

Pricing in the Multi-Line Insurer with Dependent Gamma Distributed Risks allowing for Frictional Costs of Capital *

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Abstract

This paper considers the pricing of insurance contracts for a multi-line insurer in a single period model where insurance risks have dependent gamma distributions. The pricing takes into account the impact of default, as well as financial distress costs arising from the possibility

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of default by the insurer, and allocates these to line of business in an economically meaningful manner. The costs of capital include frictional costs arising from writing insurance business and are assumed to be proportional to insurer end of period capital. In practice these must be incorporated into by-line pricing by also allocating them to line of business. We develop closed form expressions for prices and the allocation of default and frictional costs of capital to lines of business based on assumptions used in the application of standard option pricing models to insurer balance sheets. These closed form expressions can be used to implement the model in practice.

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1 Introduction

Pricing in the multi-line insurer needs to allow for the distribution of the insurance risks to be priced as well as the impact of insolvency of the insurer and the costs of capital. In order to price insurance risks it is necessary to allow for risk using a pricing principle or using financial economic theory. In a multi-line insurer, the capital of the insurer is available to support the payment of claims for all lines of business. However, limited liability means that the capital of the insurer will not be able to guarantee payment of all outstanding claims with certainty. Thus there will be a shortfall in the event of insolvency and the cost of this shortfall has to be allocated to lines of business in order to derive fair prices since these costs, although contingent, will have to be met by the policyholders.

The costs of capital include taxes, financial distress costs and agency costs. In order for the insurer to attract capital to support the writing of insurance, these prices will have to cover the additional costs of capital over and above those of the alternative marginal investment available to this capital. They will need to be allocated to line of business for fair pricing. Policyholder claims are collateralized by the premiums that they pay and the capital subscribed by the shareholders. However policyholders pay for the guarantee in the premium rates they are charged in equilibrium since the fair premium reflects the payoffs by line of business allowing for insolvency.

We use the risk capital approach to an insurer balance sheet as in Merton and Perold (1993) [8] and Myers and Read (2001) [10]. Our focus is on pricing in a multiline insurer rather than on allocation of costs of capital to lines of business which is the main focus of Myers and Read (2001) [10]. Pricing in the multi-line insurer is discussed in Phillips, Cummins and Allen (1998) [11] and Zanjani (2002) [15]. Sherris (2003) [12] discusses equilibrium insurance pricing in a mean-variance model and Sherris (2004) [13] considers the allocation of risk capital to line of business for the purposes of fair pricing. Closed form expressions for the allocation of the default option value of an insurer for use in fair pricing in a multi-line insurer are derived in Sherris and van der Hoek (2004) [14] on the assumption that claims for each line of business are log-normal and that the ratio of assets to liabilities is also log-normal. Furman and Landsman (2004) [5] consider capital allocation for a model of insurer liabilities that have dependent gamma distributions. See also Hürlimann (2001) [6].

In this paper we consider the pricing of insurance contracts in a risk

portfolio with dependent gamma distributions. This portfolio of risks can be regarded as consisting of different lines of business. Gamma distributions are used in practice to model insurance losses by line of business. The dependent gamma distribution model used has not been applied to model an insurer balance sheet previously. We use a single period model and derive closed form expressions for fair prices by line of business allowing for the insolvency risk of the insurer. We include assets as well as liabilities in our model formulation. We also show how costs of capital and financial distress costs can be allocated in this model to lines of business. We use approximations where necessary to derive the closed form expressions so that these pricing results can be implemented in practice. The results for the gamma distribution used to derive the closed form expressions and the pricing model results are new.

2 Model of the Multi-Line Insurer

We assume a single period model to derive our results. Since most non-life insurance contracts are annual contracts, this assumption, although not totally realistic, will capture the main financial aspects we are interested in modelling. The insurer writes a portfolio of n distinct insurable risks at the beginning of the period. These can be regarded as lines of business or as individual policies. We denote the end of period total claims for the i^{th} risk by L_i . Total claims at the end of the period will be $L = \sum_{i=1}^n L_i$. The insurer collects premiums at the start of the period for an amount P_i for line of business i . P_i allows for the expected losses, the risk loading, the risk of claims not being met due to the insolvency of the insurer and any costs of capital to be allocated to policyholders. Total premiums collected at the start of the period are $P = \sum_{i=1}^n P_i$.

Since we wish to allow for the impact of limited liability and costs of capital, we consider an insurer with shareholders who subscribe capital sufficient to ensure that the assets of the insurer will be a multiple of the value of the total policyholder liabilities ignoring the risk of insurer insolvency. Assets at the start of the period will then equal $V_0 = (1 + s) L_0$ where L_0 is the time 0 actuarial value of the end of period claims, $s > 0$. The value of assets at the end of the period will be denoted by V .

We will consider the costs of capital and their allocation to lines of business later. To begin with we assume the costs of capital are zero. Our model is arbitrage-free by assumption so that there exists a probability measure Q

such that the current values of the assets and liabilities are the discounted present value of end of period random payments using a risk free discount rate. We assume that there exists a risk free asset such that an investment of 1 now will return e^r at the end of the period for certain. Denote by L_{0i} the time 0 price or fair value for line of business i given by

$$L_{0i} = E^Q [e^{-r} L_i] \quad \text{for all } i = 1, \dots, n$$

and denote by V_0 the time 0 price or fair value for the assets given by

$$V_0 = E^Q [e^{-r} V]$$

where r is the risk-free continuous compounding rate of interest. We also denote the total value of the initial liabilities by $L_0 = \sum_{i=1}^n L_{0i}$. We assume complete markets so that we can observe L_{0i} for all $i = 1, \dots, n$, and V_0 from an asset market where these risks are traded. The model is based on market determined values for the parameters of the distributions for these risks as well as values for their prices of risk. This is a market consistent balance sheet approach.

To determine the fair insurance premium we must determine the payments that the policyholder will receive from the insurer and allow for the effect of limited liability in the event of insolvency. Payment of the total liability for claims will only be made if the insurer is solvent at the end of the time period. From Sherris (2004) [13] and Sherris and van der Hoek (2004) [14], the payment of claims for line of business i at the end of the period, assuming equal priority for losses by line of business in the event of insolvency, will be

$$\begin{aligned} & \frac{L_i}{L} V \quad \text{if } L > V \text{ (or } \frac{V}{L} \leq 1) \\ & L_i \quad \text{if } L \leq V \text{ (or } \frac{V}{L} > 1) \end{aligned}$$

or

$$L_i \left[1 - \left(1 - \frac{V}{L} \right)^+ \right] \tag{1}$$

The premium for line of business i will be given by

$$\begin{aligned}
P_i &= E^Q \left[e^{-r} L_i \left[1 - \left(1 - \frac{V}{L} \right)^+ \right] \right] \\
&= E^Q [e^{-r} L_i] - E^Q \left[e^{-r} L_i \left(1 - \frac{V}{L} \right)^+ \right] \\
&= L_{0i} - e^{-r} E^Q \left[L_i \left(1 - \frac{V}{L} \right)^+ \right] \\
&= L_{0i} - D_{0i}
\end{aligned} \tag{2}$$

where we denote the line of business default value for line i by D_{0i} .

An insurance policy can then be viewed as a contingent claim since if $\frac{V}{L} \leq 1$ the shareholders can exchange the assets of the company for their obligation to pay the outstanding claims. Thus policyholders have a contract for payment of claims provided the assets of the insurer exceed the total outstanding claims. Otherwise they receive their share of the assets based on the outstanding claims by line of business.

Denote the insurer default option value by D_0 so that

$$D_0 = e^{-r} \sum_{i=1}^n E^Q \left[L_i \left(1 - \frac{V}{L} \right)^+ \right] = \sum_{i=1}^n D_{0i}$$

Since

$$P = \sum_{i=1}^n P_i = L_0 - D_0$$

The initial value of capital of the insurer will be equal to

$$\begin{aligned}
C &= V_0 - P \\
&= (1 + s) L_0 - P \\
&= sL_0 + D_0
\end{aligned}$$

The beginning of period assets are

$$\begin{aligned}
V_0 &= (1 + s) L_0 \\
&= sL_0 + D_0 + L_0 - D_0 \\
&= C + P
\end{aligned}$$

with C contributed by shareholders in capital and P from policyholders in premiums.

3 The Distribution of Insurance and Asset Risks

An commonly used distribution assumption for insurance claims is the gamma distribution. We will develop a model of the underlying insurance and asset risks of the insurer such that they have gamma marginal distributions and are dependent. The distributional assumptions we use are for the Q measure probability distribution used for pricing. We follow the pricing approach developed in Sherris and van der Hoek (2004) [14] and use the techniques derived in Furman and Landsman (2004) [5] in order to determine fair prices for insurance premiums allowing for the default option value.

The model for dependent gamma risks is based on results for the distribution for the sum of independent gamma random variables derived in Mathai (1982) [7]. The model applies an approach developed in Moschopoulos (1985) [9]. These are further developed for dependent gamma random variables in Alouini, Abdi, and Kaveh (2001) [2].

Let $(X_0, \dots, X_n, X_{n+1})$ be $n + 2$ independent gamma distributed random variables with shape parameters γ_i and common rate parameter α , denoted by $G(\gamma_i, \alpha), i = 0, \dots, n + 1$. The probability density of the X_i is

$$f_{X_i}(x_i) = \frac{\alpha^{\gamma_i}}{\Gamma(\gamma_i)} e^{-\alpha x_i} x_i^{\gamma_i-1}, x_i > 0$$

Assume that the end of period claims for the n lines of business and the value of the assets V are given by

$$\begin{aligned} L_i &= \frac{\alpha_0}{\alpha} X_0 + X_i, \quad i = 1, \dots, n \\ V &= \frac{\alpha_0}{\alpha} X_0 + X_{n+1} \end{aligned} \tag{3}$$

The intuition for this model is that there is a common factor X_0 that impacts the values of each line of business claims as well as the assets. This could for example capture the effect of inflation which has an impact on insurance claims and asset values. For each line of business and for the assets there is a separate independent factor impacting the claims and asset payoffs denoted by $X_i, i = 1, \dots, n$ for the line of business i , and X_{n+1} for the assets.

Under these assumptions the claims for each line of business L_1, \dots, L_n are gamma distributed random variables, $G(\lambda_i, \alpha), i = 1, \dots, n$, where $\lambda_i = \gamma_0 + \gamma_i$, and the assets V are gamma distributed as $G(\gamma_0 + \gamma_{n+1}, \alpha)$, since

the sum of two independent gamma random variables with the same rate parameter is also gamma with shape parameter equal to the sum of the shape parameters.

The total claims liability at the end of the period is

$$L = \sum_{i=1}^n L_i = \frac{n\alpha_0}{\alpha} X_0 + X.,$$

where

$$X. = \sum_{i=1}^n X_i$$

We have that $X.$ is distributed $G(\gamma., \alpha)$, with $\gamma. = \sum_{i=1}^n \gamma_i$. However, the total claims liability is a sum of 2 gamma random variables, $\frac{n\alpha_0}{\alpha} X_0$ and $X.$, each with different rates and so the sum does not have a gamma distribution.

However, using Furman and Landsman (2004), Proposition 2 [5], we can represent L as a mixed gamma distribution with mixed shape parameter

$$L \sim G(\gamma_0 + \gamma. + \nu, \alpha),$$

where ν is a non negative integer random variable with probabilities

$$p_k = C\delta_k, \quad k \geq 0, \tag{4}$$

where

$$C = \frac{1}{n^{\gamma_0}}$$

$$\delta_k = k^{-1}\gamma_0 \sum_{i=1}^k \left(\frac{n-1}{n}\right)^i \delta_{k-i}, \quad k > 0,$$

$$\delta_0 = 1. \tag{5}$$

We will use this model to develop approximations in the form of closed form expressions for the by-line price for lines of business in a multi-line insurer that can be readily implemented.

A copula approach could also be used to model the dependence between assets and lines of business. Such an approach requires numerical techniques whereas our aim is to develop a closed form approximation that can be implemented in practice.

4 The Q Measure

Under the Q measure the expected value of claims discounted at the risk free rate is equal to the price or fair value assuming claims are default free. The variance and covariances between lines of business and assets are estimated from historical claims and asset price data.

For the model of claims and assets that we have assumed, the Q measure probability density of claims for line of business i is Gamma $G(\lambda_i, \alpha)$, $i = 1, \dots, n$ so that

$$f_{L_i}(y_i) = \frac{\alpha^{\lambda_i}}{\Gamma(\lambda_i)} e^{-\alpha y_i} y_i^{\lambda_i-1}, y_i > 0,$$

with

$$E^Q[L_i] = \frac{\lambda_i}{\alpha}$$

and

$$Var^Q[L_i] = \frac{\lambda_i}{\alpha^2}$$

In passing we also note that

$$Cov(L_i, L_j) = \frac{\gamma_0}{\alpha^2} \quad i \neq j$$

$$Cov(L_i, V) = \frac{\gamma_0}{\alpha^2}, \quad i = 1, \dots, n$$

and

$$\rho(L_i, L_j) = \frac{\gamma_0}{\sqrt{\lambda_i \lambda_j}}, \quad i, j = 1, \dots, n$$

Now under Q we have the fair value of the liability for line of business i given by

$$L_{0i} = E^Q[e^{-r}L_i] \quad \text{for all } i = 1, \dots, n$$

and similarly for the assets

$$V_0 = E^Q[e^{-r}V]$$

So

$$e^r L_{0i} = E^Q[L_i] = \frac{\lambda_i}{\alpha}$$

There are $n+4$ parameters that specify the Q measure, γ_i for $i = 0, \dots, n+1$, α_0 and α . In practice these are estimated from the market price data and

historical data on losses. We have assumed that the α parameters are the same for the gamma random variates used to construct our model of asset and liabilities. This means that the covariances are the same for all lines of business although the correlations differ by line. This assumption may be unrealistic in practice and it is possible to assume that the α 's are different. This will make the resulting pricing expressions more complex but adds little to the final results.

If we consider the continuous compounding discount rate for determining the fair value of liabilities as the risk free rate plus a margin, $r + l_i$ then

$$\begin{aligned} L_{0i} &= E^Q [e^{-r} L_i] \quad \text{for all } i = 1, \dots, n \\ &= E^P [e^{-(r+l_i)} L_i] \end{aligned}$$

and therefore

$$\begin{aligned} l_i &= \ln \left(\frac{E^P [L_i]}{E^Q [L_i]} \right) \\ &= \ln \left(\frac{\alpha}{\lambda_i} E^P [L_i] \right) \end{aligned}$$

This margin does not include any allowance for the (frictional) costs of capital. The inclusion of these frictional costs is considered later in the paper.

5 Default Option Value By Line of Business

In order to price by line of business we need to determine the default option value for each line. This is given by

$$\begin{aligned} D_{0i} &= e^{-r} E^Q \left[L_i \left(1 - \frac{V}{L} \right)^+ \right] \\ &= e^{-r} E^Q \left[\left(\frac{\alpha_0}{\alpha} X_0 + X_i \right) \left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right] \\ &= e^{-r} \frac{\alpha_0}{\alpha} E^Q \left[X_0 \left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right] \\ &\quad + e^{-r} E^Q \left[X_i \left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right], i = 1, \dots, n \end{aligned} \quad (6)$$

where $Q \sim \prod_{i=0}^{n+1} G(\gamma_i, \alpha)$.

Consider the expectation in the first term in (6).

$$\begin{aligned}
& E^Q \left[X_0 \left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right] \\
&= \int_0^\infty \dots \int_0^\infty x_0 \left(1 - \frac{\frac{\alpha_0}{\alpha} x_0 + x_{n+1}}{\frac{n\alpha_0}{\alpha} x_0 + x} \right)^+ dQ \\
&= \frac{\Gamma(\gamma_0 + 1)}{\alpha \Gamma(\gamma_0)} \int_0^\infty \dots \int_0^\infty \left(1 - \frac{\frac{\alpha_0}{\alpha} x_0 + x_{n+1}}{\frac{n\alpha_0}{\alpha} x_0 + x} \right)^+ dQ^0 \\
&= \frac{\gamma_0}{\alpha} E^{Q_0} \left[\left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right] \tag{7}
\end{aligned}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

$$\begin{aligned}
dQ &= f(x_0, \dots, x_{n+1}) dx_0 \dots dx_{n+1} \\
&= \prod_{i=0}^{n+1} \left(\frac{\alpha^{\gamma_i}}{\Gamma(\gamma_i)} e^{-\alpha x_i} x_i^{\gamma_i-1} \right) dx_0 \dots dx_{n+1}
\end{aligned}$$

and

$$dQ^0 = \left(\frac{\alpha^{\gamma_0+1}}{\Gamma(\gamma_0+1)} e^{-\alpha x_0} x_0^{\gamma_0} \right) \prod_{i=1}^{n+1} \left(\frac{\alpha^{\gamma_i}}{\Gamma(\gamma_i)} e^{-\alpha x_i} x_i^{\gamma_i-1} \alpha^{\gamma_i} \right) dx_0 \dots dx_{n+1}$$

Similarly for the expectation in the second term in (6). We can write

$$\begin{aligned}
& E^Q \left[X_i \left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right] \\
&= \frac{\gamma_i}{\alpha} E^{Q_i} \left[\left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right] \tag{8}
\end{aligned}$$

where

$$\begin{aligned}
dQ^i &= \left(\frac{\alpha^{\gamma_i+1}}{\Gamma(\gamma_i+1)} e^{-\alpha x_i} x_i^{\gamma_i} \alpha^{\gamma_i+1} \right) \prod_{j=0, j \neq i}^{n+1} \left(\frac{\alpha^{\gamma_j}}{\Gamma(\gamma_j)} e^{-\alpha x_j} x_j^{\gamma_j-1} \right) dx_0 \dots dx_{n+1} \\
i &= 1, \dots, n.
\end{aligned}$$

Thus $Q_i \sim G(\gamma_i + 1, \alpha) \prod_{j=0, j \neq i}^{n+1} G(\gamma_j, \alpha)$, $i = 1, \dots, n$.

In order to evaluate the default option value we must evaluate the expression

$$E^{Q_i} \left[\left(1 - \frac{\frac{\alpha_0}{\alpha} X_0 + X_{n+1}}{\frac{n\alpha_0}{\alpha} X_0 + X} \right)^+ \right], \quad i = 1, \dots, n$$

Note that under Q_0

$$V = \frac{\alpha_0}{\alpha} X_0 + X_{n+1} \sim G(\gamma_0 + 1 + \gamma_{n+1}, \alpha)$$

and under Q_i , $i = 1, \dots, n$

$$V = \frac{\alpha_0}{\alpha} X_0 + X_{n+1} \sim G(\gamma_0 + \gamma_{n+1}, \alpha)$$

We know that under Q_0 and Q_i , $i = 1, \dots, n$, the distribution of $\frac{n\alpha_0}{\alpha} X_0 + X$ will not be Gamma but its distribution can be represented as a mixture of Gamma random variables, so that again using Furman and Landsman (2004) Proposition 2, we have under Q_0

$$L = \frac{n\alpha_0}{\alpha} X_0 + X. \sim G(\gamma_0 + 1 + \gamma. + \tilde{\nu}, \alpha)$$

where $\tilde{\nu}$ is a non-negative integer random variable defined in

$$\tilde{p}_k = \tilde{C} \tilde{\delta}_k, \quad k \geq 0,$$

$$\tilde{C} = \frac{1}{n^{\gamma_0+1}}$$

$$\tilde{\delta}_k = k^{-1}(\gamma_0 + 1) \sum_{i=1}^k \left(\frac{n-1}{n} \right)^i \tilde{\delta}_{k-i}, \quad k > 0,$$

$$\tilde{\delta}_0 = 1.$$

Under Q_i , $i = 1, \dots, n$,

$$L = \frac{n\alpha_0}{\alpha} X_0 + X. \sim G(\gamma_0 + 1 + \gamma. + \nu, \alpha),$$

where ν is a non-negative integer random variable defined by (4) and (5).

In order to determine a closed form expression to approximate the default option value allocated to line of business i , we will determine expressions for the moments of $\ln \Lambda = \ln V - \ln L$.

Lemma 1 *If $X \sim \text{Gamma}(\gamma, \alpha)$ then $E[\ln X] = \psi(\gamma) - \ln \alpha$ and $\text{Var}[\ln X] = \frac{\Gamma''(\gamma)}{\Gamma(\gamma)} - \psi(\gamma)^2$ where $\psi(\gamma) = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)} = \frac{d}{d\gamma} \ln \Gamma(\gamma)$ is the Psi (Digamma) function, $\Gamma'(\gamma) = \frac{d}{d\gamma} \Gamma(\gamma)$, and $\psi'(\gamma) = \frac{d}{d\gamma} \psi(\gamma)$ is the Trigamma function (refer Abramowitz and Stegun, 1972 [1]).*

Proof. We have

$$\begin{aligned}
E[\ln X] &= \frac{\alpha^\gamma}{\Gamma(\gamma)} \int_0^\infty \ln x e^{-\alpha x} x^{\gamma-1} dx & (9) \\
&= \int_0^\infty \ln\left(\frac{y}{\alpha}\right) \frac{1}{\Gamma(\gamma)} e^{-y} y^{\gamma-1} dy \\
&= \frac{1}{\Gamma(\gamma)} \int_0^\infty \ln(y) e^{-y} y^{\gamma-1} dy - \ln \alpha \\
&= \frac{\Gamma'(\gamma)}{\Gamma(\gamma)} - \ln \alpha \\
&= \psi(\gamma) - \ln \alpha & (10)
\end{aligned}$$

where the second line follows from the change of variable $y = \alpha x$ and the last line by noting that

$$\begin{aligned}
\Gamma'(\gamma) &= \frac{d}{d\gamma} \int_0^\infty e^{-y} y^{\gamma-1} dy \\
&= \int_0^\infty \ln(y) e^{-y} y^{\gamma-1} dy
\end{aligned}$$

For the variance we have

$$\text{Var}[\ln X] = E[(\ln X)^2] - E[\ln X]^2$$

and

$$\begin{aligned}
E[(\ln X)^2] &= \int_0^\infty (\ln x)^2 \frac{1}{\Gamma(\gamma)} e^{-\alpha x} x^{\gamma-1} \alpha^\gamma dx \\
&= \frac{1}{\Gamma(\gamma)} \left[\int_0^\infty \ln(y)^2 e^{-y} y^{\gamma-1} dy - \int_0^\infty 2 \ln(y) \ln \alpha e^{-y} y^{\gamma-1} dy \right] + \ln \alpha^2 \\
&= \frac{\Gamma''(\gamma)}{\Gamma(\gamma)} - 2\psi(\gamma) \ln \alpha + (\ln \alpha)^2 & (11)
\end{aligned}$$

Hence

$$\begin{aligned}
Var [\ln X] &= E [(\ln X)^2] - E [\ln X]^2 \\
&= \frac{\Gamma''(\gamma)}{\Gamma(\gamma)} - 2\psi(\gamma) \ln \alpha + (\ln \alpha)^2 - [\psi(\gamma) - \ln \alpha]^2 \\
&= \frac{\Gamma''(\gamma)}{\Gamma(\gamma)} - 2\psi(\gamma) \ln \alpha + (\ln \alpha)^2 - [\psi(\gamma)^2 - 2\psi(\gamma) \ln \alpha + (\ln \alpha)^2] \\
&= \frac{\Gamma''(\gamma)}{\Gamma(\gamma)} - \psi(\gamma)^2 \tag{12}
\end{aligned}$$

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Define the function

$$\psi(\gamma; a) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \ln(a+x)x^{\gamma-1} \exp(-x)dx.$$

where $\psi(\gamma; 0)$ is simply the digamma function $\psi(\gamma)$.

Lemma 2 *If $X \sim \text{Gamma}(\gamma_X, \alpha)$, $Y \sim \text{Gamma}(\gamma_Y, \alpha)$, $Z \sim \text{Gamma}(\gamma_Z, \alpha)$ are three independent gamma variables, then the covariance of the dependent random variables $U = \ln(\beta X + Y)$, $\beta > 0$ and $W = \ln(X + Z)$ takes the form*

$$cov(U, W) = E[\psi(\gamma_Y; \alpha\beta X)\psi(\gamma_Z; \alpha X)] - E\psi(\gamma_Y; \alpha\beta X)E\psi(\gamma_Z; \alpha X), \tag{13}$$

and is approximated by

$$cov(U, W) = \frac{\beta\gamma_X}{(\gamma_Y - 1)(\gamma_Z - 1)} + o\left(\frac{\psi(\gamma_Z)}{\gamma_Y - 1}\right) + o\left(\frac{\psi(\gamma_Y)}{\gamma_Z - 1}\right), \quad \gamma_Y, \gamma_Z \rightarrow \infty. \tag{14}$$

Proof. Notice that conditional on $X = x$ the random variables U and W are independent. From (9) we have

$$\begin{aligned}
E(U \cdot W | X = x) &= E(U | X = x)E(W | X = x) \\
&= \frac{\alpha^{\gamma_Y}}{\Gamma(\gamma_Y)} \int_0^\infty \ln(\beta x + y) e^{-\alpha y} y^{\gamma_Y-1} dy \\
&\quad \times \frac{\alpha^{\gamma_Z}}{\Gamma(\gamma_Z)} \int_0^\infty \ln(x + z) e^{-\alpha z} z^{\gamma_Z-1} dz \\
&= \psi(\gamma_Y; \alpha\beta x)\psi(\gamma_Z; \alpha x),
\end{aligned}$$

and applying the expression for covariance gives (13). Now, using a Taylor expansion for the function $\psi(\gamma; a)$ with respect to a we get

$$\psi(\gamma_Z; \alpha x) = \psi(\gamma_Z) + \frac{1}{(\gamma_Z - 1)}\alpha x + \frac{1}{2}\alpha^2 x^2 R_Z, \quad (15)$$

where

$$\begin{aligned} |R_Z| &= \frac{1}{\Gamma(\gamma_Z)} \int_0^\infty \frac{1}{(\theta + z)^2} e^{-z} z^{\gamma_Z - 1} dz \leq \frac{1}{\Gamma(\gamma_Z)} \int_0^\infty e^{-z} z^{\gamma_Z - 3} dz \\ &= \frac{1}{(\gamma_Z - 1)(\gamma_Z - 2)}, \gamma_Z > 2, \end{aligned} \quad (16)$$

and $\theta \in [0, \alpha x]$ and $\alpha x > 0$. Similarly

$$\psi(\gamma_Y; \alpha \beta x) = \psi(\gamma_Y) + \frac{1}{(\gamma_Y - 1)}\alpha \beta x + \frac{1}{2}\beta^2 \alpha^2 x^2 R_Y, \quad (17)$$

where

$$|R_Y| \leq \frac{1}{(\gamma_Y - 1)(\gamma_Y - 2)}, \gamma_Y > 2. \quad (18)$$

From (15) and (17) it follows that

$$\begin{aligned} E\psi(\gamma_Y; \alpha \beta X) &= \psi(\gamma_Y) + \frac{\beta \gamma_X}{(\gamma_Y - 1)} + O\left(\frac{1}{\gamma_Y^2}\right), \gamma_Y \rightarrow \infty, \\ E\psi(\gamma_Z; \alpha X) &= \psi(\gamma_Z) + \frac{\gamma_X}{(\gamma_Z - 1)} + O\left(\frac{1}{\gamma_Z^2}\right), \gamma_Z \rightarrow \infty, \end{aligned}$$

and therefore

$$\begin{aligned} E\psi(\gamma_Y; \alpha \beta X)E\psi(\gamma_Z; \alpha X) &= \psi(\gamma_Y)\psi(\gamma_Z) + \frac{\gamma_X \psi(\gamma_Y)}{(\gamma_Z - 1)} \\ &\quad + \frac{\beta \gamma_X \psi(\gamma_Z)}{(\gamma_Y - 1)} + \frac{\beta \gamma_X^2}{(\gamma_Y - 1)(\gamma_Z - 1)} \\ &\quad + o\left(\frac{\psi(\gamma_Z)}{\gamma_Y - 1}\right) + o\left(\frac{\psi(\gamma_Y)}{\gamma_Z - 1}\right). \end{aligned} \quad (19)$$

On the other hand

$$\begin{aligned}
\psi(\gamma_Y; \alpha\beta x)\psi(\gamma_Z; \alpha x) &= \psi(\gamma_Y)\psi(\gamma_Z) + \frac{\beta\alpha x\psi(\gamma_Z)}{(\gamma_Y - 1)} + \frac{\alpha x\psi(\gamma_Y)}{(\gamma_Z - 1)} \\
&\quad + \frac{\beta\alpha^2 x^2}{(\gamma_Y - 1)(\gamma_Z - 1)} + R, \tag{20}
\end{aligned}$$

where

$$\begin{aligned}
R &= \frac{1}{2}\psi(\gamma_Z)\beta^2\alpha^2 x^2 R_Y + \frac{1}{2}\frac{1}{(\gamma_Z - 1)}\beta^2\alpha^3 x^3 R_Y \\
&\quad + \frac{1}{2}\psi(\gamma_Y)\alpha^2 x^2 R_Z + \frac{1}{2}\frac{1}{(\gamma_Y - 1)}\beta\alpha^3 x^3 R_Z + \frac{1}{4}\beta^2\alpha^4 x^4 R_Y R_Z \tag{21}
\end{aligned}$$

Then from (20) and (21) we have, taking into account (16) and (18),

$$\begin{aligned}
E[\psi(\gamma_Y; \alpha\beta X)\psi(\gamma_Z; \alpha X)] &= \psi(\gamma_Y)\psi(\gamma_Z) + \frac{\beta\gamma_X\psi(\gamma_Z)}{(\gamma_Y - 1)} + \frac{\gamma_X\psi(\gamma_Y)}{(\gamma_Z - 1)} \\
&\quad + \frac{\beta\gamma_X(\gamma_X + 1)}{(\gamma_Y - 1)(\gamma_Z - 1)} + o\left(\frac{\psi(\gamma_Z)}{\gamma_Y - 1}\right) + o\left(\frac{\psi(\gamma_Y)}{\gamma_Z - 1}\right). \tag{22}
\end{aligned}$$

Substituting (19) and (22) into (13) we get (14). ■

From Lemma 2 it follows that we can use the following approximate formula for the covariance between U and W for large γ_Y, γ_Z

$$\text{cov}(U, W) \approx \frac{\beta\gamma_X}{(\gamma_Y - 1)(\gamma_Z - 1)}. \tag{23}$$

Using Lemma 1 we have that under Q_0

$$\begin{aligned}
\mu_\Lambda^0 &= E^{Q_0}[\ln \Lambda] \\
&= E^{Q_0}[\ln V] - E^{Q_0}[\ln L] \\
&= \psi(\gamma_0 + \gamma_{n+1} + 1) - \left(\sum_{k=0}^{\infty} \psi(\gamma_0 + 1 + \gamma. + k) \tilde{p}_k \right) \tag{24}
\end{aligned}$$

and under $Q_i, i = 1, \dots, n$

$$\begin{aligned}
\mu_\Lambda^i &= E^{Q_i}[\ln \Lambda] \\
&= E^{Q_i}[\ln V] - E^{Q_i}[\ln L] \\
&= \psi(\gamma_0 + \gamma_{n+1}) - \left(\sum_{k=0}^{\infty} \psi(\gamma_0 + 1 + \gamma. + k) p_k \right) \tag{25}
\end{aligned}$$

Also under Q_0

$$\begin{aligned}
\sigma_{\Lambda^0}^2 &= Var^{Q_0} [\ln \Lambda] \\
&= \frac{\Gamma''(\gamma_0 + \gamma_{n+1} + 1)}{\Gamma(\gamma_0 + \gamma_{n+1} + 1)} - \psi(\gamma_0 + \gamma_{n+1} + 1)^2 \\
&\quad + \sum_{k=0}^{\infty} \left[\frac{\Gamma''(\gamma_0 + 1 + \gamma. + k)}{\Gamma(\gamma_0 + 1 + \gamma. + k)} - \psi(\gamma_0 + 1 + \gamma. + k)^2 \right] \tilde{p}_k \\
&\quad - 2Cov^{Q_0} [\ln V, \ln L] \tag{26}
\end{aligned}$$

From Lemma 2, putting $\beta = n$, $X = \frac{\alpha_0}{\alpha} X_0 \smile G(\gamma_0 + 1, \alpha)$, $Y = X. \smile G(\gamma., \alpha)$, $Z = X_{n+1} \smile G(\gamma_{n+1}, \alpha)$, gives

$$Cov^{Q_0} [\ln V, \ln L] = E[\psi(\gamma.; n\alpha_0 X_0)\psi(\gamma_{n+1}; \alpha_0 X_0)] - E\psi(\gamma.; n\alpha_0 X_0)E\psi(\gamma_{n+1}; \alpha_0 X_0)$$

and from the approximation formula (23) we have for large $\gamma.$ and γ_{n+1}

$$Cov^{Q_0} [\ln V, \ln L] \approx \frac{n(\gamma_0 + 1)}{(\gamma. - 1)(\gamma_{n+1} - 1)}.$$

Under Q_i $i = 1, \dots, n$, we have

$$\begin{aligned}
\sigma_{\Lambda^i}^2 &= Var^{Q_i} [\ln \Lambda] \\
&= \frac{\Gamma''(\gamma_0 + \gamma_{n+1})}{\Gamma(\gamma_0 + \gamma_{n+1})} - \psi(\gamma_0 + \gamma_{n+1})^2 \\
&\quad + \sum_{k=0}^{\infty} \left[\frac{\Gamma''(\gamma_0 + 1 + \gamma. + k)}{\Gamma(\gamma_0 + 1 + \gamma. + k)} - \psi(\gamma_0 + 1 + \gamma. + k)^2 \right] p_k \\
&\quad - 2Cov^{Q_i} [\ln V, \ln L], i = 1, \dots, n. \tag{27}
\end{aligned}$$

With $X = \frac{\alpha_0}{\alpha} X_0 \smile G(\gamma_0, \alpha)$, $Y = X. \smile G(\gamma. + 1, \alpha)$ we have from Lemma 2 that

$$Cov^{Q_i} [\ln V, \ln L] = E[\psi(\gamma.+1; n\alpha_0 X_0)\psi(\gamma_{n+1}; \alpha_0 X_0)] - E\psi(\gamma.+1; n\alpha_0 X_0)E\psi(\gamma_{n+1}; \alpha_0 X_0)$$

or using the approximation gives

$$Cov^{Q_i} [\ln V, \ln L] \approx \frac{n\gamma_0}{\gamma.(\gamma_{n+1} - 1)}, i = 1, \dots, n.$$

Having determined the mean and variance of the log ratio based on the underlying Gamma distributions, we follow Sherris and van der Hoek (2004) [14], and develop an approximation by assuming that $\frac{V}{L}$ is lognormal with the mean and variance derived for the log ratio. We will then be able to determine an approximate closed form expression for the by line default option value. The accuracy of this approximation is based on our assumption that the lognormal distribution for the ratio of the assets to the liabilities is a good practical approximation for our Gamma distribution assumptions for lines of business and asset values. In practice the log-normal assumption made here is an often used assumption and is the basis for the application of standard option pricing models to insurer balance sheets.

We assume that $\Lambda = \frac{V}{L}$ is lognormal with

$$\ln \Lambda = \ln V - \ln L \sim Normal(\mu_\Lambda, \sigma_\Lambda^2) \quad (28)$$

where

$$\mu_\Lambda = E[\ln \Lambda] = E[\ln V] - E[\ln L]$$

and

$$\sigma_\Lambda^2 = Var[\ln \Lambda] = Var[\ln V] + Var[\ln L] - 2Cov[\ln V, \ln L]$$

under the relevant distributions for Q^0 and Q^i were obtained before.

Lemma 3 *If Λ is lognormal with*

$$\ln \Lambda = \ln V - \ln L \sim Normal(\mu_\Lambda, \sigma_\Lambda^2)$$

then

$$\begin{aligned} E[(1 - \Lambda)^+] &= E[(1 - \Lambda) I_{(1 - \Lambda > 0)}] \\ &= \Phi(0) - e^{\mu_\Lambda + \frac{1}{2}\sigma_\Lambda^2} \Phi\left(\frac{-(\mu_\Lambda + \sigma_\Lambda^2)}{\sigma_\Lambda}\right) \\ &= \frac{1}{2} - e^{\mu_\Lambda + \frac{1}{2}\sigma_\Lambda^2} \Phi\left(\frac{-(\mu_\Lambda + \sigma_\Lambda^2)}{\sigma_\Lambda}\right) \end{aligned}$$

where $I_{(1 - \Lambda > 0)} = 1$ if $1 - \Lambda > 0$ and 0 otherwise.

Proof. Follows from standard properties of the normal distribution. ■

The value of the default option value for line of business $i \in \{1, \dots, n\}$ is

$$e^{-r} \frac{\alpha_0 \gamma_0}{\alpha^2} E^{Q_0} \left[\left(1 - \frac{V}{L} \right)^+ \right] \\ + e^{-r} \frac{\gamma_i}{\alpha} E^{Q_i} \left(1 - \frac{V}{L} \right)^+,$$

and under our log normal approximating assumption it is given by

$$D_{0i} = e^{-r} \frac{\alpha_0 \gamma_0}{\alpha^2} \left[\frac{1}{2} - e^{\mu_{\Lambda^0} + \frac{1}{2} \sigma_{\Lambda^0}^2} \Phi \left(\frac{-(\mu_{\Lambda^0} + \sigma_{\Lambda^0}^2)}{\sigma_{\Lambda^0}} \right) \right] \\ + e^{-r} \frac{\gamma_i}{\alpha} \left[\frac{1}{2} - e^{\mu_{\Lambda^i} + \frac{1}{2} \sigma_{\Lambda^i}^2} \Phi \left(\frac{-(\mu_{\Lambda^i} + \sigma_{\Lambda^i}^2)}{\sigma_{\Lambda^i}} \right) \right], i = 1, \dots, n. \quad (29)$$

These expressions provide closed form expressions for computation of by-line prices based on our dependent gamma distribution assumptions and log-normal approximations that are commonly used in the application of standard option pricing models to insurer balance sheets.

6 Frictional Costs of Capital and Allocation to Lines of Business

There are various costs of capital that by-line pricing should include arising from the frictional costs associated with capital. These include the transactions costs of raising the capital, any additional insurer tax costs, agency costs as well as financial distress costs. We adopt the assumptions used in Estrella (2004) [4] for costs of capital and show how these can be allocated to line of business. We assume that total costs are determined based on the insurer balance sheet and we then allocate these to lines of business based on the allocation of the default option cost. This approach takes into account the total balance sheet frictional costs rather than the marginal approach of Myers and Read (2001) [10]. Since these costs are assumed to be contingent on the total balance sheet, this approach is considered realistic for practical application of the model.

Frictional costs such as those arising from transactions costs and other costs such as additional taxation and agency costs are assumed to occur as

a percentage of the end of period surplus and are only incurred provided the insurer is solvent. If we denote the value of these as C_c then

$$C_c = E^Q [c_c e^{-r} (V - L)^+]$$

where c_c is the costs of capital as a percentage of the surplus, provided it is positive. We then have

$$\begin{aligned} C_c &= E^Q \left[c_c e^{-r} L \left(\frac{V}{L} - 1 \right)^+ \right] \\ &= \sum_{i=1}^n E^Q [c_c e^{-r} L_i (\Lambda - 1)^+] \end{aligned}$$

If we denote the costs of capital allocated to line of business $i \in \{1, \dots, n\}$ by C_c^i we then have

$$\begin{aligned} C_c^i &= c_c e^{-r} E^Q [L_i (\Lambda - 1)^+] \\ &= c_c e^{-r} E^Q [L_i [(1 - \Lambda)^+ - (1 - \Lambda)]] \\ &= c_c D_{0i} - c_c e^{-r} E^Q [L_i (1 - \Lambda)] \\ &= c_c D_{0i} - c_c L_{\Lambda i} \end{aligned} \tag{30}$$

Consider the last term in this expression. Changing measure in the manner of (7) and (8) we get

$$\begin{aligned} L_{\Lambda i} &= e^{-r} E^Q [L_i (1 - \Lambda)] \\ &= e^{-r} \left[\frac{\gamma_0}{\alpha} E^{Q_0} (1 - \Lambda) + \frac{\gamma_i}{\alpha} E^{Q_i} (1 - \Lambda) \right] \\ &= e^{-r} \left[\frac{\gamma_0}{\alpha} (1 - E^{Q_0} \Lambda) + \frac{\gamma_i}{\alpha} (1 - E^{Q_i} \Lambda) \right], i = 1, \dots, n. \end{aligned}$$

Under assumption (28) we have

$$E^{Q_i} [\Lambda] = \exp(\mu_{\Lambda}^i + \frac{1}{2} \sigma_{\Lambda^i}^2), i = 0, 1, \dots, n,$$

where $\mu_{\Lambda}^i, \sigma_{\Lambda^i}^2, i = 0, 1, \dots, n$, were derived in formulas (24), (25), (26), and (27).

We derive an approximation for the expectation of Λ so that the assumption (28) of log normality of Λ is not required. Define the function

$$\phi(\gamma; a) = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} \frac{1}{a+x} x^{\gamma-1} \exp(-x) dx, \gamma > 1$$

so that $\phi(\gamma; 0) = \frac{1}{\gamma-1}$.

Lemma 4 *If $X \sim \text{Gamma}(\gamma_X, \alpha)$, $Y \sim \text{Gamma}(\gamma_Y, \alpha)$, $Z \sim \text{Gamma}(\gamma_Z, \alpha)$ are three independent gamma variables, then for $\beta > 0$,*

$$\begin{aligned} E\left(\frac{X+Z}{\beta X+Y}\right) &= (\gamma_X + \gamma_Z)E\phi(\gamma_Y; \beta\alpha X) \\ &= (\gamma_X + \gamma_Z)\left(\frac{1}{\gamma_Y - 1} - \frac{\beta\gamma_X}{(\gamma_Y - 1)(\gamma_Y - 2)} + o\left(\frac{\beta\gamma_X}{(\gamma_Y - 1)(\gamma_Y - 2)}\right)\right), \\ \gamma_Y &\rightarrow \infty. \end{aligned} \quad (31)$$

Proof. Conditional on $X = x$, the nominator and denominator in (31) are independent so we can write

$$\begin{aligned} E\left(\frac{X+Z}{\beta X+Y} \middle| X = x\right) &= E(X+Z|X=x)E\left(\frac{1}{\beta X+Y} \middle| X=x\right) \\ &= \alpha\left(x + \frac{\gamma_Z}{\alpha}\right)\phi(\gamma; \alpha\beta x). \end{aligned}$$

Then the full expectation has the form

$$E\left(\frac{X+Y}{\beta X+Z}\right) = (\gamma_X + \gamma_Z)E\phi(\gamma_Y; \beta\alpha X).$$

Using a Taylor expansion for the function $\phi(\gamma; a)$ with respect to a we get

$$\phi(\gamma_Y; \beta\alpha x) = \frac{1}{\gamma_Y - 1} - \beta\alpha x \frac{1}{(\gamma_Y - 1)(\gamma_Y - 2)} + \frac{1}{2}\beta^2\alpha^2 x^2 R_Y,$$

where

$$\begin{aligned} 0 < R_Y &= \frac{2}{\Gamma(\gamma_Y)} \int_0^\infty \frac{1}{(\theta + y)^3} e^{-y} y^{\gamma_Y - 1} dy \leq \frac{2}{\Gamma(\gamma_Z)} \int_0^\infty e^{-y} z^{\gamma_Y - 4} dz \\ &= \frac{2}{(\gamma_Y - 1)(\gamma_Y - 2)(\gamma_Y - 3)}, \quad \gamma_Y > 3, \end{aligned} \quad (32)$$

where $\theta \in [0, \alpha\beta x]$ and $\alpha\beta x > 0$. Then

$$\begin{aligned} E\left(\frac{X+Z}{\beta X+Y}\right) &= (\gamma_X + \gamma_Z)\left(\frac{1}{\gamma_Y - 1} - \frac{\beta\alpha}{(\gamma_Y - 1)(\gamma_Y - 2)} EX + \frac{1}{2}\beta^2\alpha^2 E(X^2 R_Y)\right) \\ &= (\gamma_X + \gamma_Z)\left(\frac{1}{\gamma_Y - 1} - \frac{\beta\gamma_X}{(\gamma_Y - 1)(\gamma_Y - 2)} + o\left(\frac{\beta\gamma_X}{(\gamma_Y - 1)(\gamma_Y - 2)}\right)\right), \\ \gamma_Y &\rightarrow \infty, \end{aligned}$$

because from (32) one obtains

$$\frac{1}{2}\beta^2 a^2 E(X^2 R_Y) \leq \frac{\beta^2 \gamma_X (\gamma_X + 1)}{(\gamma_Y - 1)(\gamma_Y - 2)(\gamma_Y - 3)}$$

■

Using Lemma 4 and the appropriate expressions for X, Y and Z given after Lemma 2 we immediately obtain

$$E^{Q_0}[\Lambda] \approx (\gamma_0 + 1 + \gamma_{n+1}) \left(\frac{1}{\gamma. - 1} - \frac{n(\gamma_0 + 1)}{(\gamma. - 1)(\gamma. - 2)} \right),$$

and

$$E^{Q_i}[\Lambda] \approx (\gamma_0 + \gamma_{n+1}) \left(\frac{1}{\gamma.} - \frac{n(\gamma_0 + 1)}{\gamma.(\gamma. - 1)} \right), \quad i = 1, \dots, n$$

In practice for an insurer balance sheet of reasonable size $\gamma. = \sum_{i=1}^n \gamma_i$ would be expected to be large so that the approximations will be reasonable for practical applications.

Following Estrella (2004) [4], we assume that financial distress costs are a percentage of the absolute value of the shortfall of assets over liabilities in the event of insolvency. The value of these costs will then be related to the default option value by line of business. Denoting the value of these financial distress costs by C_f we have

$$\begin{aligned} C_f &= E^Q [c_f e^{-r} (L - V)^+] \\ &= E^Q [c_f e^{-r} L (1 - \Lambda)^+] \end{aligned}$$

where c_f is the financial distress costs as a percentage of the asset shortfall in the event of insolvency.

We can determine the allocation of these to line of business i as

$$\begin{aligned} C_f^i &= c_f e^{-r} E^Q [L_i (1 - \Lambda)^+] \\ &= c_f D_{0i} \end{aligned} \tag{33}$$

where C_f^i is the allocation of the financial distress costs to line of business i .

An expression for the price by line of business, including the frictional costs of capital, for our multi-line insurer is given by

$$\begin{aligned} P_i + C_c^i + C_f^i &= e^{-r} \frac{\lambda_i}{\alpha} - c_c L_{\Lambda i} - (1 - c_f - c_c) D_{0i} \\ &= e^{-r} \frac{\lambda_i}{\alpha} - c_c (L_{\Lambda i} - D_{0i}) - (1 - c_f) D_{0i} \end{aligned} \tag{34}$$

Closed form expressions for L_{Λ_i} and D_{0i} are given in this paper under a dependent gamma distribution model.

7 Conclusion

We have developed a pricing model for an insurer with multiple lines of business where we assume that lines of business and assets have dependent gamma distributions and where we allow for the default option value and its allocation to line of business. We include the financial distress costs of capital in the pricing by an allocation to line of business using the default option value. In order to do this we consider frictional costs of capital and how they relate to the variables in the pricing. On the assumption that financial distress costs are a percentage of the asset shortfall in the event of insolvency these can be uniquely allocated to line of business. Frictional costs that arise as a percentage of the end of period surplus, assumed to include taxation, moral hazard and adverse selection costs, agency costs (managerial perquisites) and transactions costs, can also be allocated to line of business in the same manner. We derive closed form expressions that are used to develop by-line prices allowing for dependent gamma distributed risks and an assumption that is the basis of the application of standard option pricing models to an insurer balance sheet. Closed form expressions are derived for the allocation of frictional costs to lines of business in the model.

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