Insurance and Asset Pricing in Incomplete Markets with Heavy Tailed Risks

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Abstract

A model for pricing risks in incomplete markets using prices for traded assets and allowing for heavy tailed risks is developed. The approach used is based on an approximation that collapses to the CAPM for multinormal portfolios. The pricing result is derived as an approximation using elliptical distributions and a modification of a previously developed incomplete markets insurance pricing result based on the Esscher transform. The result allows relative pricing of a portfolio of insurance or asset risks compared to a traded portfolio of risks.

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1 Introduction

Theoretical models of equilibrium asset pricing were originally developed assuming mean-variance preferences for investors or multinormal distributions of returns. Practical applications of portfolio models often assume multinormal returns. The original CAPM (Sharpe [15], Lintner [11], Mossin [13]) was developed in the mean-variance preference model framework. Asset pricing models, such as the CAPM, have been shown to hold for more general asset return distributions. In general the elliptical class of distributions has been shown to remain consistent with a form of CAPM (Owen and Rabinovitch [14], Ingersoll [7]). Berk [1] shows when mean-variance maximization is consistent with expected utility maximization.

In insurance risk and option pricing models, normal distributions of returns are not satisfactory and so log returns are often modeled with a normal distribution. More recently, the family of elliptical distributions has generated interest in actuarial research for models for a portfolio of risks $X_1, ..., X_n$ considered as n random variables. Premium principles for pricing insurance risks have also been considered using the elliptical family.

Insurance markets are incomplete in the sense that individuals can not trade insurance contracts on assets or property nor can they hold arbitrary amounts of insurance coverage. Various premium principles have been proposed for pricing insurance contracts. Wang (2002) [16] has developed a distortion function approach to insurance pricing and illustrated how it is consistent with the CAPM under certain distributional assumptions. Landsman (2003) [9] has generalized Wang's approach to elliptical distributions.

The elliptical family is richer than the traditional multivariate normal model, because it contains many important distributions such as the generalized Student, the exponential power models and others as well as the normal distribution. The elliptical family also includes heavy tailed distributions that have become popular in the modelling of stock daily returns (see McDonald [12]). An important advantage of the elliptical family is that they have a variance-covariance structure similar to the Normal family and they preserve the property of symmetric marginals. However, the elliptical family is more general than the normal distribution and is not specified by only the expectation and variance-covariance structure.

In incomplete markets, the Esscher premium principle in insurance has been adapted to pricing financial contracts. The approach used in this paper is a form of generalization of the Esscher approach. The main benefit of the approach is that it captures the effect of heavy tails of the asset distributions in the pricing formula and is asymptotically equivalent to a form of CAPM result.

In this paper we demonstrate how a result proposed by Landsman (2003) [9] for insurance premium principles can be applied to incomplete market pricing of assets or insurance. The main idea of the result is that a traded portfolio of risks, which is assumed to have a joint elliptical distribution, is used to price a portfolio of non-traded risks, also assumed to have a joint elliptical distribution but different to that of the traded asset portfolio. This allows a market price of risk to be determined from the traded asset portfolio, which can then be used in the pricing of the non-traded risks. The pricing result reflects the market price of risk and the distribution of the assets to be priced through the density generator of the elliptical distribution. This approach also allows for heavy tails in the distribution of the risks.

2 Esscher Premium, Asset Pricing and CAPM

The Esscher premium principle determines the net insurance premium for a loss X as

$$¦(X) = \frac{E\left(Xe^{\lambda X}\right)}{E\left(e^{\lambda X}\right)}, \quad \lambda > 0$$

Gerber and Shiu (1994) [5] developed the Esscher transform approach to option pricing. This is equivalent to using an exponential form for the state price density.

Following the work of Bühlmann (1980, 1984) on economic premium principles, Wang (2002) [16] defines the exponential tilting of a risk X_1 induced by a reference portfolio X_2, as

$$H_\lambda(X_1, X_2) = \frac{E(X_1 \exp(\lambda X_2))}{E(\exp(\lambda X_2))}, \quad \lambda > 0,$$

Landsman (2003) [9] showed that, provided the risks have a finite variance-covariance structure, the Esscher premium and exponential tilting are asymptotically equivalent to a (co)variance premium principle

$$H_\lambda(X_1, X_2) = \mu_{X_1} + \lambda\rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2} + o(\lambda), \quad \lambda \to 0$$

For the case of the multinormal distribution for the risks, the result is exact.

Observe that with λ defined as the market price of risk

$$\lambda = \frac{E(R_M) - r_f}{\sigma_M}$$

and X_2 taken as the market portfolio M, this result recovers an (asymptotic) version of the CAPM

$$H_\lambda(X_1, M) = \mu_{X_1} + \frac{cov(X_1, M)}{\sigma_M}[E(R_M) - r_f] + o(\lambda) \quad \lambda \to 0$$

which is exact for multinormal distribution of risks.

3 Pricing with Elliptical Tilting - the Main Idea

An important weakness of the standard CAPM result and exponential tilting is that it fails to take into account the deviation of a risk from the normal distribution in the tail of the distribution. To make the premium more sensitive to the shape of the underlying distribution, Landsman (2003) [9] introduced *elliptical tilting* as a natural generalization of exponential tilting for the elliptical family.

Let $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)$ be a vector of expectations, $\boldsymbol{\Sigma}$ be some $n \times n$ positive definite matrix and $g_n(x)$ some nonnegative function. Then the random vector $\mathbf{X} = (X_1, ..., X_n)$ is said to have an $n-$variate elliptical distribution with vector of expectations $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$, and density generator g_n if it's density is represented as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{(2\pi)^{n/2}\sqrt{|\boldsymbol{\Sigma}|}} g_n\left[\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right]$$

and we write $\mathbf{X} = (X_1, ..., X_n) \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. The normalizing constant c_n is explicitly determined by

$$c_n = \Gamma(n/2)\left[\int_0^\infty x^{n/2-1} g_n(x)dx\right]^{-1}, \tag{1}$$

which is assumed to be finite.

More details on the Elliptical family and the conditions for the generator g_n are given in Kelker (1970) [8], Fang, et al. (1987) [4], Embrechts et al

(1999) [6], and Landsman and Valdez (2003) [10]. In the latter paper results for the Tail-VaR risk measure are derived for the elliptical family.

Let $E_2(\mu,\Sigma,g_2)$ be a bivariate elliptical distribution. In Landsman (2003) [9] the elliptical tilting of risk X_1 induced by X_2 with $E_2(\mu,\Sigma,g_2)$, was defined as follows

$$
\begin{aligned}
&H_\lambda(X_1,X_2;E_2(\mu,\Sigma,g_2)) &(2)\\
&= \frac{1}{C}\int\!\!\int_{R^2} x_1 \frac{g_2\left[\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)-\lambda x_2\right]}{g_2\left[\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right]} f_{\mathbf{X}}(x_1,x_2)dx_1dx_2,\ \ \lambda>0,
\end{aligned}
$$

where

$$
C=\int_{R^2}\frac{g_2\left[\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)-\lambda x_2\right]}{g_2\left[\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right]} f_{\mathbf{X}}(x_1,x_2)dx_1dx_2.
$$

It can be shown that $H_\lambda(X_1,X_2;E_2(\mu,\Sigma,g_2))$ coincides with Wang's exponential tilting if $E_2(\mu,\Sigma,g_2)$ is bivariate normal. This immediately follows from (2), because for the normal family $g_2(x)=\exp(-x)$.

In general the vector $\mathbf{X}=(X_1,X_2)$ does not need to have the same distribution as used for elliptical tilting, $E_2(\mu,\Sigma,g_2)$, but if $\mathbf{X}$ is also distributed $E_2(\mu,\Sigma,g_2)$ then it can be shown that

$$
H_\lambda(X_1,X_2;E_2(\mu,\Sigma,g_2))=\mu_1+\lambda\rho_{12}\sigma_1\sigma_2,
$$

where $\sigma_i=\sqrt{\sigma_{ii}}, i=1,2$

$$
\rho_{12}=\frac{\sigma_{12}}{\sigma_1\sigma_2},
$$

and $\sigma_{ij}, i,j=1,2$ are the elements of the matrix Σ. Of course ρ_{12} coincides with the correlation coefficient between X_1, X_2 if the covariance of $\mathbf{X}$ exists.

If the vector $\mathbf{X}$ is elliptically distributed with the same variance-covariance structure but with a different generator h_2, so that $\mathbf{X}\sim E_2(\mu,\Sigma,h_2)$, then Landsman (2003) [9] proved the following asymptotic representation

$$
H_\lambda(X_1,X_2;E_2(\mu,\Sigma,g_2))=\mu_1+\lambda\beta_{g_2}(h_2)\rho_{12}\sigma_1\sigma_2+o(\lambda), \lambda\to 0,
$$

where

$$
\beta_{g_2}(h_2)=\frac{1}{4}c_2\int_0^{\infty} xJ_2(\frac{1}{2}x))h_2(x)dx \qquad (3)
$$

and

$$J_2(u) = -\frac{d}{du}\log g_2(u) \tag{4}$$

(c_2 is given in (1)).

In this case, the elliptical family $E_2(\mu, \Sigma, g_2)$ used for elliptical tilting can be regarded as a reference distribution that can be used in the pricing of risks from other distributions. In this terminology, exponential tilting can be considered to be elliptical tilting with respect to a normal reference distribution. In other words, the relative difference between the reference distribution and the distribution of the vector $\mathbf{X} = (X_1, X_2)$ is reflected in the difference between elliptical tilting and the standard exponential tilting.

In this paper, a generalization of the Esscher insurance pricing approach to elliptical tilting is used to derive pricing results for assets relative to a reference portfolio. This is a natural extension of the covariance premium principle in insurance and recovers the CAPM in asset pricing under the assumption of multi-normal distributions.

The main result follows from the representation

$$GVP(X_1, X_2; E_2(\mu, \Sigma, g_2)) = \mu_1 + \lambda\beta_{g_2}(h_2)\rho_{12}\sigma_1\sigma_2, \tag{5}$$

called the generalized variance premium.

The approach in this paper is to use this pricing formula and to calibrate it to a reference portfolio that is assumed to have an elliptical distribution. Using the calibrated parameters from the traded prices, the pricing formula can be applied to another portfolio of risks that are also assumed to have a known elliptical distribution. The important aspect of the pricing formula is that the market price of risk is calibrated to the traded prices along with the relative impact of the elliptical distribution on prices. Thus the formula allows for the heavy tails of the underlying distribution through the $\beta_{g_2}(h_2)$ function.

In insurance an application for this formula would be where an insurer would calibrate the pricing formula to the traded assets that it holds in its portfolio. It would then use the formula to price its insurance contracts including an estimate of the market price of risk and allowing for the heavy tails of the insurance risks. It could also use the approach to price any non-traded assets. A key assumption is that of the elliptical distribution and also the use of the approximation. In some senses this is an extension of the CAPM to allow for heavy tails. However this is a more general approach than the CAPM pricing results that assume normal distributions. We should also note that the formula should be applied to expected returns or expected growth rates of insurance liabilities. Thus the formula is assumed to apply to the log of prices and the log of liability values.

4 Pricing with Elliptical Distributions

In this section we will consider a general form of pricing result, and in the following section we will discuss the special case of the generalized Student-t model, scaled by the shape (power parameter) p. The generalized Student-t distribution has found empirical support in the asset modelling literature.

For the elliptical assumption the multivariate distribution of a portfolio of risks is determined not only by the expectations and variance-covariance structure, but also by the density generator of the elliptical distribution. Different assets or portfolios of assets will in general have different density generators reflecting their different price characteristics. For example, the generalized Student family has the same form of the density generator

$$g_n(u) = \left(1 + \frac{u}{k_p}\right)^{-p},$$

where

$$c_n = \frac{\Gamma(p)}{\Gamma(p - n/2)}(2\pi k_p)^{-n/2},$$

n -is a dimension of the portfolio, k_p is a normalizing constant, with different $p \in (1/2, \infty)$ for the shape parameter. Varying p allows fitting the tail behavior to a wide range of shapes of distributions, some of them being so heavy tailed that the variance is infinite ($p < 1.5$).

The density generators can vary by the form of the generator as well as by a parameter such as in the generalized Student case. Since the generator reflects an important characteristic of the joint distribution, it is assumed that prices are affected by both the density generator as well as by the variance-covariance structure of the distribution. This is a departure from the standard mean-variance preference assumption underlying the CAPM.

Let $\mathbf{X} = (X_1, ..., X_n)$ be a multivariate portfolio of risks with a distribution from the elliptical family $E_n(\mu, \Sigma, g_n)$. In fact we have, along with g_n, a sequence $(g_1, ..., g_{n-1})$ of density generators of the marginal distributions of this family. Now we use one of the important properties of the elliptical family of distribution, which motivates its use in asset pricing and portfolio theory: for some $m \times n$ matrix of rank $m \leq n$, A, and some m dimensional column-vector b, we have

$$AX + b \sim E_m\left(A\mu + b, A\Sigma A^T, g_m\right), \tag{6}$$

This means that the sum or aggregate of a portfolio of assets $S = \sum_{i=1}^{n} X_i$ also has a distribution from the same elliptical family. If we set $\mu_{i,S} = (\mu_i, \mu_S)^T$, $\mu_S = \sum_{j=1}^{n} \mu_j$

$$\Sigma_{i,S} = \begin{pmatrix} \sigma_i^2 & \sigma_{iS} \\ \sigma_{iS} & \sigma_S^2 \end{pmatrix},$$

with $\sigma_i^2 = \sigma_{ii}$, $\sigma_{iS} = \sum_{j=1}^{n} \sigma_{ij}$, $\sigma_S^2 = \sum_{i,j=1}^{n} \sigma_{ij}$ then we obtain the result that the bivariate vector (X_i, S) is distributed $E_2(\mu_{i,S}, \Sigma_{i,S}, g_2)$. The reason that the elliptical family can generalize standard multi-normal portfolio and asset pricing results should be clear. It has aggregation properties similar to those of the mulitnormal distribution.

Now assume that a particular density generator $\mathfrak{g}_n$ is chosen as a reference generator for an $n-$ dimensional portfolio. As noted previously together with $\mathfrak{g}_n$ we have also a sequence of generators $(\mathfrak{g}_1, ..., \mathfrak{g}_{n-1})$ for all the marginal distributions of dimensions from $n - 1$ to 1.

The generalized (co)variance premium principle (5) is that for individual risks of the $X-$ portfolio

$$GVP(X_i, S; E_2(\mu_{i,S}, \Sigma, \mathfrak{g}_2)) = \mu_i + \lambda\beta_{\mathfrak{g}_2}(g_2)\rho_{iS}\sigma_i\sigma_S, i = 1, ..., n. \tag{7}$$

Consider a portfolio with weights $\alpha = (\alpha_1, ..., \alpha_n)$ for the vector of the portfolio proportions and denote by P_M the market return (price) of the portfolio $M = \alpha^T X.$ Then we have for the expected return (price) of the portfolio the equation

$$P_M = \mu_M + \lambda\beta_{\mathfrak{g}_2}(g_2)\sigma_M^2,$$

where $\mu_M = \alpha^T\mu$, $\sigma_M^2 = \alpha^T\Sigma\alpha$ and we can then determine λ in (7) as

$$\lambda_M = \frac{P_M - \mu_M}{\beta_{\mathfrak{g}_2}(g_2)\sigma_M^2} \tag{8}$$

This is a portfolio based price of risk consistent with the market price of risk in standard portfolio theory but taking into account the heavy tail characteristics of the elliptical distribution other than expected value and variance.

We can then use this λ_M to price other asset portfolios. Let $Y = (Y_1, ..., Y_m)$ be a portfolio of risks from the elliptical family with another sequence of density generators $\bar{g} = (\bar{g}_1, ..., \bar{g}_m)$, i.e, $Y \sim E_m\left(\mu, \Sigma, \bar{g}_m\right)$. Then for any risk $Y_i, i = 1, 2, ..., m$ we can substitute (8) into (5) and obtain

$$P_{Y_i} = \mu_i + \frac{P_M - \mu_M}{\sigma_M^2} \frac{\beta_{\tilde{g}_2}(\bar{g}_2)}{\beta_{\tilde{g}_2}(g_2)} \rho_{iS}\sigma_i\sigma_S, \ i = 1, ..., n.$$

Now if the reference generator itself $\tilde{g}_2 = g_2$ (it may happen , for example, when the reference portfolio of risks have the same generator as for X-portfolio), then it has been shown by Landsman (2003) [9] that $\beta_{\tilde{g}_2}(g_2) = \beta_{g_2}(g_2) = 1$, and formulas for λ and for expected returns (prices) simplify to:

$$\lambda_M = \frac{P_M - \mu_M}{\sigma_M^2},$$

and

$$P_{\tilde{X}_i} = \mu_i + \frac{P_M - \mu_M}{\sigma_M^2} \beta_{\tilde{g}_2}(\bar{g}_2) \rho_{iS}\sigma_i\sigma_S, i = 1, ..., n. \tag{9}$$

In the next section we consider the special case of multivariate Student models, that have been used in daily stock return modelling. We also consider the selection of the reference distribution for this situation.

5 Multivariate Student Pricing Model.

In this section we illustrate the ideas presented in the previous section, by assuming that the asset portfolio distribution belongs to the Student family. We consider the multivariate Student distribution in it's most general form given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_{n,p}}{(2\pi)^{n/2}\sqrt{|\Sigma|}} \left[1 + \frac{(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)}{2k_{n,p}}\right]^{-p}, \tag{10}$$

where

$$c_{n,p} = \frac{\Gamma(p)}{\Gamma(p - n/2) k_{n,p}^{n/2}}, \tag{11}$$

and $k_{n,p}$ is the normalized constant (see details in Landsman and Valdez (2003) [10]).

For any given expectation vector μ and "covariance" matrix Σ, the Student distribution has a shape parameter p, which when varied will change the tail behavior of the distribution. The multivariate normal distribution is simply the limiting case of the Student when $p = \infty$. The density generator of this family

$$g_n(u) = \frac{1}{(1 + \frac{u}{k_{n,p}})^p} \tag{12}$$

The following Theorem shows that the density generator of the Student family really depends on the dimension of this family:

Theorem 1 *Suppose $g_n(u)$ is a density generator of an n-variate Student family with shape (power) parameter p, having form (12). Then the density generator of it's $n-1$-variate has power parameter equal to $p - 1/2$, given by*

$$g_{n-1}(u) = \frac{1}{(1 + \frac{u}{k_{n-1,p}})^{p-1/2}}, \tag{13}$$

where $k_{n-1,p} = k_{n,p}$.

5.1 Univariate Generalized Student-t distribution

From (10) the univariate Generalized Student-t distribution has the form

$$f_X(x) = \frac{1}{\sigma\sqrt{2k_{1,p}}B(1/2, p-1/2)} \left[1 + \frac{(x-\mu)^2}{2k_{1,p}\sigma^2}\right]^{-p}, \quad p > 1/2. \tag{14}$$

If $p \leq 1/2$ the density does not exist. For $1/2 < p \leq 1$ the expectation does not exist. If $1 < p \leq 3/3$, the expectation exists, but the variance is still infinite. For $p > 3/2$, the Generalized Student-t distribution has finite variance. For this case if we put $k_{1,p} = p - 3/2$, the variances corresponded to densities (14), all are equal to

$$V(X) = \sigma^2, \quad p > 3/2.$$

So, let us put

$$k_{1,p} = \begin{cases} p - 3/2, & p > 3/2 \\ 1/2, & 1/2 < p \leq 3/2 \end{cases}. \tag{15}$$

The distribution with density (14) is called the univariate generalized Student-t distribution (UGST). This distribution is discussed by many authors, for example McDonald (1996) [12].

5.2 Generalized (co)variance premium principle with Student family

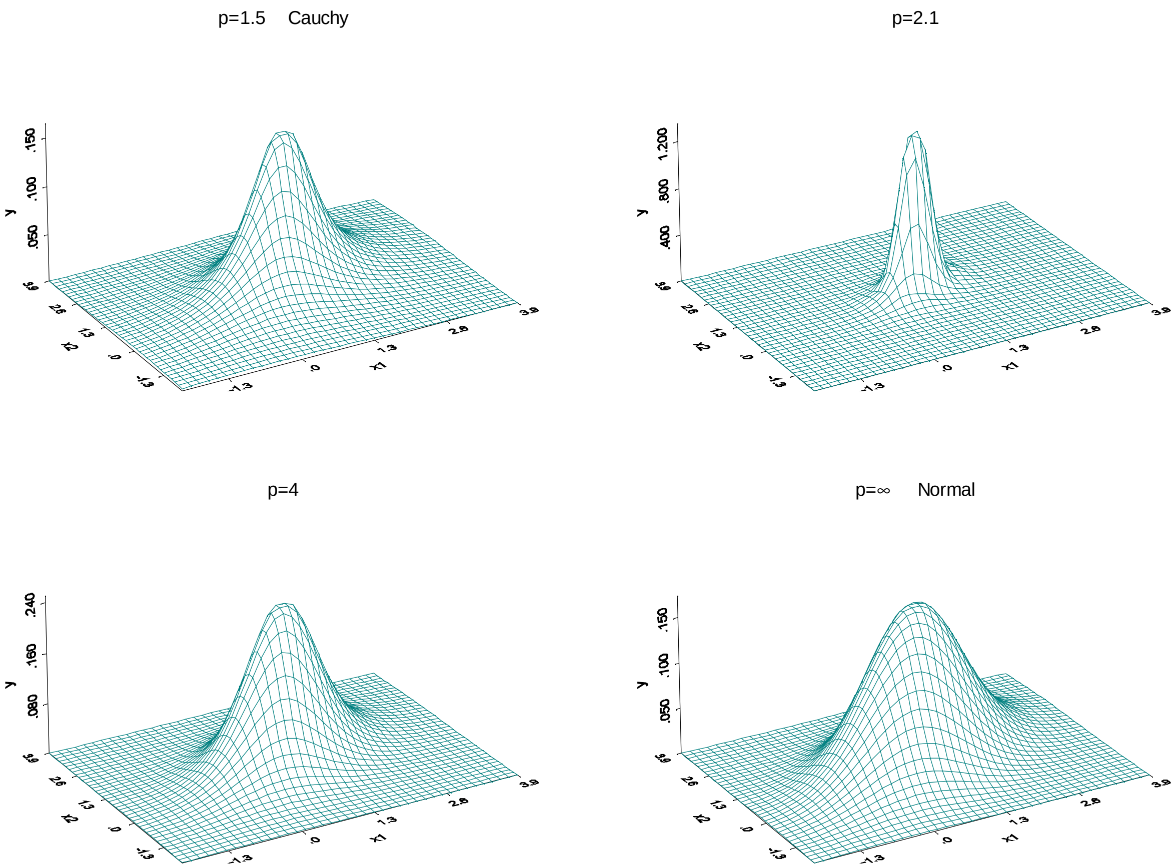
Suppose $\mathbf{X} = (X_1, ..., X_n)$ is a multivariate portfolio of risks having a Student distribution with vector of expectations μ, covariance structure matrix Σ, and density generator from (12). From Theorem 1 it immediately follows that the covariance exists only if $p > (n+2)/2$. If we chose

$$k_{n,p} = \begin{cases} p - (n+2)/2, & p > (n+2)/2 \\ 1/2, & n/2 < p \leq (n+2)/2 \end{cases}, \tag{16}$$

in (12) it follows that, from the same theorem and (15) $\Sigma = COV(\mathbf{X})$, when $p > 2$. Denote the density generator (12) with $k_{n,p}$ from (16) by $g_{n,p}$. For the case $n = 2$ we have

$$k_{2,p} = \begin{cases} p - 2, & p > 2 \\ 1/2, & 1 < p \leq 2 \end{cases}.$$

Figure 1: Densities of bivariate GST for $p = 1.5, 2.1, 4, 1$ and $\rho = 0.4$



In Figure 1 we show GST density's graphs with common vector of expectations $\mu = (0.9, 0.9)$ and matrix

$$\S = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \rho = 0.4,$$

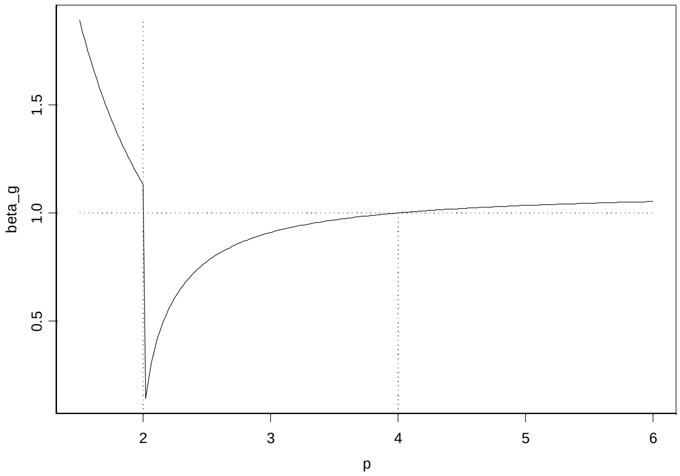


Figure 2: Varying $\beta_{2,p_0}(p)$ in $p; p_0 = 4$

but di¤erent parameters $p = 1.5, 2.1, 4, 1$.

We can determine the form of the coe¢ cient $\beta_{g_2}(g_2)$ in the generalized (co)variance premium formula (7) for the Student family. In Figure 2 the reference density generator has shape (power) parameter $p_0 = 4$. We can see in the Figure that $p = 2$ is a breaking point around which the behavior of $\beta_{2,p_0}(p)$ is di¤erent. For this reason we have selected $p_0 = 2$ as a shape parameter for the reference density generator.

For the reference density generator with $p_0 = 2$ the graph of the coe¢ cient $\beta_{2,p_0}(p)$ is given in Figure 3.

6 Application to Financial Asset Data

To illustrate the application of the pricing result we analyze actual stock market data. Consider a portfolio of 15 stocks from the Dow Jones stock index (3MComp., Alcoa Inc., Boeing Co., Caterpillar Inc., Exxon Mobil Corp., Fedex Corp., General MTRS Corp., HONEYWELL INTL Inc., United Technologies Corp., United Parcel SVC Inc. CL B, International Paper Co., Disney Walt Co Disney Com., Du Pont E I De Nemours&Co, Eastman KO-

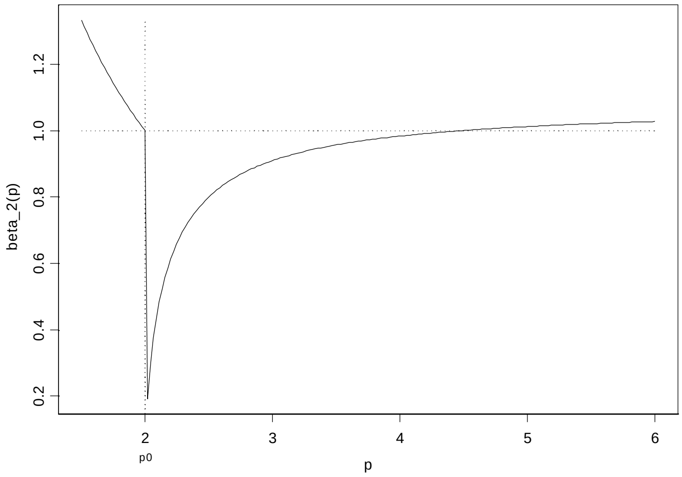


Figure 3: Varying $\beta_{2,p_0}(p)$ in p; $p_0 = 2$

DAK Co., Wal Mart Stores Inc.,) and denote these by **X** $= (X_1, ..., X_n)$ with $n = 15$. Using daily stock returns it is easily verified using Anderson-Darling test statistics (AD), that, with a high level of significance, **X** can be considered as generated from a multivariate GST distribution with power parameter $p_n = 12.86$. The density generator of this distribution will be used as a reference generator for pricing other assets.

Consider, for example, a portfolio **Y** $= (Y_1, ..., Y_m)$ of $m = 10$ stock daily returns of high technology companies from NASDAQ/Computers (ABOBE Sys. Inc., Compuware Corp., NVIDIA Corp., Peoplesoft Inc., Veritas Software Co., Sandisk Corp., Microsoft Corp., Symantec Corp., Citrix Sys Inc., Intuit Inc.). A good fit for this second portfolio was found to be multivariate GST with power parameter $p_m = 7.34$.

Now we can use formula (9) for pricing risks in portfolio **Y**. In fact, denoting the distribution of these portfolios as **X** $\sim E_n(\mu, \Sigma, g_n)$ and **Y** $\sim E_m(\overline{\mu}, \overline{\Sigma}, \overline{g}_m)$, where g_n and $\overline{g}_m$ are GST density generators (12) with p equal p_n and p_m respectively we can evaluate k_{n,p_n} and k_{m,p_m} from (16). Now Theorem 1 states that g_2 and $\overline{g}_2$ are Student density generators with $p_0 = p_n - n/2 + 1 = 8.36$ and $p = p_m - m/2 + 1 = 3.34$ respectively. We can

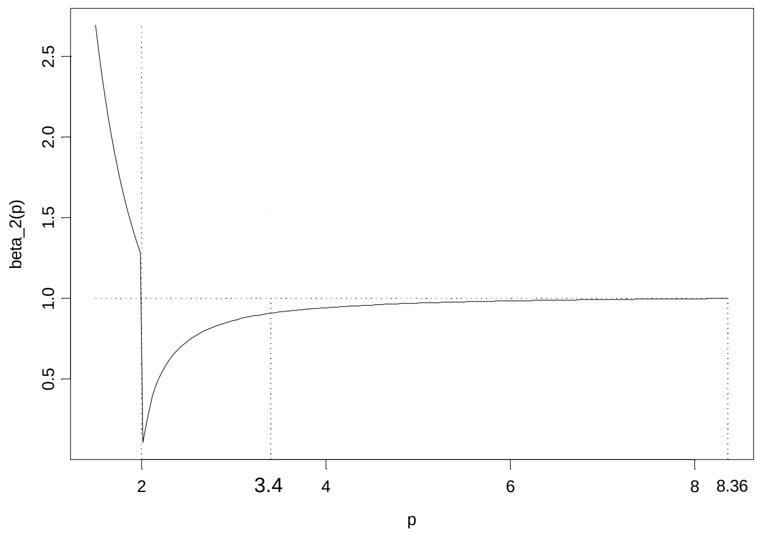


Figure 4: Varying $\beta_{2,p_0}(p)$ in $p; p_0 = 8.36$

then derive the coefficient $\beta_{\bar{g}_2}(\bar{g}_2)$ in (9) by formula (??) to get

$$\beta_{\bar{g}_2}(\bar{g}_2) = \beta_{g_{2,p_0}}(g_{2,p}) = 0.91$$

This is illustrated in Figure 4.

The expected return (price) formula we have derived can then be used to price the stocks from the NASDAQ's subset $\mathbf{Y}$, using the market price of the portfolio from Dow Jones's subset $\mathbf{X}$, and taking into account the distribution of both the reference portfolio and the portfolio of risks to be priced, to get

$$P_{Y_i} = \mu_i + 0.91 \frac{P_M - \mu_M}{\sigma_M^2} \rho_{iS} \sigma_i \sigma_S, i = 1, ..., n.$$

Thus we have a pricing formula that prices one set of stocks in comparison to another set of stocks taking into account the price of risk calibrated to the first portfolio as well as the heavy tails of the distributions. Thus the pricing formula takes into account the departure from the normal distribution assumption.

As an application in insurance, an insurance company could use its asset portfolio as the reference portfolio used to determine the price of risk and the risks to be priced could be the insurance portfolios. The result of such an application is yet to be investigated.

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