

Risk Measures and Insurance Premium Principles

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Abstract

Risk measures based on distorted probabilities have been recently developed in actuarial science and applied to insurance rate making. An example is the proportional hazards transform. A risk measure should satisfy the properties of risk aversion and diversification for both insurance and asset allocation decisions. The risk measure based on distorted probabilities is not consistent with the change of measure used in financial economics for pricing. This measure is also not consistent in its treatment of insurance and investment risks. We propose a risk measure that has the properties of risk aversion and diversification, is additive and consistent in its treatment of insurance and investment risks.

1 Introduction

Models are used in actuarial science for both quantifying risks and for pricing risks. Quantifying risk requires a risk measure to convert a random future gain or loss into a certainty equivalent that can then be used to order different risks and for decision making purposes. In order to quantify risk it is necessary to specify the probability distributions of the risks involved and to apply a preference function to these probability distributions. Thus this process involves both statistical assumptions and economic assumptions. The resulting risk measure should satisfy desirable properties. These properties include risk aversion, which is fundamental to insurance, and diversification, which is fundamental to portfolio theory and investment selection.

In order to determine a price or premium for a risk it is necessary to convert the random future gain or loss into financial terms. As well as a probability distribution for the gain or loss, a premium or pricing principle is also required. Pricing a risk uses this premium principle to convert the random gain or loss into a premium or price. Thus the components of a model required to price risks are a statistical model for the risks, an economic model for preferences and pricing or premium principles to convert risk measures into monetary terms.

Prices or premiums must also satisfy some basic properties. The model used should produce consistent and sensible results. These include consistency with observed behavior in financial or insurance markets and the consistent treatment of different risks. It is also important to have a consistent treatment of asset and liability (or insurance) gains and losses.

Many different premium principles have been proposed in actuarial science. Goovaerts [4] discusses various premium principles and examines various properties that these premium principles should satisfy. In financial economics the principle of no-arbitrage is an important requirement for a consistent financial model. Panjer et al [8] cover both equilibrium pricing and no-arbitrage pricing in financial models. Wang [10] proposes a premium principle based on a proportional transformation of the hazard function. This premium principle corresponds to the certainty equivalent of the dual theory of expected utility developed by Yaari [14]. These approaches to pricing insurance contracts treat insurance losses as positive random variables and produce premiums that are higher than the expected value of the insurance loss.

2 Assumptions and notation

Random gains and losses, arising from insurance losses and investment decisions, are denoted by the random variables X_i . For each random gain or loss, denote the probability distribution function by $F_i(x) = \Pr\{X_i \leq x\}$ and the decumulative distribution function by $\bar{F}_i(x) = \Pr\{X_i > x\}$. The inverse of the decumulative distribution function is defined as

$$\bar{F}^{-1}(q) = \inf\{x : \bar{F}(x) \leq q\}, \quad 0 \leq q < 1, \quad \bar{F}^{-1}(1) = 0$$

We take the perspective of an individual so that for an insurance loss X_i will be non-positive. The pay-off on an insurance policy in the event of a claim is denoted by Y_i , and will be non-negative. For an investment in an asset, X_i will be bounded below for limited liability securities. Usually insurance will cover the loss of value of an asset arising from predefined insurable events. In this case an individual will usually also hold the asset and will be subject to investment risk arising from fluctuations caused by economic and other factors. The total risk for the asset includes both economic fluctuations in value and losses from insurable events.

We assume that any model used to quantify risk and to determine prices and premiums should have the properties that individual risk preferences exhibit risk aversion and portfolio diversification. Both of these properties are considered essential properties for a financial model used for pricing and asset allocation. Most real world financial decisions including the purchase of insurance and investment decisions are consistent with risk aversion and portfolio diversification. Portfolio diversification implies risk aversion for expected utility models. However risk aversion is not sufficient to imply portfolio diversification for general preference functions (Dekel [2]).

We define risk aversion and diversification as follows:

Definition 1 *Risk aversion - preferences exhibit risk aversion when the expectation of a risk is preferred to the risk i.e. actuarially fair gambles are unacceptable.*

Definition 2 *Diversification - preferences exhibit portfolio diversification if a convex combination of risks is preferred to any single risk, assuming all risks are identical.*

A preference relation \succeq is assumed to exist over probability distributions with the symbol \succ indicating strict preference and \sim indicating indifference. For example $F_1 \succ F_2$ indicates that the random gain or loss X_1 with probability distribution function F_1 is strictly preferred to the random gain or loss X_2 , with probability distribution function F_2 .

3 Expected Utility

An axiomatic approach to the derivation of a risk preference function for ordering risks using expected utility was given by von Neumann and Morgenstern [9]. The axioms are also found in Wang and Young [12]. The key axiom is the so-called independence axiom. This axiom states that if $X \succ Y$ and Z is any risk then

$$\{(\alpha, X), (1 - \alpha, Z)\} \succ \{(\alpha, Y), (1 - \alpha, Z)\}$$

for all α such that $0 \leq \alpha \leq 1$ where $\{(\alpha, X), (1 - \alpha, Z)\}$ is the probabilistic mixture with

$$F_{\{(\alpha, X), (1 - \alpha, Z)\}}(x) = \alpha F_X(x) + (1 - \alpha) F_Z(x)$$

or equivalently

$$\bar{F}_{\{(\alpha, X), (1 - \alpha, Z)\}}(x) = \alpha \bar{F}_X(x) + (1 - \alpha) \bar{F}_Z(x).$$

We can write this as

$$\alpha \bar{F}_X(x) + (1 - \alpha) \bar{F}_Z(x) \succ \alpha \bar{F}_Y(x) + (1 - \alpha) \bar{F}_Z(x).$$

Assume that an individual has initial wealth W . From the axioms it is possible to show that there exists a utility function u such that $F_1 \succ F_2$ if and only if $E[u(W + X_1)] > E[u(W + X_2)]$. Properties and applications of utility functions are covered in the survey paper by Gerber and Pafumi [3].

Since we assume risk aversion, $u(X)$ is an increasing concave function of X and is at least twice differentiable with $u'(X) > 0$ and $u''(X) < 0$. Utility functions are only unique up to a positive affine transformation so that $u^*(X) = au(X) + b$ will produce the same ordering of risks as would $u(X)$. Utility functions can be standardized by taking $u(k) = 0$ and $u'(k) = 1$ for some point k .

For an insurance loss X_i will be negative. If we write the loss as a positive random variable $Y = -X$ then $E[u(X)] = E[u(-Y)]$.

For risk aversion, we have

$$E[u(X)] \leq u(E[X])$$

since u is concave, and this holds for all risks where X is negative for insurance losses. Alternatively we have

$$u^{-1}[E[u(X)]] \leq E[X]$$

for risk aversion. For insurance losses, taking $Y = -X$ where Y is a non-negative random variable, this is the same as

$$-u^{-1}[E[u(-Y)]] \geq E[Y]$$

The certainty equivalent of a risk is defined as follows.

Definition 3 *If an individual is indifferent between a risk and receiving an amount with certainty then this certain amount is the certainty equivalent for the risk.*

Denote the certainty equivalent of risk X by C_X . For an individual with current wealth W we have

$$u(W + C_X) = E[u(W + X)]$$

so that

$$C_X = u^{-1}[E[u(W + X)]] - W$$

For an insurance risk, if we consider the loss amount $Y = -X$ and π_Y as the certainty equivalent then

$$u(W - \pi_Y) = E[u(W - Y)]$$

or

$$\pi_Y = W - u^{-1}[E[u(W - Y)]]$$

Note that

$$-C_{-Y} = \pi_Y$$

Jensen's inequality can be used to show that for any random variable X and for concave u

$$u(W + E[X]) \geq E[u(W + X)]$$

so that

$$E[X] \geq C_X.$$

We also have

$$\pi_Y \geq E[Y]$$

for Y a non-negative loss random variable.

Consider a portfolio of risks $\{X_1, \dots, X_n\}$ given by $\sum_{i=1}^n \alpha_i X_i$. For portfolio diversification we require an individual to have a preference for holding the portfolio to holding any of the individual risks in the portfolio assuming all the risks are identical. Thus for diversification we require

$$C_{Portfolio} \succ C_{X_i}$$

where

$$C_{X_1} = C_{X_2} = \dots = C_{X_n}$$

For expected utility we have diversification ([2]).

For pricing, consider selecting a portfolio of risks $\{X_i : i = 1, \dots, n\}$ and insurance policies with positive loss payments $\{Y_i : i = n + 1, \dots, m\}$ to maximize expected utility of wealth. Assume that risk or policy i has price P_i . For investment risks X_i , P_i is the amount of initial investment and for insurance policies that pay Y_i , P_i is the premium.

The problem is to

$$\max E \left[u \left(\sum_{i=1}^n \alpha_i X_i + \sum_{i=n+1}^m \alpha_i Y_i \right) \right]$$

$$\text{subject to } W = \sum_{i=1}^m \alpha_i P_i$$

The Lagrangian will be

$$E \left[u \left(\sum_{i=1}^n \alpha_i X_i + \sum_{i=n+1}^m \alpha_i Y_i \right) \right] - \lambda \left[W - \sum_{i=1}^m \alpha_i P_i \right]$$

and differentiating with respect to α_i we have for optimality,

$$\lambda P_i = -E [u' (W^*) X_i] \text{ for } i = 1, \dots, n.$$

$$\lambda P_i = -E [u' (W^*) Y_i] \text{ for } i = n + 1, \dots, m.$$

where

$$W^* = \sum_{i=1}^n \alpha_i^* X_i + \sum_{i=n+1}^m \alpha_i^* Y_i$$

Assume that X_1 involves no risk from holding investments or insurance risk. This is a risk free use of current wealth involving no exposure to economic risk or exposure to insurable events. Then

$$P_1 = X_1 = k$$

where k is a constant, so that the optimal value of the Lagrange multiplier is

$$\hat{\lambda} = -E [u' (W^*)]$$

We then have

$$P_i = \frac{E [u' (W^*) X_i]}{E [u' (W^*)]} \text{ for } i = 1, \dots, n.$$

$$P_i = \frac{E [u' (W^*) Y_i]}{E [u' (W^*)]} \text{ for } i = n + 1, \dots, m.$$

We can therefore express the price or premium for risk X_i as

$$P_i = E [\Psi X_i]$$

and the premium for the insurance contracts as

$$P_i = E [\Psi Y_i]$$

where

$$\Psi = \frac{u' \left(\sum_{i=1}^n \alpha_i^* X_i + \sum_{i=n+1}^m \alpha_i^* Y_i \right)}{E \left[u' \left(\sum_{i=1}^n \alpha_i^* X_i + \sum_{i=n+1}^m \alpha_i^* Y_i \right) \right]}$$

Now use the random variable Ψ as a Radon-Nikodym derivative to change the probability measure to Q so that under this altered probability measure we have $P_i = E_Q [X_i]$. Thus under this new probability measure we have that

each risk is priced at its expectation so that we can refer to this Q measure as the *risk neutral measure*. See also [3] for a derivation of a similar result in the context of a Pareto optimal risk exchange.

Premiums for insurance contracts using this change of measure are additive for all risks. Thus they will satisfy our requirement for consistent treatment of risks.

4 Dual Utility Theory and Distortion Functions

The dual theory of Yaari [14] develops an alternative to expected utility using an axiomatic approach where the independence axiom for expected utility is replaced with the dual independence axiom. The dual independence axiom states that if $X \succ Y$ and Z is any risk then

$$\alpha \overline{F}_X^{-1}(q) + (1 - \alpha) \overline{F}_Z^{-1}(q) \succ \alpha \overline{F}_Y^{-1}(q) + (1 - \alpha) \overline{F}_Z^{-1}(q)$$

for all α such that $0 \leq \alpha \leq 1$.

These axioms imply that there exists a continuous nondecreasing function g , such that $F_1 \succ F_2$ if and only if

$$-\int_0^1 g(q) d\overline{F}_1^{-1}(q) > -\int_0^1 g(q) d\overline{F}_2^{-1}(q)$$

Letting $q = \overline{F}_i(x)$ we have that

$$-\int_0^1 g(q) d\overline{F}_i^{-1}(q) = \int_0^\infty g(\overline{F}_i(x)) dx$$

Yaari [14] notes that $U_{X_i} = \int_0^\infty g(\overline{F}_i(x)) dx$ is a utility which assigns the certainty equivalent to a random variable. Thus a decision maker would be indifferent between receiving U_{X_i} for certain and the risk X_i .

Wang ([10], [11]) proposes pricing insurance risks using a distortion function based on the proportional hazards transform. For an insurance risk Y , a non-negative random variable, with decumulative distribution function $\overline{F}_Y(x) = \Pr\{Y > x\}$, Wang proposes the premium principle $H_r(X) = \int_0^\infty (\overline{F}_i(x))^r dx$ with $0 \leq r \leq 1$. $H_r(X)$ is used to calculate risk-adjusted premiums. Note that $g(x) = x^r$, $0 \leq r \leq 1$, is a concave function.

Wang and Young ([12]) define a distortion function as a nondecreasing function with $g(0) = 0$ and $g(1) = 1$ such that, for a non negative random variable Y the certainty equivalents $H_g[Y]$ and $H_g[-Y]$ are given by

$$H_g(Y) = \int_0^\infty g(\overline{F}_i(x))dx = \int_0^1 \overline{F}_Y^{-1}(q)dg(q)$$

and

$$H_g(-Y) = -H_{\tilde{g}}(Y)$$

where \tilde{g} is the distortion function defined by $\tilde{g}(q) = 1 - g(1 - q)$, $0 \leq q \leq 1$. Note that \tilde{g} is convex if g is concave.

$H_g(Y)$ satisfies the following properties (see[12])

- If g is concave then $H_g(Y) \geq E(Y)$.
- $H_g(aY + b) = aH_g(Y) + b$, for $a, b \geq 0$.
- For concave g , $H_g(Y_1 + Y_2) \leq H_g(Y_1) + H_g(Y_2)$

This third property is referred to as sub-additivity. $H_g(Y)$ will be additive in the special case of comonotonic risks. Risks X_1 and X_2 are comonotonic if there exists a risk Z and nondecreasing real-valued functions f and h such that $X_1 = f(Z)$ and $X_2 = h(Z)$ ([12]). The concept of comonotonic risks is an extension of perfect correlation.

If we were to apply this certainty equivalent approach to investment risks then, given risk aversion, we require $H_g[X] \leq E[X]$ which means that g should be convex. For an insurance loss $Y = -X$, a non-negative random variable we have, according to the definition of Wang and Young ([12]),

$$H_g[Y] = H_g[-X] = -H_{\tilde{g}}(X) \tag{1}$$

In this case \tilde{g} is convex for risk aversion so that g will be concave. We then have that $H_{\tilde{g}}(X) \leq E[X]$ or $-H_{\tilde{g}}(X) \geq -E[X] = E[Y]$ and therefore $H_g[Y] \geq E[Y]$.

Note that for g convex we have

$$H_g\left[\sum_{i=1}^n \alpha_i X_i\right] \geq H_g[X_i]$$

if $H_g[X_i] = H_g[X_1]$ for $i = 1, \dots, n$. Thus we have diversification for investment risks.

If we take insurance losses as non-negative random variables, Y_i , then it is necessary to use a concave g consistent with the convex \tilde{g} used for investment risks X_i . However, diversification does not hold for concave g . As a result, the distortion risk measure is not suitable for asset-liability management. Liability, or loss risks, and asset risks are not treated consistently since there is diversification for asset risks but not for liability risks.

Except for comonotonic risks, $H_g(Y)$ is not additive. We require the premium principle to be additive in the same way as prices of financial assets are additive. This effectively assumes that insurance premiums are determined in a market with perfect information, no transactions costs or other imperfections and that there is no-arbitrage in insurance markets. This is consistent with the notion of equilibrium in an insurance market and with pricing of financial assets under perfect market assumptions. In this paper we do not address issues in insurance pricing that arise from relaxing these assumptions.

For the risk measure $H_g[X]$ we have risk aversion and diversification properties provided g is convex. However, we would not want to use this as a premium principle since we would like premiums to be additive. This additivity property only holds for $H_g[X]$ for comonotonic risks whereas we would like this property to hold in general.

Motivated by the distortion function approach used in the risk measure $H_g[X]$, we propose a method of determining premiums which has the property that premiums for a portfolio of risks are additive. We also define a certainty equivalent using this pricing risk measure using a concave function similar to the utility function. This certainty equivalent has the desired properties of risk aversion, diversification and a more consistent ordering of risks similar to that for expected utility.

5 Change of Measure for Premiums using Distortion Functions

The distortion function approach to insurance premiums of Wang ([10], [11]) uses the distribution of losses as positive random variables. The premium

derived is equivalent to a certainty equivalent in the dual expected utility theory. If we were to apply this certainty equivalent to investment risks then we would need to use a convex \tilde{g} function equivalent to the concave g used for insurance pricing.

The insurance premium is greater than or equal to the expected loss, consistent with risk aversion. However, if the distortion function approach to insurance losses is used for pricing then insurance premiums are not additive except for comonotonic risks.

We require a premium principle for a portfolio of risks which has the property of risk aversion and a risk measure consistent with the use of a concave utility function applied to insurance losses.

For the payoffs from a portfolio of insurance contracts where the amount paid is a positive random variable $\{Y_1, \dots, Y_n\}$ with distributions (F_1, \dots, F_n) given by $Y_\alpha = \sum_{i=1}^n \alpha_i Y_i$. we define a premium

$$\pi_r(Y_\alpha) = E_{P_{n,r}} Y_\alpha,$$

where $P_{n,r}$ is the probability measure corresponding to the distribution $F_{n,r} = \prod_{i=1}^n F_i^r$, $0 < r \leq 1$, the product of each distribution raised to the power of r .

It is clear that such a premium principle will be additive for the portfolio of risks and

$$\pi_r(Y_\alpha) \geq EY_\alpha.$$

In order to treat investment and insurance risks consistently, we define a risk measure of a portfolio as follows

$$U_r(Y_\alpha) = u^{-1}(E_{P_{n,r}} u(Y_\alpha)),$$

where u is concave.

Theorem 1 *Let u be a concave, increasing and twice differentiable nonlinear function, $u'(0) > 0$, and let the expectations*

$$E_{P_{n,r}} Y_j$$

be continuous functions of r at point $r = 1$ from the left hand side. Then there exists an r^ with $0 < r^* < 1$, such that for any $r^* \leq r < 1$ and any $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\sum_{i=1}^k \alpha_i = 1$,*

$$U_r(Y_\alpha) \leq EY_\alpha,$$

i.e. the risk measure $U_r(\cdot)$ has the risk aversion property.

Proof. From concavity of u it follows that $u(x) \leq u(0) + u'(0)x$. Suppose, without loss of generality, that $u(0) = 0$. Then, from the conditions of the theorem, the function

$$\Phi(\alpha_1, \dots, \alpha_n, r) = u^{-1}(E_{P_{n,r}} u(\sum_{i=1}^k \alpha_i Y_i))$$

as a function of r will be continuous at the point $r = 1$ from the left hand side for any $\alpha = (\alpha_1, \dots, \alpha_n)$. This means that

$$\Phi(\alpha_1, \dots, \alpha_n, r) \xrightarrow{r \rightarrow 1-0} \Phi(\alpha_1, \dots, \alpha_n, 1) = u^{-1}(Eu(\sum_{i=1}^k \alpha_i Y_i)),$$

and this convergence is uniform for $\alpha \in H = \{\sum_{i=1}^k \alpha_i = 1\}$, because the hyperplane H is compact. Then for $\Delta > 0$ there exists $0 < r^* < 1$ such that for any $\alpha \in H$

$$|\Phi(\alpha_1, \dots, \alpha_n, r) - u^{-1}(Eu(\sum_{i=1}^k \alpha_i Y_i))| < \frac{\Delta}{2} \quad (2)$$

On the other hand for concave and nonlinear u , the function

$$\Delta(\alpha) = \pi_{r|_{r=1}}(Y_\alpha) - \Phi(\alpha_1, \dots, \alpha_n, 1) > 0, \text{ for any } \alpha \in H,$$

is continuous in α , and as $\alpha \in H$, a compact set,

$$\inf_{\alpha \in H} \Delta(\alpha) = \Delta(\alpha^*) > 0, \alpha^* \in H.$$

Then

$$\begin{aligned} EY_\alpha - U_r(Y_\alpha) &= \pi_{r|_{r=1}}(Y_\alpha) - \Phi(\alpha_1, \dots, \alpha_n, \alpha) \\ &\geq \Delta(\alpha) - |\Phi(\alpha_1, \dots, \alpha_n, 1) - \Phi(\alpha_1, \dots, \alpha_n, \alpha)| \\ &\geq \Delta(\alpha^*) - \Delta(\alpha^*)/2 > 0, \text{ for } r^* < r < 1, \end{aligned}$$

setting in (2) $\Delta = \Delta(\alpha^*)$. \square

The risk measure proposed will also have a diversification property for insurance losses where these losses are taken as negative (non-positive) random variables.

6 Consistency of risk measure for investment and risk measure for losses in preference relations.

Suppose we consistently define a risk measure for investment, $Rm_I(X)$, and a risk measure for non-negative losses, $Rm_L(Y)$. In the previous sections we considered some specific definitions of $Rm_I(X)$ and $Rm_L(Y)$. These were:

1) Expected utility: for concave u

$$Rm_I(X) = u^{-1}[Eu(W + X)] - W, \quad Rm_L(Y) = W - u^{-1}[Eu(W - Y)] \quad (3)$$

2) Distortion Functions: for concave g

$$Rm_I(X) = H_{\tilde{g}}(X), \quad Rm_L(Y) = H_g(Y)$$

where $\tilde{g}(q) = 1 - g(1 - q)$ is convex.

3) Change of measure: for $r^* \leq r < 1$

$$Rm_I(X) = U_r(X) = u^{-1}(E_{P_{n,r}} u(X)), \quad Rm_L(Y) = E_{P_{n,r}}(Y) \quad (4)$$

The measures $Rm_I(X)$ and $Rm_L(Y)$ will be consistent with respect to preference relations if they produce equivalent preferences, i.e. for two non-negative r.v.'s. from the portfolio

$$X \underset{Rm_I}{\prec} Y \quad (5)$$

and

$$X \underset{Rm_L}{\prec} Y \quad (6)$$

should hold simultaneously. Notice that the measures given as expected utility are very close to our requirement of consistency in preference relations. In fact, if we define $Rm_I(X)$ and $Rm_L(Y)$ by formulae (3) we get from (5) that

$$W + X \underset{u}{\prec} W + Y \quad (7)$$

where $\underset{u}{\prec}$ is the preference given by $Eu(\cdot)$, and (6) is equivalent to

$$W - Y \underset{u}{\succ} W - X \quad (8)$$

It is clear that (7) and (8) can be considered to be equivalent for any function u , symmetric with respect to W , i.e. $u(W + X) = -u(W - X)$.

We can also consider a less restrictive form of consistency of measures for investment and for losses. For instance we might require the equivalence of (5) and (6) for investments or losses that are ordered in *second stochastic dominance (or stop loss order)*.

Definition 4 (Kaas, van Heerwaarden, Goovaerts, [7], Ch. 3, Wang [13])
A risk X is smaller than a risk Y in second stochastic dominance (SSD) - ($X \prec_{SSD} Y$), if for all $x \geq 0$

$$\int_x^\infty \bar{F}_X(u) du \leq \int_x^\infty \bar{F}_Y(u) du, \quad (9)$$

where $\bar{F}_X(x) = 1 - F_X(x)$ is the tail of the distribution of X .

Definition 5 *Two measures $Rm_I(X)$ and $Rm_L(Y)$ are consistent in preference relations with respect to SSD if for $X \prec_{SSD} Y$ preferences (5) and (6) hold simultaneously.*

Let us show first that, generally speaking, measures constructed using the distortion function are not consistent in preference relations with respect to SSD.

Example 1 . *Let us take $g(t) = 1 - (1 - t)^m$, m -integer, $m > 1$, then $\tilde{g}(t) = t^m$. It is clear that $g(t)$ is concave and $\tilde{g}(t)$ is convex and*

$$Rm_I(X) = H_{\tilde{g}}(X) = \int_0^\infty \bar{F}_X(x)^m dx = E(\min(X_1, \dots, X_m)), \quad (10)$$

$$Rm_L(Y) = H_g(Y) = \int_0^\infty (1 - F_Y(x))^m dx = E(\max(Y_1, \dots, Y_m)), \quad (11)$$

where (X_1, \dots, X_m) are i.i.d with distribution $F_X(x)$, and (Y_1, \dots, Y_m) are i.i.d. with distribution $F_Y(x)$. Suppose that

$$\bar{F}_X(x) = I_{(-\infty, 0)} + (1 - x)I_{[0, 1]},$$

and

$$\bar{F}_Y(x) = I_{(-\infty, 1/4)} + (-4x + 2)I_{[1/4, 1/2]}$$

are two uniform distributions on $[0, 1]$ and $[1/4, 1/2]$ respectively. In this case, as a function of x ,

$$\int_x^1 (\bar{F}_Y(u) - \bar{F}_X(u)) du$$

increases from $1/8$ to $1/6$ on the interval $[0, 1/3]$ and for $1/3 < x \leq 1$ it decreases to 0. It is nonnegative on the whole interval $[0, 1]$. This means that $X \underset{SSD}{\prec} Y$. On the other hand using the formulae for expectation of extremal statistics of a uniform distribution on the interval $[a, b]$ (see, for example, David [1], Ch.3)

$$E(\min(X_1, \dots, X_m)) = a + \frac{1}{m+1}(b-a)$$

$$E(\max(X_1, \dots, X_m)) = b - \frac{1}{m+1}(b-a)$$

Then from (10) and (11) it follows that

$$Rm_I(X) = \frac{1}{m+1} < Rm_I(Y) = \frac{1}{4} + \frac{1}{4} \frac{1}{m+1}.$$

At the same time

$$Rm_L(X) = 1 - \frac{1}{m+1} > Rm_L(Y) = \frac{1}{2} - \frac{1}{4} \frac{1}{m+1}$$

which shows that Rm_I and Rm_L are not consistent with respect to SSD for the distortion measure.

Now consider risk measures Rm_I and Rm_L defined by (4), constructed by the change of measure approach proposed in this paper. We show that they can be consistent in preference relation with respect to SSD under the change of measure.

Definition 6 $X \underset{r-SSD}{\prec} Y$ in r -SSD, if for any $x \geq 0$

$$\int_x^\infty \overline{F_X(u)^r} du \leq \int_x^\infty \overline{F_Y(u)^r} du. \quad (12)$$

Then setting $x = 0$ we get from (4) that

$$Rm_L(X) \leq Rm_L(Y), \quad (13)$$

i.e

$$X \underset{Rm_L}{\prec} Y \quad (14)$$

From (12) it immediately follows that (see Wang (1998))

$$E_{P_{n,r}} u(-X) \geq E_{P_{n,r}} u(-Y)$$

Therefore

$$-X \underset{Rm_I}{\succ} -Y \quad (15)$$

If $u(x)$ is an odd function then from (15) we can write

$$X \underset{Rm_I}{\prec} Y \quad (16)$$

and then $Rm_I(X)$ and $Rm_L(Y)$ are consistent in preferences with respect to r -SSD. If u is not an odd function then we have only (14) and (15).

Notice that from Kaas, van Heerwaarden, Goovaerts, [7], Ch. 3, for some risk measures (14) is in fact the same as (15), and $Rm_I(X)$ and $Rm_L(Y)$ are then formally consistent.

Remark 1 *Second order stochastic dominance is also defined in the economic and finance literature. A risk Y is preferred to a risk X if (Huang et al [6] Chapter 2, Hadar and Russel [5]),*

$$\int_0^x F_Y(u) du \leq \int_0^x F_X(u) du \quad (17)$$

for all $x \geq 0$. If this holds then (17) is equivalent to

$$Eu(X) \leq Eu(Y)$$

for any increasing and concave utility function u i.e. for risk aversion. For the equivalent definition of r -SSD based on the approach suggested in this paper, we have (13) and (16) simultaneously.

Notice, that (9) and (17) are not equivalent. In fact, in Example 1 $F_X(x) = xI_{[0,1]} + I_{(1,\infty)}$ and $F_Y(x) = (4x - 1)I_{[1/4,1/2]} + I_{(1/2,\infty)}$. Then

$$\int_0^x (F_X(u) - F_Y(u))dy \quad (18)$$

as a function of x increases from 0 to $1/32$ for $0 \leq x \leq 1/4$, and decreases from $1/32$ to $-5/32$ for $1/4 < x \leq 1$. Thus there is a change in the sign of (18). Measures constructed using the distortion function are not consistent with respect to SSD as in (17). As an example consider X distributed uniformly on $[0, 1]$ and Y distributed uniformly on $[1/2, 3/4]$.

7 Conclusions

Insurance premiums should be derived from a risk measure where preferences have the properties of risk aversion and diversification. Under standard arbitrage-free and perfect markets assumptions, equilibrium premiums should be additive. We should also be able to apply the risk measure to both sides of the balance sheet and maintain a consistent ordering of risks in so doing.

A risk measure based on distorted probabilities, recently proposed as a premium principle in actuarial science, does not satisfy this additive property for equilibrium insurance premiums. The premiums based on distorted probabilities are in general sub-additive and only additive for comonotonic risks. Thus for a portfolio of different classes of insurance risk, the premiums for the different classes will not be consistent with equilibrium.

We propose a premium principle for a portfolio of insurance risks using a change of measure such that the premiums for the risks in the portfolio will always be additive. This premium principle will have the risk aversion property so that premiums will exceed the expected value of the losses. We also propose a certainty equivalent risk measure based on a concave utility function and the change of measure used for pricing that appears to order investment risks consistently with insurance risks.

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