a}, \mathbf{b} \in [0, 1]^n$ with $\mathbf{a} \leq \mathbf{b}$ the volume of the hyperrectangle with corners \mathbf{a} and \mathbf{b} is positive, i.e.

$$\sum_{i_1=1}^2 \cdots \sum_{i_n=1}^2 (-1)^{i_1+\cdots+i_n} C(u_{i_1}, \dots, u_{i_n}) \geq 0$$

where $u_{i_1} = a_i$ and $u_{i_2} = b_i$.

So essentially a n -copula is a n -dimensional distribution function on $[0, 1]^n$ with standard uniform marginal distributions. The next theorem, due to Sklar, is central to the theory of copulas and forms the basis of the applications of that theory to statistics.

Theorem 1 Sklar (1959) *Let $\mathbf{X}' = (X_1, \dots, X_n)$ be a random vector with joint distribution function $F_{\mathbf{X}}$ and marginal distribution functions F_{X_i} , $i = 1, \dots, n$. Then there exists a copula C such that for all $\mathbf{x} \in \mathbb{R}^n$*

$$F_{\mathbf{X}}(\mathbf{x}) = C(F_{X_1}(x_1), \dots, F_{X_n}(x_n)). \tag{1}$$

If F_{X_1}, \dots, F_{X_n} are all continuous then C is unique, otherwise C is uniquely determined on $\text{Ran } F_{X_1} \times \cdots \times \text{Ran } F_{X_n}$. Conversely, given a copula C and marginal distribution functions F_{X_1}, \dots, F_{X_n} , the function $F_{\mathbf{X}}$ as defined by (1) is a joint distribution function with margins F_{X_1}, \dots, F_{X_n} .

As is clear, Sklar's theorem separates a joint distribution into a part that describes the dependence structure (the copula) and parts that describe the marginal behavior (the marginal distributions). For further details on copula functions we refer to Nelsen (1999) and Joe (1997).

In this paper we join the marginal default time distributions by a Student t copula. The Student t copula is the dependence function of the multivariate Student t distribution, which we quickly define for reference. For more details about the multivariate Student t distribution we refer to the excellent book by Kotz *et al.* (2000).

Definition 2 A random vector $\mathbf{X}' = (X_1, \dots, X_n)$ is said to have a (non-singular) multivariate Student t distribution with ν degrees of freedom and (positive definite) dispersion matrix R , denoted $\mathbf{X} \sim t_n(\nu, R)$, if its density is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{(\nu\pi)^n |R|}} \left(1 + \frac{\mathbf{x}'R^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+n}{2}}.$$

The Student t copula function, denoted $C_{\nu,R}$, can be derived directly from (1) and is given by

$$C_{\nu,R}(\mathbf{u}) = \int_{-\infty}^{t_{\nu}^{-1}(u_1)} \cdots \int_{-\infty}^{t_{\nu}^{-1}(u_n)} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{(\nu\pi)^n |R|}} \left(1 + \frac{\mathbf{x}'R^{-1}\mathbf{x}}{\nu}\right)^{-\frac{\nu+n}{2}} d\mathbf{x}$$

where t_{ν}^{-1} denotes the quantile function of the Student t distribution with ν degrees of freedom. As $\nu \rightarrow \infty$, the Student t distribution converges to the normal distribution and the Student t copula converges to the normal copula, being the standard market model. The Student t distribution exhibits heavier tails than the normal distribution, in fact its tails ultimately behave like Pareto laws, see Beirlant *et al.* (2004). Compared to the Gaussian copula the student t copula has nonzero tail dependence resulting in extreme values that occur in clusters.

In this paper we equip the Student t copula model with a latent factor structure. Such a structure not only has an economic interpretation but also facilitates the computation of the loss distribution to a great extent as conditioning on the underlying factors results in independent default times. The Student t distribution belongs to the class of multivariate normal variance mixtures and hence a Student t random vector \mathbf{X} can be represented as

$$\mathbf{X} \stackrel{\mathcal{D}}{=} \sqrt{\frac{\nu}{W}} \mathbf{Y}, \tag{2}$$

where $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution, $\mathbf{Y} \sim N_n(\mathbf{0}, R)$ with R a positive definite correlation matrix, and $W \sim \chi_{\nu}^2$ independently of \mathbf{Y} . We assume that all components of \mathbf{Y} depend on the same latent factors, represented by a d vector \mathbf{Z} , in the following way

$$Y_i = \mathbf{a}'_i \mathbf{Z} + \sqrt{1 - \|\mathbf{a}_i\|^2} \varepsilon_i, \quad i = 1, \dots, n \tag{3}$$

where \mathbf{a}_i are d vectors of factor loadings satisfying $\|\mathbf{a}_i\| < 1$, $\mathbf{Z} \sim N_d(\mathbf{0}, \mathbf{I}_d)$ and $\varepsilon_1, \dots, \varepsilon_n$ are $N(0, 1)$ random variables, independent from each other and independent from \mathbf{Z} . Moreover, \mathbf{Z}

and $\varepsilon_1, \dots, \varepsilon_n$ are assumed to be independent from W . Under (3), using the basic properties of normal random vectors, $\mathbf{Y} \sim N_n(\mathbf{0}, \mathbf{A}\mathbf{A}' + \mathbf{D})$, with $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)'$ and \mathbf{D} a diagonal matrix with elements $\{1 - \|\mathbf{a}_i\|^2\}_{i=1, \dots, n}$. Note that conditional on \mathbf{Z} , Y_1, \dots, Y_n are independent and that conditional on \mathbf{Z} and W , X_1, \dots, X_n are independent. Factor model (3) may also be combined with the skewed t copula and the grouped t copula, both being asymmetric generalizations of the Student t copula model, as proposed by Demarta and McNeil (2005).

To gain intuition for the Student t factor model as defined by (2) and (3), consider the random variables X_1, \dots, X_n as latent default times. These are related to the original default times in the following way

$$\begin{aligned} T_i \leq t_i &\Leftrightarrow F_{T_i}(T_i) \leq F_{T_i}(t_i) \\ &\Leftrightarrow U_i \leq F_{T_i}(t_i) \\ &\Leftrightarrow X_i \leq t_\nu^{-1}(F_{T_i}(t_i)), \end{aligned} \quad (4)$$

where U_i is a random variable uniformly distributed on $(0, 1)$, $i = 1, \dots, n$.

The model is calibrated to observable market prices of credit default swaps, i.e. the default thresholds are chosen so that they produce risk neutral default probabilities implied by quoted credit default swap spreads: $C_i = t_\nu^{-1}(F_{T_i}(t_i))$.

Using (2), (3) and (4) the joint default time distribution can be written in the following form

$$\begin{aligned} P(T_1 \leq t_1, \dots, T_n \leq t_n) &= P(X_1 \leq t_\nu^{-1}(F_{T_1}(t_1)), \dots, X_n \leq t_\nu^{-1}(F_{T_n}(t_n))) \\ &= \int_{\mathbb{R}^d} \int_0^\infty \prod_{i=1}^n \Phi\left(\frac{\sqrt{\frac{w}{\nu}}C_i - \mathbf{a}'_i \mathbf{z}}{\sqrt{1 - \|\mathbf{a}_i\|^2}}\right) f_{\mathbf{Z}}(\mathbf{z}) f_W(w) dw d\mathbf{z}, \end{aligned} \quad (5)$$

with Φ denoting the cumulative standard normal distribution, $f_{\mathbf{Z}}$ is the joint density of the latent factors \mathbf{Z} :

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{(2\pi)^{d/2}} \exp(-\mathbf{z}'\mathbf{z}/2), \quad \mathbf{z} \in \mathbb{R}^d,$$

and f_W is the density function of the χ_ν^2 distribution:

$$f_W(w) = \frac{\exp(-w/2)w^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)}, \quad w > 0, \nu > 0.$$

The above integral has no analytic expression and hence for practical purposes must be computed numerically. The normal factors can be integrated out by a Gauss-Hermite quadrature and the chi-square factor by a Gauss-Laguerre quadrature. Note that the product in the integrand of (5) contains the univariate conditional default time distributions. Indeed,

$$\begin{aligned}
P(T_i \leq t_i | \mathbf{Z} = \mathbf{z}, W = w) &= P(X_i \leq t_\nu^{-1}(F_{T_i}(t_i)) | \mathbf{Z} = \mathbf{z}, W = w) \\
&= P\left(\sqrt{\frac{\nu}{W}}(\mathbf{a}_i' \mathbf{Z} + \sqrt{1 - \|\mathbf{a}_i\|^2} \varepsilon_i) \leq t_\nu^{-1}(F_{T_i}(t_i)) | \mathbf{Z} = \mathbf{z}, W = w\right) \\
&= P\left(\varepsilon_i \leq \frac{\sqrt{\frac{W}{\nu}} t_\nu^{-1}(F_{T_i}(t_i)) - \mathbf{a}_i' \mathbf{Z}}{\sqrt{1 - \|\mathbf{a}_i\|^2}} | \mathbf{Z} = \mathbf{z}, W = w\right) \\
&= \Phi\left(\frac{\sqrt{\frac{W}{\nu}} C_i - \mathbf{a}_i' \mathbf{z}}{\sqrt{1 - \|\mathbf{a}_i\|^2}}\right). \tag{6}
\end{aligned}$$

3 Random recovery and portfolio loss distribution

The joint default time distribution introduced in the previous section describes the joint default behavior of the debtors underlying the CDO structure and hence completely determines the CDO cash flows. In general, at default only a fraction of the notional can be recovered. This is the so-called recovery rate, which will be denoted by R . The recovery rates are assumed to be random and follow the cumulative Gaussian recovery model proposed by Andersen and Sidenius (2004):

$$R_i = \Phi(\mu_i + \mathbf{b}_i' \mathbf{Z} + \xi_i), \tag{7}$$

where μ_i is a location parameter, \mathbf{b}_i is a d vector of factor loadings and $\xi_i \sim N(0, \sigma_{\xi_i}^2)$, $i = 1, \dots, n$. Further, the error terms ξ_1, \dots, ξ_n are assumed to be independent from each other and also independent from \mathbf{Z} , W and $\varepsilon_1, \dots, \varepsilon_n$. The loss given default of obligor i can then be written as

$$L_i = N_i(1 - R_i), \tag{8}$$

with N_i denoting the notional of the i -th CDS, $i = 1, \dots, n$.

Some remarks:

- conditional on \mathbf{Z} the losses L_1, \dots, L_n are independent,
- conditional on \mathbf{Z} the latent default time X_i and the loss given default L_i are independent,
- conditional on \mathbf{Z} and W all components of the model, i.e. $X_1, \dots, X_n, L_1, \dots, L_n$, are independent,
- in (7), next to Φ , other continuous and strictly increasing functions $C_i : \mathbb{R} \rightarrow [0, 1]$ can be used to model the dependence of losses on the latent factors.

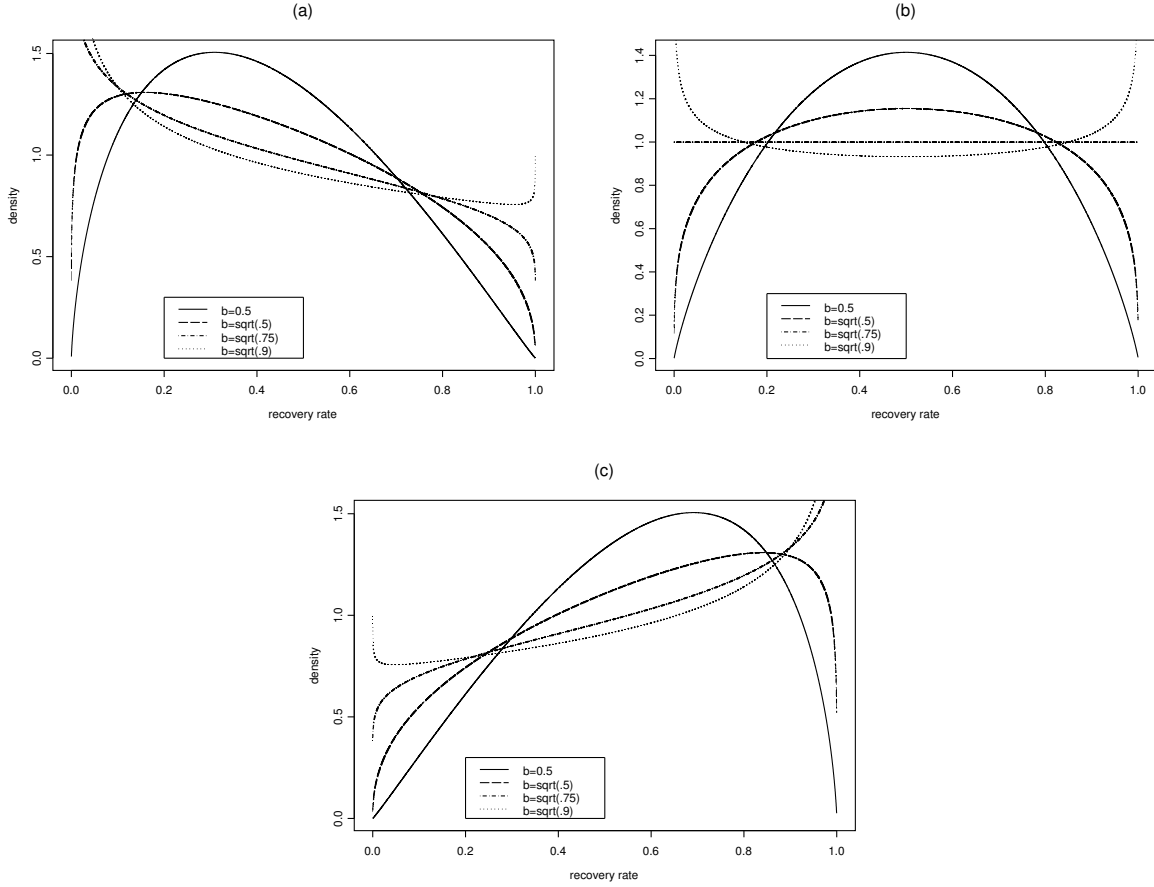


Figure 1: Recovery rate density functions with $\sigma_\xi^2 = 0.25$ and (a) $\mu = -0.25$, (b) $\mu = 0$ and (c) $\mu = 0.25$.

Model (7) is capable to produce a wide variety of distributions. This is illustrated in Figure 1 where we show the recovery rate density function given by

$$f_R(r) = \frac{\phi\left(\frac{\Phi^{-1}(r)-\mu}{\sqrt{\mathbf{b}'\mathbf{b}+\sigma_\xi^2}}\right)}{\phi(\Phi^{-1}(r))\sqrt{\mathbf{b}'\mathbf{b}+\sigma_\xi^2}},$$

in which ϕ is the standard normal density function, for some parameter settings.

We now derive the portfolio loss distribution. Let $L(T)$ denote the losses accumulated over $[0, T]$. Clearly

$$L(T) = \sum_{i=1}^n L_i I_{[0, T]}(T_i),$$

where $I_{\mathcal{A}}(x) = 1$ if $x \in \mathcal{A}$ and 0 otherwise. Because of the conditional independence of $L_1, \dots, L_n, T_1, \dots, T_n$, the conditional portfolio loss distribution is obtained as the convolution product of the individual obligors' loss distributions

$$P(L(T) \leq \ell | \mathbf{Z} = \mathbf{z}, W = w) = \left(F_{\tilde{L}_1 | \mathbf{z}, w} * \dots * F_{\tilde{L}_n | \mathbf{z}, w} \right) (\ell), \quad 0 \leq \ell \leq N, \quad (9)$$

where $N = \sum_{i=1}^n N_i$,

$$\begin{aligned} F_{\tilde{L}_i | \mathbf{z}, w}(\ell) &= P(\tilde{L}_i \leq \ell | \mathbf{Z} = \mathbf{z}, W = w) \\ &= 1 - P(T_i \leq T | \mathbf{Z} = \mathbf{z}, W = w)(1 - P(L_i \leq \ell | \mathbf{Z} = \mathbf{z}, W = w)), \end{aligned} \quad (10)$$

with $P(T_i \leq T | \mathbf{Z} = \mathbf{z}, W = w)$ as given by (6) and

$$P(L_i \leq \ell | \mathbf{Z} = \mathbf{z}, W = w) = \Phi \left(\frac{\mu_i + \mathbf{b}'_i \mathbf{z} - \Phi^{-1} \left(1 - \frac{\ell}{N_i} \right)}{\sigma_{\xi_i}} \right).$$

The unconditional portfolio loss distribution is obtained by integrating out the latent factors:

$$P(L(T) \leq \ell) = \int_{\mathbb{R}^d} \int_0^\infty P(L(T) \leq \ell | \mathbf{Z} = \mathbf{z}, W = w) f_{\mathbf{Z}}(\mathbf{z}) f_W(w) dw d\mathbf{z}. \quad (11)$$

In practice, loss distribution (11) is approximated by expressing the individual obligor's loss given default distributions in terms of numbers of loss units. Consider a loss unit u and let K_i denote the loss given default of obligor i expressed in loss units. We have for $\ell = 1, \dots, \ell_i^{\max}$,

$$\begin{aligned} P(K_i = 0 | \mathbf{Z} = \mathbf{z}, W = w) &= P(L_i \leq 0 | \mathbf{Z} = \mathbf{z}, W = w) \\ &= 0, \\ P(K_i = \ell | \mathbf{Z} = \mathbf{z}, W = w) &= P(L_i \leq \ell u | \mathbf{Z} = \mathbf{z}, W = w) \\ &\quad - P(L_i \leq (\ell - 1)u | \mathbf{Z} = \mathbf{z}, W = w), \end{aligned} \quad (12)$$

where $\ell_i^{\max} = \lceil N_i / u \rceil$. Now, using (12), (11), can be computed recursively. Let $K(T)$ denote the portfolio loss over $[0, T]$ expressed in loss units and let $P^{(i)}$ denote the distribution of $K(T)$ for the first i obligors. Then

$$\begin{aligned} &P^{(i)}(K(T) = \ell | \mathbf{Z} = \mathbf{z}, W = w) \\ &= \sum_{k=0}^{\min(\ell_i^{\max}, \ell)} P^{(i-1)}(K(T) = \ell - k | \mathbf{Z} = \mathbf{z}, W = w) P(\tilde{K}_i = k | \mathbf{Z} = \mathbf{z}, W = w), \end{aligned}$$

where $P(\tilde{K}_i = k | \mathbf{Z} = \mathbf{z}, W = w)$ is the discrete analogue of (10). The recursion starts from the boundary case of an empty portfolio for which $P^{(0)}(K(T) = \ell | \mathbf{Z} = \mathbf{z}, W = w) = \delta_{\ell, 0}$.

Alternatively the loss distribution can be computed using the fast Fourier transform. To do so we start with the characteristic function of $K(T)$:

$$E\left(e^{itK(T)}\right) = E\left(\prod_{j=1}^n e^{itK_j I_{[0,T]}(T_j)}\right) = E\left[\prod_{j=1}^n E\left(e^{itK_j I_{[0,T]}(T_j)} \mid \mathbf{Z}, W\right)\right]$$

with

$$\begin{aligned} & E\left(e^{itK_j I_{[0,T]}(T_j)} \mid \mathbf{Z}, W\right) \\ &= \sum_{\ell=0}^{\ell_j^{\max}} \left(e^{it\ell} P(T_j \leq T \mid \mathbf{Z} = \mathbf{z}, W = w) + P(T_j > T \mid \mathbf{Z} = \mathbf{z}, W = w) \right) P(K_j = \ell \mid \mathbf{Z} = \mathbf{z}, W = w) \\ &= 1 - P(T_j \leq T \mid \mathbf{Z} = \mathbf{z}, W = w) \left(1 - \sum_{\ell=0}^{\ell_j^{\max}} e^{it\ell} P(K_j = \ell \mid \mathbf{Z} = \mathbf{z}, W = w) \right) \end{aligned}$$

A formal expansion of the product yields a characteristic function of the form

$$E\left(e^{itK(T)}\right) := \sum_{\ell=0}^{\ell^{\max}} e^{it\ell} P(K(T) = \ell), \quad (13)$$

where $\ell^{\max} = \sum_{j=1}^n \ell_j^{\max}$. Finally, applying an inverse Fourier transform to the sequence $E(e^{i2\pi k K(T)/(\ell^{\max}+1)})$, $k = 0, \dots, \ell^{\max}$, yields the loss distribution.

4 Convex order approximations

Let us denote by $V_i := N_i(1 - R_i)I_{[0,T]}(T_i)$ the loss over $[0, T]$ associated with name i . Consider the vector (V_1, \dots, V_n) of correlated random variables. We are interested in the distribution of $L(T) = \sum_{i=1}^n V_i$. The determination of the distribution function of this sum is time consuming.

As suggested by Kaas et al. (2000), one approach to approximate the distribution of $L(T)$ consists in approximating this sum with $E[L(T) \mid \Lambda]$ for some arbitrary random variable Λ . On the other hand, replacing the copula of (V_1, \dots, V_n) by the comonotonic copula yields an upper bound in the convex order. Applying this result from Kaas et al. (2000) to the aggregated loss over $[0, T]$ gives

$$\sum_{i=1}^n E[V_i \mid \Lambda] \leq_{cx} L(T) \leq_{cx} \sum_{i=1}^n F_{V_i}^{-1}(U),$$

with U a standard uniform random variable. The upper bound changes the original copula, but keeps the marginal distributions unchanged. The lower bound on the other hand, changes both the copula and the marginals involved. Since convex order implies stop-loss order, it is easy to compute bounds for CDO tranche premiums. Remark that $E[V_i \mid \Lambda] = N_i(1 - R_i)P(T_i \leq T \mid \Lambda)$.

We now approximate the real reference credit portfolio with a portfolio consisting of a large number of equally weighted identical instruments (having the same term structure of default probabilities, recovery rates, and correlations to the common factor). In other words, we assume that the portfolio is homogeneous, i.e. $\mathbf{a}_i = \mathbf{a}$, $C_i = C$ and $R_i = R$ for all i . Denoting the total portfolio notion by N , we then set $N_i = \frac{N}{n}$ for all i . The loss fraction on the portfolio notional over $[0, T]$ is then given by

$$L_n^{\text{HP}}(T) = (1 - R) \frac{1}{n} \sum_{i=1}^n I_{[0, T]}(T_i).$$

We are interested in approximating the distribution function of this portfolio loss fraction. This proves to be possible if the homogeneous portfolio gets very large (i.e. $n \rightarrow \infty$). By the strong law of large numbers, we obtain:

$$P \left[\lim_{n \rightarrow \infty} L_n^{\text{HP}}(T) = (1 - R)P(T_i \leq T | \Lambda) \middle| \Lambda \right] = 1 \quad \text{a.s.}$$

and taking expectations on both sides gives:

$$L_n^{\text{HP}}(T) \xrightarrow{\text{a.s.}} (1 - R)P(T_i \leq T | \Lambda) \quad \text{as } n \rightarrow \infty.$$

The conditional probability that the i^{th} issuer defaults is given by

$$P(T_i \leq T | \Lambda) = \Phi \left(\frac{\Lambda}{\sqrt{1 - \|\mathbf{a}\|^2}} \right),$$

with $\Lambda = \sqrt{\frac{W}{\nu}} C - \mathbf{a}'\mathbf{Z}$. Note that in the Gaussian factor model Λ equals $C - \mathbf{a}'\mathbf{Z}$.

For large homogeneous portfolios, we then make the approximation: $L_n^{\text{HP}}(T) \approx h(\eta)$ where $h : \mathbb{R} \rightarrow [0, 1]$ is given by

$$h(x) = (1 - R) \Phi \left(\frac{x}{\sqrt{1 - \|\mathbf{a}\|^2}} \right).$$

Note that in effect we are replacing the random variable $L_n^{\text{HP}}(T)$ by its lower bound in convex order $E[L_n^{\text{HP}}(T) | \Lambda]$. The distribution of $L_n^{\text{HP}}(T)$ is directly given by that of Λ .

Under the assumption that the individual default probability is less than 50% (which should be satisfied in all practical cases) the cumulative distribution function F_Λ and the density function f_Λ of the variable Λ are given by

$$\begin{aligned} F_\Lambda(t) &= P[\Lambda \leq t] = \int_0^\infty \Phi \left(\frac{t}{\mathbf{a}} - \frac{C}{\mathbf{a}} \sqrt{\frac{w}{\nu}} \right) \gamma_{\frac{1}{2}, \frac{\nu}{2}}(w) dw, \\ f_\Lambda(t) &= \frac{1}{\mathbf{a} \sqrt{\pi} 2^{\frac{\nu+1}{2}} \Gamma(\frac{\nu}{2})} \int_0^{+\infty} e^{-\frac{1}{2\|\mathbf{a}\|^2} (t - \sqrt{\frac{w}{\nu}} C)^2} w^{\frac{\nu}{2} - 1} e^{-\frac{w}{2}} dw, \end{aligned}$$

with γ the density of the gamma distribution.

Schloegl and O’Kane (2005) derived an extremely efficient formula to calculate the density function by solving the integral explicitly in terms of a finite sum over incomplete gamma functions.

Finally we have that $P[L_n^{\text{HP}}(T) \leq \theta] = F_\Lambda(h^{-1}(\theta))$ for any $\theta \in [0, 1]$.

5 Conclusion

The calculation of loss distributions of the portfolio of reference instruments over different time horizons is the central problem of pricing synthetic CDOs. The factor copula approach for modelling correlated defaults has become very popular. Unfortunately, computationally intensive Monte Carlo simulation techniques have to be used if the correlation structure is assumed to be completely general. While the Gaussian copula model, introduced to the credit field by Li (2000), has become an industry standard, its theoretical foundations, such as credit spread dynamics may be questioned.

In this paper dependence between default times is modelled through Student t copulas. A factor approach is used leading to semi-analytic pricing expressions that ease model risk assessment. It is assumed that defaults of different titles in the credit portfolio are independent conditional on a common market factor. In this paper recursions and Fourier methods are used to compute the conditional default distribution. We presented an extension to the student t factor model in which the loss amounts — or equivalently, the recovery rates — associated with defaults are random. We detailed the model properties and compared the semi-analytic pricing approach with large portfolio approximation techniques.

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