Multivariate Stochastic Volatility Models with Correlated Errors

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Abstract

We develop a Bayesian approach for parsimoniously estimating the correlation structure of the errors in a multivariate stochastic volatility model. Since the number of parameters in the joint correlation matrix of the return and volatility errors is potentially very large, we impose a prior that allows the off-diagonal elements of the inverse of the correlation matrix to be identically zero. The model is estimated using a Markov chain simulation method that samples from the posterior distribution of the volatilities and parameters. We illustrate the approach using both simulated and real examples. In the real examples, the method is applied to equities at three levels of aggregation: returns for firms within the same industry, returns for different industries and returns aggregated at the index level. We find pronounced correlation effects only at the highest level of aggregation.

Keywords: Bayesian estimation; Correlation matrix; Leverage; Markov chain Monte Carlo; Model averaging.
JEL classification: C11, C15, C30

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1 Introduction

A number of studies in financial economics report that conditional stock return volatility responds asymmetrically to positive and negative return shocks. Well known examples include French et al. (1987), Schwert (1990), Campbell and Hentschel (1992), Cheung and Ng (1992), Engle and Ng (1993), Glosten et al. (1993), Braun et al. (1995), Bekaert and Harvey (1997) and Bekaert and Wu (2000). Several explanations are proposed in the literature for the existence of an asymmetric relation between volatility and returns. The most widely cited of these is due to Black (1976) and Christie (1982) who suggest that the asymmetry reflects changes in financial leverage. They argue in particular that when a firm experiences a positive (negative) return, it becomes less (more) leveraged due to the rise (fall) in the market value of its equity, thereby making it less (more) risky and decreasing (increasing) its volatility. In other words, return shocks are negatively correlated with volatility shocks.

Despite the long history of studying leverage effects in finance, econometricians are only now beginning to allow for this type of asymmetry in stochastic volatility models because, until recently, the latent nature of the volatility process made the estimation of such models difficult using existing econometric methods. Often the only feasible strategy was to either formulate a method of moments estimator, or develop a linear state-space representation of the model and estimate it via quasi maximum likelihood (QML) using the Kalman filter. Harvey and Shephard (1996) used the QML approach in one of the first studies of leverage effects in a stochastic volatility framework. Over the last few years, however, their approach has been supplanted by simulation-based methods for fitting stochastic volatility models (see, e.g., Shephard and Pitt (1997), Durbin and Koopman (1997), Kim et al. (1998), Sandmann and Koopman (1998), and Chib et al. (2002)). As these methods gain wider recognition, more studies of models with leverage effects are slowly making their way into the literature (e.g., Jacquier et al. (2004)).

Our article considers a multivariate stochastic volatility model that accommodates leverage effects by allowing a general correlation structure between the errors in the return and volatility equations. Unlike univariate models, which are relatively straightforward to handle, multivariate stochastic volatility models still pose significant computational challenges to applied researchers because it is usually necessary to estimate a large number of parameters. One way to overcome this problem is to assume that the volatility process for each asset has a factor structure, as in Chib et al. (2006), who successfully fit a multivariate model with up to 50 stocks. However, the tractability of imposing a factor model is not without cost because there is no guarantee that a factor structure with a small number of factors is empirically plausible, and it may be difficult to interpret when dealing with correlated errors.

The main contribution of this paper is to develop a general methodology for parsimo-
niously dealing with large dimensional correlation matrices in multivariate stochastic volatility models. Although we consider one particular correlation structure in the article, our methods apply to other multivariate models, and in particular multivariate factor models. Our methodology is Bayesian and uses ideas of Bayesian subset selection and model averaging, which were first developed for the linear regression model, to achieve a parsimonious representation of the joint correlation matrix of the errors for the return and volatility equations. To see why we adopt this methodology, suppose we want to fit a model involving 10 stocks. Under our model — a direct multivariate generalization of the univariate log-linear specification — there are 190 unique parameters in the joint correlation matrix of the errors. With this many free parameters, the estimation methods developed for the univariate case are not as effective and it is necessary to explicitly control for the parameter-rich nature of the model in some way. Our approach imposes a prior on the correlation matrix that allows the off-diagonal elements of the inverse of the correlation matrix to be zero, which is equivalent to allowing the partial correlations of the vector of returns and log volatilities to be zero.

This approach to parsimoniously estimating a covariance matrix is called covariance selection and is proposed by Wong et al. (2003). The basic idea of covariance selection is due to Dempster (1972), who suggests zeroing out elements of the inverse of the covariance matrix (based on the data) to obtain a parsimonious, and hence more efficient, estimate of the covariance matrix. The approach by Wong et al. (2003) is adapted by Pitt et al. (2005) to estimate a correlation matrix. In our case, the time varying covariance matrix is factored into a product of variances and a correlation matrix, with the log variances following independent stochastic volatility processes. Such a decomposition of variances and a correlation matrix is similar to the decomposition in Barnard et al. (2000), but they do not allow elements of the correlation matrix or its inverse to be zero and the variances do not have a time series structure. The computations are carried out using a Markov chain Monte Carlo simulation method.

We illustrate the methodology using both simulated and real data. The simulation results show that our methodology works well and recovers the correct degree of parsimony both when the cross-correlation between the return errors and the volatility errors is zero and when it is present. We also fit the model to real daily stock returns for the period January 1988 to December 2001, which we analyze at three different levels of aggregation. First, we use the returns on several sets of individual stocks that are grouped according to industry. Second, we use the returns on a set of narrowly-defined portfolios constructed by giving equal weights to the firms within the industry groups. Finally, we use the returns on a set of broadly-based portfolios that correspond to different market indexes. This covers all the three types of return data that are used in the finance literature to investigate leverage effects.

The paper is organized as follows. Section 2 introduces the stochastic volatility model and its multivariate extension, discusses the prior assumptions and the Markov chain Monte
Carlo sampling scheme. Section 3 presents some empirical results from applying our method to simulated data and section 4 presents the results from applying our method to equity returns data aggregated at three different levels. We conclude with a short discussion in section 5. An appendix gives details of the sampling scheme.

2 Model, prior and sampling scheme

2.1 Model description

To simplify the exposition, we assume a zero mean for the returns process throughout this paper. The basic univariate stochastic volatility model in our article is

\[ y_t = \exp \left( h_t / 2 \right) e_t, \quad (2.1) \]
\[ h_t = \mu + \phi(h_{t-1} - \mu) + \psi a_t, \quad (2.2) \]

where \( y_t \) is the return at time \( t \) and \( \exp(h_t / 2) \) is the volatility at time \( t \). We assume that the log variance \( h_t \) follows a first order autoregressive process with mean \( \mu \) and persistence parameter \( \phi \). We also assume that \( (e_t, a_t)' \) is bivariate normal with zero mean, \( \text{var}(e_t) = \text{var}(a_t) = 1 \), and with \( \rho \) the correlation between \( e_t \) and \( a_t \). Jacquier et al. (2004) estimate a similar model, but also allow for fat-tailed errors. These authors note that when the correlation \( \rho < 0 \), positive and negative shocks \( e_t \) have asymmetric effects on volatility. Specifically, if \( e_t \) is negative, then \( a_t \) is likely to be positive with a consequent increase in volatility, whereas a positive \( e_t \) is likely to result in a negative value of \( a_t \) and hence a decrease in volatility. Chib et al. (2002) estimate such a model with \( \rho = 0 \), but allow for jumps and fat-tailed errors.

We use the following generalization of the univariate model (2.1) and (2.2) due to Harvey, Ruiz and Shephard (1994) and Danielsson (1998). Let \( y_t = (y_{t1}, \ldots, y_{tp})' \) be the \( p \times 1 \) vector of returns at time \( t \) and \( h_t \) be the corresponding vector of log variances, such that

\[ y_t = S_t^{1/2} e_t, \quad (2.3) \]
\[ h_t = \mu + \Phi(h_{t-1} - \mu) + \Psi^1 a_t, \quad (2.4) \]

where \( S_t = \text{diag}(\exp(h_{t1}), \ldots, \exp(h_{tp})) \), \( \mu = (\mu_1, \ldots, \mu_p)' \), and \( \Phi \) and \( \Psi \) are diagonal matrices whose diagonals are the vectors \( \phi = (\phi_1, \ldots, \phi_p)' \) and \( \psi = (\psi_1^2, \ldots, \psi_p^2)' \). Let \( r_t = (e_t, a_t)' \). We assume that \( r_t \) is multivariate normal with zero mean, and covariance matrix \( C \), with \( C \) a correlation matrix, that is, it has ones on the diagonal. This means that \( e_t \sim N(0, C_1) \) and \( a_t \sim N(0, C_2) \), where \( C_1 \) is the \( p \times p \) upper left submatrix of \( C \) and \( C_2 \) is the \( p \times p \) lower right submatrix of \( C \). Let \( C_{12} = \text{cov}(e_t, a_t) \), which is the covariance matrix of the errors in the observation and volatility equations.

A number of authors investigate the following modification of the univariate stochastic volatility model, for example Harvey and Shephard (1996), Meyer and Yu (2000) and Yu
(2005); (2.1) stays the same, but the volatility evolution equation (2.2) is
\[ h_{t+1} = \mu + \phi(h_t - \mu) + \psi a_t, \] (2.5)
which means that a shock to \( e_t \) affects \( h_{t+1} \) rather than \( h_t \). Yu (2005) argues that this modification is appealing from a theoretical perspective because it implies that \( y_t \) is a martingale difference sequence. In the multivariate version of the model the observation equation is still (2.4), but the volatility evolution equation is
\[ h_{t+1} = \mu + \Phi(h_t - \mu) + \Psi^1 a_t. \] (2.6)
Although we do not fit this model as part of the empirical analysis, it is straightforward to adapt the covariance selection approach proposed in our article to this model.

An alternative approach to estimating a multivariate stochastic volatility model is to impose a factor structure. The simplest specification of the factor model is
\[
\begin{align*}
  y_t &= BF_t + \Sigma_{t}^{\frac{1}{2}} e_t, & e_t &\sim \text{MVN}_p(0, I), \\
  F_t &\sim \text{MVN}_k(0, S_t), \\
  h_t &= \mu + \Phi(h_{t-1} - \mu) + \Psi^1 a_t, & a_t &\sim \text{MVN}_k(0, I),
\end{align*}
\] (2.7) \( (2.8) \) \( (2.9) \)
where \( F_t = (F_{t1}, \ldots, F_{tk})' \) are \( k \) underlying time varying factors with zero mean and covariance matrix \( S_t = \text{diag}(\exp(h_{t1}), \ldots, \exp(h_{tk})) \) and \( B = (B_1, \ldots, B_k) \) is a \( p \times k \) matrix of factor loadings. The log variances, \( h_{ti}, i = 1, \ldots, k \) are assumed to be independent and \( \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_p^2) \) is a constant matrix, giving a time-varying covariance matrix for the process as \( BS_tB' + \Sigma \). The number of factors used is typically small compared to the number of series considered. See Pitt and Shephard (1999), Jacquier et al. (1999), Chib et al. (2006). However, these authors do not consider correlated errors.

We could incorporate correlated errors into the model (2.7) – (2.9) by allowing \( e_t \) and \( a_t \) to be correlated, but it would be difficult to interpret these correlations in terms of individual returns and volatilities because there are now two sources of volatility, from the \( e_t \) and also from the factors. Thus our approach may be preferred to the factor approach in many applications. We note that although the factor model allows for changing correlations, it assumes that the matrix of factor loadings is constant. It does not necessarily follow, therefore, that a factor model with time-varying correlations will outperform a non-factor model that assumes constant correlations. This is an empirical question that we do not attempt to answer in this article.

The development of new multivariate models continues to be an active area of research. Asai and McAleer (2005) propose a dynamic asymmetric leverage (DAL) model that accommodates threshold effects, i.e., it allows volatility to undergo discrete shifts depending on whether the return for the previous period is above or below some threshold value. This type of specification offers a potential avenue for extending our covariance selection methodology. Additional examples of multivariate stochastic volatility models are
provided in McAleer (2005), Asai and McAleer (2006), and Yu and Meyer (2006). To simplify the discussion we henceforth refer to any model that allows for correlation between the errors in the observation equation and volatility transition equation as a stochastic volatility model with leverage. Model selection criteria plus subject matter knowledge can be used to select which of the alternative models is best for any particular data set.

2.2 Prior specification

The joint prior specification for \( \Theta = (\mu, \phi, \psi, C) \) assumes that they are independent of each other and that the elements of \( \phi \) and \( \psi \) are independent of each other. That is

\[
p(\mu, \phi, \psi, C) = p(\mu)p(\phi)(\prod_{i=1}^{p} p(\phi_i))(\prod_{i=1}^{p} p(\psi_i^2)).
\]

The prior on \( \phi_i, i = 1, \ldots, p, \) is uniform on \((-1, 1)\), which ensures the stationarity of the volatility process. The prior on \( \mu \) is taken proportional to a constant, which is the usual noninformative prior on log variances. The prior on \( \psi_i^2 \) is assumed to be inverse gamma with parameters \( (\alpha, \beta) \),

\[
\frac{(\psi_i^2)^{-(1+\alpha)} \exp(-\beta/\psi_i^2)^{\beta\alpha}}{\Gamma(\alpha)}.
\]

We set \( \alpha = 10e^{-10} \) and \( \beta = 10e^{-3} \), making the prior uninformative. In our work it is more convenient to work with \( v_i = (\psi_i^2)^{-\frac{1}{2}}, i = 1, \ldots, p, \) and generate \( V = (v_1, \ldots, v_p)' \) as a block. The prior for \( V \) is

\[
p(V) \propto \prod_{i=1}^{p} v_i^{2\alpha-1} \exp(-\beta V'V).
\]

The elements of the correlation matrix \( C \) are not generated directly but are parameterized in terms of a matrix \( D \), where \( C^{-1} = TDT \) with \( D \) a correlation matrix and \( T \) a diagonal matrix, such that

\[
T_i = (D^{-1})^{\frac{1}{2}}_{ii}, i = 1, \ldots, 2p,
\]

because \( C \) is a correlation matrix. We perform element selection on the off-diagonal elements of the matrix \( D \), allowing them to be set to zero explicitly. The prior for \( D \) is proposed by Wong et al. (2003) and the important details are repeated here for completeness.

To describe the prior, let \( J_{ij} = 1 \) if \( D_{ij} \neq 0 \) and \( J_{ij} = 0 \) if \( D_{ij} \equiv 0 \), for \( i = 1, \ldots, 2p, j < i \). That is, \( J_{ij} \) is a binary variable that indicates whether the element \( D_{ij} \) in the strict lower triangle of \( D \) is identically zero or not. Let \( J = \{J_{ij}, i = 1, \ldots, 2p, j < i \} \), i.e., the
ensemble of all the \( J_{ij} \) and let \( J_{\{i\}} \) be \( J \) with \( J_{ij} \) excluded. Let \( S(J) \) be the number of elements in \( J \) that are 1, and let \( S(J_{\{i\}}) \) be the number of elements in \( J_{\{i\}} \) that are 1. Let \( D_{\{J=1\}} = \{ D_{ij} : J_{ij} \in J \text{ and } J_{ij} = 1 \} \) and \( D_{\{J=0\}} = \{ D_{ij} : J_{ij} \in J \text{ and } J_{ij} = 0 \} \). Let \( C_{2p} \) be the class of all \( 2p \times 2p \) correlation matrices. Let

\[
V(J) = \int_{D \in C_{2p}} dD_{\{J=1\}}
\]

be the volume of the correlation matrix for given configuration \( J \), let \( m = 2p(2p - 1)/2 \), and let

\[
V(l) = \left( \frac{m}{l} \right)^{-1} \sum_{J:S(J)=l} V(J)
\]

be the average volume of all matrices having exactly \( l \) nonzero elements. Wong et al. (2003) propose the following hierarchical prior for \( D \).

\[
p(D \mid J) = V(J)^{-1} dD_{\{J=1\}} I(D \in C_{2p}),
\]

\[
p(J \mid S(J) = l) = \frac{V(J)}{V(l)} \left( \frac{m}{l} \right)^{-1},
\]

\[
p(S(J) = l \mid \varphi) = \left( \frac{m}{l} \right) \varphi^l (1 - \varphi)^{m-l},
\]

where \( 0 \leq \varphi \leq 1 \) and is interpreted as the probability that \( J_{ij} = 1 \). Thus \( p(D \mid J) \) is uniform, the prior for \( J \) given \( S(J) = l \) is uniform up to a volume adjustment, and the prior for \( S(J) \), given \( \varphi \), is binomial. Wong et al. (2003) assume that \( \varphi \) is uniform on \((0,1)\) and show that with this prior,

\[
p(S = l) = \frac{1}{m + 1},
\]

i.e., \( S \) is uniformly distributed in model size.

To simplify the computations, we approximate (2.10) by

\[
p(J \mid S(J) = l) = \left( \frac{m}{l} \right)^{-1}
\]

in our applications. This simplification has a minor effect on the empirical results for the size of correlation matrices used in the article.

2.3 Markov chain Monte Carlo sampling scheme

The MCMC sampling scheme used to estimate the parameters of the multivariate model is now outlined, with the technical details given in the appendix. Let \( y = (y_1, \ldots, y_n)' \) and \( h = (h_1, \ldots, h_n)' \). Step 0 is an initialization step and the sampling scheme cycles through the following 5 steps using 1000 iterations for burn-in and 10000 iterations for inference.
0. Initialize $\Theta$ and $h$.

The parameters $\Theta$ and the volatility $h$ are initialized by fitting univariate stochastic volatility models to each series. Given the other parameters, $C$ is initialized to the estimate of the correlation matrix of the $r_t$, where $r_t$ is defined in section 2.1.

1. $\phi \mid y, h, C, \mu, \psi$

The vector $\phi$ is generated using the Metropolis-Hastings method with a multivariate Gaussian proposal density such that each $\phi_i$ constrained to lie between $-1$ and $1$.

2. $\mu \mid y, h, C, \phi, \psi$

The conditional distribution of $\mu$ is multivariate Gaussian.

3. $V \mid y, h, C, \phi, \mu$

The conditional distribution of $V$ is not standard, so $V$ is generated using the Metropolis-Hastings method with a proposal density that is multivariate-$t$, and centered around the mode of the conditional density.

4. $h \mid y, \Theta$

The Markovian structure of the model can be exploited to simplify the posterior of $h$ to a series of univariate conditional densities,

$$p(h_t \mid h_{t-1}, h_{t+1}, y_{t-1}, y_{t+1}, C, \phi, \mu, \psi), \ t = 1, \ldots, n.$$ 

The $h_t$ can be generated one at a time, but experimentation shows that this strategy converges slowly and mixes poorly. Instead, $h$ is generated in blocks as in Shephard and Pitt (1997). Let $h_{a:b} = (h_a, h_{a+1}, \ldots, h_b)$ denote the block of log variances that we wish to generate and $h_{\setminus a:b}$ denote all of $h$ excluding $h_{a:b}$. The block size is randomly chosen to be 1, 2 or 3 with equal probability at each stage. The conditional distribution of the block, $p(h_{a:b} \mid h_{\setminus a:b}, y, \Theta)$, is not tractable and the Metropolis-Hastings method is used with a Gaussian proposal density.

5. $C \mid y, h, \phi, \mu, \psi$

The parsimonious correlation matrix estimation method of Wong et al. (2003), as modified by Pitt et al. (2005), is used to estimate $C$. The details are in the appendix.

2.4 Inference

Inference on the parameters follows the standard Bayesian approach of looking at functionals of the posterior distributions. For example, the posterior mean estimate of the log variance is given by

$$\hat{h}_t = \frac{1}{L} \sum_{l=1}^{L} h_t^{[l]} \quad i = 1, \ldots, p, \quad t = 1, \ldots, n.$$ 

8
where $h_t^{[l]}$ is the $l$th iterate from the inference stage. Similarly, the posterior mean estimate of $\psi_i$ is

$$\hat{\psi}_i = \frac{1}{L} \sum_{t=1}^L \left( v_i^{[t]} \right)^{-2} \quad i = 1, \ldots, p,$$

and the posterior mean estimate of $\phi_i$ is

$$\hat{\phi}_i = \frac{1}{L} \sum_{t=1}^L \phi_i^{[t]} \quad i = 1, \ldots, p.$$

where $v_i^{[t]}$ and $\phi_i^{[t]}$ are the $l$th iterate from the inference stage.

3 Simulated empirical examples

This simulation study illustrates the ability of our method to recover the model parameters in the multivariate system and, in particular, any leverage effects that may exist. Four correlated series were generated, using the same parameters $\phi_i = 0.9$, $\mu_i = -9.0$ and $v_i^2 = 0.1$ for each. We consider two simulated examples, the first without leverage effects and the second with leverage effects. The correlation matrix in Table 3.1, which contains leverage effects, is used for the second simulation study. For the first simulation study, the lower left and upper right quadrants of the matrix are set to 0, giving a correlation matrix with no leverage effects. There are 1000 observations in each simulation study.

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</table>

Table 3.1: Simulated example: The true correlation matrix used for simulations.

3.1 No leverage example

Figure 3.1 displays the performance of the Markov chain Monte Carlo simulations. The first column contains plots of the sample paths of parameters $\mu_1$, $\phi_1$ and $v_1^2$. The second column contains plots of the corresponding correlograms, and the third column contains plots of the histograms of the parameters.
Figure 3.1: No leverage example: Plots of the iterates for $\mu_1$, $\phi_1$ and $\psi_1^2$ from the MCMC output.
The autocorrelations of the iterates of $\mu_1$ die out very quickly, but the autocorrelations of the iterates of $\phi_1$ and $\psi_1^2$ take appreciably longer. The plots for the parameters of the other three series are similar. Correlograms for the iterates of the log variances also show good mixing, but are not reproduced here.

Figure 3.2 presents plots of the true (line) and estimated (dashed) log variances when there are no leverage effects. The general shape of the log variances is recovered, but as expected, the estimates of the log variances are smoother than the true variances.

Table 3.2 presents the posterior mean estimates of $\mu$, $\phi$ and $\psi^2$. Table 3.3 contains the posterior mean estimate of the correlation matrix, $C$. The matrix $C$ is presented in compact form to conserve space. The estimated correlation elements correspond closely to their true values. The estimate $\hat{C}_1$ shows less variability than the estimates $\hat{C}_{12}$ and $\hat{C}_2$. We conjecture that this is because the returns are observed, whereas the volatilities are not.

Figure 3.3 presents image plots displaying the structure of the correlation matrix $C$. Let $J_{ij} = 1$ if $D_{ij} \neq 0$ and let $J_{ij} = 0$ otherwise. Let $J$ be the matrix of the $J_{ij}$. Panel (a) of Figure 3.3 is a shaded ‘image’ plot of the estimated probabilities that $J_{ij} = 1$, with the shading being lightest for 0 and darkest for 1. These probabilities are estimated from the output of the MCMC scheme. Panel (b) and (c) are shaded ‘images’ of the absolute values of the elements of the estimate of matrices $D$ and $C$. The shading is the same as for panel (a). The block nature of the correlation matrix is clearly evident in panel (c),
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<th>$\phi_i$</th>
<th>$\psi_i^2$</th>
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<td>0.11(0.02)</td>
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Table 3.2: No leverage example: Posterior mean estimates of the parameters $\mu$, $\phi$ and $\psi$ with the posterior standard errors in parenthesis.

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</tr>
</tbody>
</table>

Table 3.3: No leverage example: The posterior mean estimate of $C$ (times 100), in compact form, with the posterior standard errors (times 100) in parenthesis. The top four rows show $\hat{C}_1$, the middle four $\hat{C}_{12}$, and the bottom four $\hat{C}_2$. 
Figure 3.3: No leverage example: (a) Shaded ‘image’ of the estimated probability of \( J_{ij} = 1 \). Panels (b) and (c) are shaded ‘images’ of the absolute values of the elements of the posterior mean estimate of matrices \( D \) and \( C \).

highlighting the ability of our methodology to pick up parsimony where it exists.

3.2 Leverage example

This section summarizes the results for the second simulated example, when leverage effects are present. Tables 3.4 and 3.5 present the posterior mean estimates of the parameters \( \mu, \phi, \psi \) and \( C \). The parameter estimates for \( \mu, \phi \) and \( \psi \) are quite similar for both examples, but leverage effects are now identified in the estimate of the correlation matrix \( C \), as well as in the image plots in Figure 3.4.

<table>
<thead>
<tr>
<th>Series</th>
<th>( \mu_i )</th>
<th>( \phi_i )</th>
<th>( \psi_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>-8.82(0.10)</td>
<td>0.89(0.02)</td>
<td>0.12(0.02)</td>
</tr>
<tr>
<td>Two</td>
<td>-8.90(0.09)</td>
<td>0.86(0.03)</td>
<td>0.12(0.03)</td>
</tr>
<tr>
<td>Three</td>
<td>-8.91(0.11)</td>
<td>0.89(0.02)</td>
<td>0.12(0.02)</td>
</tr>
<tr>
<td>Four</td>
<td>-8.90(0.10)</td>
<td>0.88(0.02)</td>
<td>0.11(0.02)</td>
</tr>
</tbody>
</table>

Table 3.4: Leverage example: Posterior mean estimates of the parameters \( \mu, \phi \) and \( \psi \) with the posterior standard errors in parenthesis.

The acceptance rates for the sampling of \( h \) are high for both examples. For the example with no leverage, the acceptance rate is 99.2% for a block size of one, 98.0% for a block size of 2 and 96.5% for a block size of 3. We obtain similarly high acceptance rates for the leverage example.

4 Equity returns example

We now apply our methods to equity returns data aggregated at 3 different levels. The first level is the market index level. There are eight return series in the following order: return
Table 3.5: Leverage example: The posterior mean estimate of $C$ (times 100), using the same compact form as in Table 3.3, with the posterior standard errors (times 100) in parenthesis.

\[
\begin{array}{cccc}
- & 62(2) & 62(2) & 61(2) \\
62(2) & - & 60(2) & 60(2) \\
62(2) & 60(2) & - & 64(2) \\
61(2) & 60(2) & 64(2) & - \\
-19(7) & -5(8) & -5(8) & -9(8) \\
1(8) & -22(7) & -5(7) & 4(8) \\
-4(7) & -1(8) & -26(7) & -1(10) \\
6(8) & -1(8) & -14(8) & -21(7) \\
- & 73(6) & 66(7) & 74(5) \\
73(6) & - & 54(8) & 62(7) \\
66(7) & 54(8) & - & 61(8) \\
74(5) & 62(7) & 61(8) & - \\
\end{array}
\]

Figure 3.4: Leverage example: (a) Shaded ‘image’ of the estimated probability of $J_{ij} = 1$. Panels (b) and (c) are shaded ‘images’ of the absolute values of the elements of the posterior mean estimate of matrices $D$ and $C$.


The second level of aggregation is at the industry level. We consider eight narrowly-focused industries in the US market: Petroleum, Food products, Pharmaceuticals, Banks, Industrial equipment, Aerospace, Electric utilities, and Department/discount stores. The return for each industry is based on an equally-weighted portfolio of eight well-known and actively traded stocks within that industry.
The third level of aggregation is at the individual firm level. For each of the industries specified above, we consider the return on each firm within that industry. The sample period for all the equity data is from January 4, 1988 to December 29, 2001, giving a total of 3285 observations. All the data is mean corrected so the returns have zero means.

4.1 Individual firms return data

For conciseness, we show only the results for the eight firms in the Petroleum industry. The results for the firms in the other seven industries are similar.

Table 4.1 presents the estimates for the parameters $\mu_i$, $\phi_i$ and $\psi_i^2$. Table 4.2 presents the posterior mean estimate of the matrix $C$ in compact form and shows that the data exhibits little leverage effect, which is confirmed by the image plots in Figure 4.1. Figure 4.1 also shows that the matrix of partial correlations of the log variances is quite sparse, although the corresponding matrix of correlations $C_2$ is not. This shows the power of our method to detect parsimony in the partial correlations.

<table>
<thead>
<tr>
<th>Petroleum Industry</th>
<th>$\mu_i$</th>
<th>$\phi_i$</th>
<th>$\psi_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exxon Mobil</td>
<td>-8.77(0.05)</td>
<td>0.90(0.02)</td>
<td>0.06(0.01)</td>
</tr>
<tr>
<td>Phillips Petroleum</td>
<td>-8.27(0.04)</td>
<td>0.81(0.03)</td>
<td>0.14(0.02)</td>
</tr>
<tr>
<td>Sunoco</td>
<td>-8.45(0.05)</td>
<td>0.71(0.04)</td>
<td>0.39(0.05)</td>
</tr>
<tr>
<td>Unocal</td>
<td>-8.10(0.04)</td>
<td>0.81(0.03)</td>
<td>0.11(0.02)</td>
</tr>
<tr>
<td>Royal Dutch Petroleum</td>
<td>-8.96(0.08)</td>
<td>0.95(0.01)</td>
<td>0.050(0.01)</td>
</tr>
<tr>
<td>Kerr Mcgee</td>
<td>-8.56(0.05)</td>
<td>0.85(0.03)</td>
<td>0.13(0.03)</td>
</tr>
<tr>
<td>Amerada Hess</td>
<td>-8.41(0.04)</td>
<td>0.77(0.04)</td>
<td>0.13(0.02)</td>
</tr>
<tr>
<td>Occidental Petroleum</td>
<td>-8.48(0.05)</td>
<td>0.74(0.03)</td>
<td>0.34(0.03)</td>
</tr>
</tbody>
</table>

Table 4.1: The Petroleum industry: Estimates of the parameters, $\mu_i$, $\phi_i$ and $\psi_i^2$, with the posterior standard errors times 100 in parenthesis.

4.2 The eight industries

The methodology is applied to the equity data, aggregated at the industry level. Tables 4.3 and 4.4 summarize the parameter estimates and show that the mean level of the log variances is lower for the industry level data than for the firm level data. The persistence of the log variances is similar, ranging from 0.85 to 0.95 for the different series. As at the firm level, little leverage is detected, although more of the cross-correlations in $C_{12}$ are now negative. The image plots for the industry level data are given in Figure 4.2.
Figure 4.1: The Petroleum industry: (a) Shaded ‘image’ of the estimated probability of $J_{ij} = 1$. Panels (b) and (c) are shaded ‘images’ of the absolute values of the elements of the posterior mean estimates of matrices $D$ and $C$.

Figure 4.2: The eight industries: (a) Shaded ‘image’ of the estimated probability of $J_{ij} = 1$. Panels (b) and (c) are shaded ‘images’ of the absolute values of the elements of the posterior mean estimates of matrices $D$ and $C$ respectively.
Table 4.2: The Petroleum Industry: The posterior mean estimate of $C$ (times 100), in compact form, with the posterior standard errors (times 100) in parenthesis. The top eight rows show $\hat{C}_1$, the middle eight $\hat{C}_{12}$, and the bottom eight $\hat{C}_2$.  

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4.3 The eight market indices

Tables 4.5 and 4.6 report the results for the equity return data at the index level and show that the mean level of the log variances is lower and the persistence of the log variances is higher than for the first two levels of aggregation.

Figure 4.3 presents plots of the estimated log variances for the eight stock indices, and suggests that the log variances for all the series display high correlation. The variances also seem to increase in the latter part of the series.

Table 4.6 presents the estimate of $C$ in compact form, and suggests that there is a large amount of leverage, not only within series, but also between series. The SP500-V and SP500-E series exhibit the least amount of leverage among all the series considered.
<table>
<thead>
<tr>
<th>Industry</th>
<th>$\mu_i$</th>
<th>$\phi_i$</th>
<th>$\psi_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petroleum</td>
<td>-9.20(0.05)</td>
<td>0.85(0.02)</td>
<td>0.145(0.021)</td>
</tr>
<tr>
<td>Food products</td>
<td>-9.20(0.05)</td>
<td>0.86(0.02)</td>
<td>0.116(0.014)</td>
</tr>
<tr>
<td>Pharmaceuticals</td>
<td>-8.89(0.06)</td>
<td>0.88(0.01)</td>
<td>0.138(0.018)</td>
</tr>
<tr>
<td>Banks</td>
<td>-8.94(0.08)</td>
<td>0.93(0.01)</td>
<td>0.095(0.013)</td>
</tr>
<tr>
<td>Industrial</td>
<td>-9.20(0.05)</td>
<td>0.87(0.02)</td>
<td>0.112(0.02)</td>
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<tr>
<td>Aerospace</td>
<td>-9.40(0.06)</td>
<td>0.88(0.02)</td>
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<tr>
<td>Electric</td>
<td>-9.88(0.07)</td>
<td>0.91(0.01)</td>
<td>0.12(0.02)</td>
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<tr>
<td>Stores</td>
<td>-8.84(0.06)</td>
<td>0.90(0.01)</td>
<td>0.11(0.02)</td>
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</tbody>
</table>

Table 4.3: The eight industries: Estimates of the parameters, $\mu_i$, $\phi_i$ and $\psi_i^2$, with the posterior standard errors given in parenthesis.

<table>
<thead>
<tr>
<th>Industry</th>
<th>$C$ (times 100)</th>
<th>$\mu_i$</th>
<th>$\phi_i$</th>
<th>$\psi_i^2$</th>
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</thead>
<tbody>
<tr>
<td>Petroleum</td>
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<td>56(5)</td>
<td>72(4)</td>
<td>69(6)</td>
</tr>
<tr>
<td>Food</td>
<td>56(5)</td>
<td>56(5)</td>
<td>72(4)</td>
<td>69(6)</td>
</tr>
<tr>
<td>Pharmaceuticals</td>
<td>56(5)</td>
<td>56(5)</td>
<td>72(4)</td>
<td>69(6)</td>
</tr>
<tr>
<td>Banks</td>
<td>56(5)</td>
<td>56(5)</td>
<td>72(4)</td>
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Table 4.4: The eight industries: The posterior mean estimate of $C$ (times 100) using the same compact form as in Table 4.2, with the posterior standard errors (times 100) in parenthesis.

18
<table>
<thead>
<tr>
<th>Index</th>
<th>$\mu_i$</th>
<th>$\phi_i$</th>
<th>$\psi_i^2$</th>
</tr>
</thead>
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<tr>
<td>NYSE-V</td>
<td>-9.50(0.09)</td>
<td>0.93(0.01)</td>
<td>0.087(0.01)</td>
</tr>
<tr>
<td>NYSE-E</td>
<td>-10.19(0.07)</td>
<td>0.93(0.01)</td>
<td>0.07(0.01)</td>
</tr>
<tr>
<td>AMEX-V</td>
<td>-9.96(0.09)</td>
<td>0.93(0.01)</td>
<td>0.08(0.01)</td>
</tr>
<tr>
<td>AMEX-E</td>
<td>-10.43(0.06)</td>
<td>0.93(0.01)</td>
<td>0.093(0.01)</td>
</tr>
<tr>
<td>NASDAQ-V</td>
<td>-8.71(0.13)</td>
<td>0.96(0.004)</td>
<td>0.066(0.01)</td>
</tr>
<tr>
<td>NASDAQ-E</td>
<td>-9.79(0.11)</td>
<td>0.96(0.004)</td>
<td>0.067(0.01)</td>
</tr>
<tr>
<td>SP500-V</td>
<td>-9.24(0.09)</td>
<td>0.93(0.01)</td>
<td>0.09(0.01)</td>
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<tr>
<td>SP500-E</td>
<td>-9.37(0.08)</td>
<td>0.94(0.01)</td>
<td>0.06(0.01)</td>
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</table>

Table 4.5: The eight stock indices: Estimates of the parameters, $\mu_i$, $\phi_i$ and $\psi_i^2$, with the posterior standard errors given in parenthesis.

Figure 4.3: The eight stock indices: Plot of the estimated log variances for the eight indices.
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<td>-</td>
<td>70 (1)</td>
<td>87 (.4)</td>
<td>61 (1)</td>
<td>67 (1)</td>
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<td>66 (1)</td>
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<tr>
<td>99 (.1)</td>
<td>87 (1)</td>
<td>75 (1)</td>
<td>61 (1)</td>
<td>82 (1)</td>
<td>62 (1)</td>
<td>-</td>
<td>0.97 (.2)</td>
<td>-</td>
<td>81 (1)</td>
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<td>-</td>
<td>81 (1)</td>
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<tr>
<td>98 (.1)</td>
<td>92 (.3)</td>
<td>80 (1)</td>
<td>67 (1)</td>
<td>82 (1)</td>
<td>67 (1)</td>
<td>97 (.2)</td>
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Table 4.6: The eight stock indices: The posterior mean estimate of $C$ (times 100), using the same compact form as in Table 4.2, with the posterior standard errors (times 100) in parenthesis.
Figure 4.4: The 8 stock indices: (a) Shaded ‘image’ of the estimated probability of $J_{ij} = 1$. Panels (b) and (c) are shaded ‘images’ of the absolute values of the elements of the posterior mean estimates of matrices $D$ and $C$.

Figure 4.4 shows that the matrix of partial correlations $D$ is quite sparse, even though the correlation matrix $C$ is quite full, suggesting that our methodology estimates the correlation matrix parsimoniously.

Overall our results indicate that leverage effects are negligible at the firm level, they start to appear at the industry level, but are still not significant, and it is only at the index level that they become significant. We note, however, that Yu (2005) obtains larger estimates of the absolute value of the leverage correlation when he uses the modified volatility transition equation (2.5) rather than our transition equation to fit the univariate stochastic volatility model to index returns. Perhaps similar findings would emerge in the multivariate setting considered here. We leave this question to future research.

5 Discussion

Our article presents a general approach for estimating the correlation matrix of the errors in a multivariate stochastic volatility model when correlation is allowed between errors in the observation equations and the volatility equations. In particular, our methods allow for a parsimonious representation of the correlation structure through covariance selection on the partial correlations. The power of our method to detect structure parsimoniously is best illustrated by the analysis of the index data, where there is parsimony in the partial correlation matrix, but not in the correlation matrix.

The analysis of the real data produces some interesting findings. The firm- and industry-level results show surprisingly little evidence of asymmetric volatility. For the most part our posterior mean estimates of the joint correlation matrix of the return and volatility errors is close to block diagonal, but the evidence of asymmetric volatility is much stronger for broadly-based indices. This suggests that leverage effects are largely a feature of market-wide rather than firm-specific returns and volatility.
Acknowledgment

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Appendix  Details of the generation process

This appendix shows how to generate $\Theta = (\phi, \psi, \mu, C)$ and $h$. To do so it is necessary to give some definitions and derive some preliminary results. Let $A$ and $B$ be two matrices having the same dimensions or two vectors having the same number of elements. Define $C = A \odot B$ as the matrix with $C_{ij} = A_{ij}B_{ij}$ or the vector with $C_i = A_iB_i$. For a vector $a$, let $\text{diag}(a)$ be the diagonal matrix having the elements of $a$ on the diagonal. For a square matrix $A$, let $\text{diagv}(A)$ be the vector consisting of the diagonal elements of $A$. It is not hard to check that if $B$ and $C$ are square matrices then

$$
\text{tr}(\text{diag}(a)B) = a'\text{diagv}(B) \quad \text{tr}(\text{diag}(a)B\text{diag}(a)C) = a'(B \odot C)a
$$

For $t > 1$, let

$$
r_t = \begin{pmatrix} S_t^{-\frac{1}{2}}y_t \\ \Psi^{-\frac{1}{2}}(h_t - \mu - \Phi(h_{t-1} - \mu)) \end{pmatrix} = \begin{pmatrix} e_t \\ a_t \end{pmatrix}.
$$

For $t = 1$, let

$$
r_t = \begin{pmatrix} S_t^{-\frac{1}{2}}y_t \\ \Psi^{-\frac{1}{2}}(h_t - \mu) \end{pmatrix} = \begin{pmatrix} e_t \\ \Psi^{-\frac{1}{2}}(h_t - \mu) \end{pmatrix}.
$$

Because $h_t$ is stationary

$$
h_t = \mu + \Psi^{\frac{1}{2}}a_t + \Psi^{\frac{1}{2}}\Phi a_{t-1} + \ldots
$$

Let $C_{2,*} = \text{var}(\Psi^{-\frac{1}{2}}(h_t - \mu))$. Then $C_{2,*} = C_2 + \Phi C_{2,*} \Phi$ implying

$$(C_{2,*})_{ij} = (C_2)_{ij}/(1 - \phi_i \phi_j).$$

Let $C_* = \text{var}(r_1)$ and write it as

$$
C_* = \begin{pmatrix} C_1 & C_{12} \\ C_{21} & C_{2,*} \end{pmatrix},
$$

$\Omega = C^{-1}$ and $\Omega_* = C_*^{-1}$, and partition $\Omega$ and $\Omega_*$ as

$$
\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}, \quad \Omega_* = \begin{pmatrix} \Omega_{11,*} & \Omega_{12,*} \\ \Omega_{21,*} & \Omega_{22,*} \end{pmatrix},
$$

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where each of the submatrices is \( p \times p \). From above,

\[
 p(y_t, h_t|y_s, h_s, 1 \leq s < t, \Theta) = \det(S_t)^{-\frac{1}{2}} \det(\Psi)^{-\frac{1}{2}} p(r_t|\Theta) = \exp(-1'h_t/2) \left( \prod_{i=1}^{p} \psi_i^2 \right)^{-\frac{1}{2}} p(r_t|\Theta)
\]

with

\[
p(r_t|\Theta) \propto \begin{cases} 
\det(\Omega)^{\frac{1}{2}} \exp\left(-\frac{1}{2}r_t'\Omega r_t\right), & \text{if } t > 1 \\
\det(\Omega_s)^{\frac{1}{2}} \exp\left(-\frac{1}{2}r_t'\Omega_s r_t\right), & \text{if } t = 1
\end{cases}
\]

Generating \( \phi | y, h, C, \mu, \psi \)

\[
p(\phi|y, h, \Theta|\phi) \propto p(y, h|\Theta) p(\phi) \propto p(r_1|\Theta) \prod_{t=2}^{n} p(r_t|\Theta)p(\phi)
\]

\[
\propto p(r_1|\Theta) \prod_{t=2}^{n} \exp\left(-\frac{1}{2}r_t'\Omega r_t\right) p(\phi).
\]

It is computationally inconvenient to generate \( \phi \) directly from its conditional density because \( p(r_1|\Theta) \) is a complex function of \( \phi \). Instead, we generate \( \phi \) using a Metropolis-Hastings proposal

\[
q(\phi|\Theta, \phi, y, h) \propto \exp\left(-\frac{1}{2} \sum_{t=2}^{n} r_t'\Omega r_t\right).
\]

For \( t > 1 \),

\[
r_t'\Omega r_t = -2 \left( e_{12}' \Omega_{12} + \tilde{h}_t' \Omega_{22} \right) \Phi \tilde{h}_{t-1} + \tilde{h}_{t-1}' \Omega_{22} \Psi^{-\frac{1}{2}} \Phi \tilde{h}_{t-1} + \cdots
\]

where \( \tilde{h}_t = \Psi^{-\frac{1}{2}}(h_t - \mu) \) and \( + \cdots \) means additive terms that do not depend on \( \phi \). Hence,

\[
\sum_{t=2}^{n} r_t' \Omega^{-1} r_t = -2 \text{tr} \left( \Phi \sum_{t=2}^{n} \tilde{h}_{t-1} \left( e_{12}' \Omega_{12} + \tilde{h}_t' \Omega_{22} \right) \right) + \text{tr} \left( \Phi \Omega_{22} \Phi \sum_{t=2}^{n} \tilde{h}_{t-1} \tilde{h}_t' \right) + \cdots
\]

\[
= -2 \text{tr} (\Phi T) + \text{tr} (\Phi \Omega_{22} \Phi K) + \cdots
\]

\[
= \phi' M \phi - 2\phi' T v + \cdots
\]

where

\[
T = \sum_{t=2}^{n} \tilde{h}_{t-1} \left( e_{12}' \Omega_{12} + \tilde{h}_t' \Omega_{22} \right), \quad K = \sum_{t=2}^{n} \tilde{h}_{t-1} \tilde{h}_t'.
\]
$T_v = \text{diag}(T)$ and $M = \Omega_{22} \odot K$. The proposal density for $\phi$ is multivariate Gaussian with mean $M^{-1}T_v$ and covariance $M^{-1}$, and with $\phi$ constrained so that each $\phi_i$ lies in the open interval $(-1, 1)$.

**Generating $\mu \mid y, h, C, \phi$**

The conditional distribution of $\mu$ is multivariate Gaussian and is derived in a similar way to that of $\phi$. We have

$$p(\mu \mid y, h, C, \phi, \psi) \propto \exp \left( -\frac{1}{2} \sum_{t=2}^{n} r'_t C^{-1} r_t - \frac{1}{2} r'_1 C_*^{-1} r_1 \right) p(\mu),$$

which we simplify to expressions involving only $\mu$. For $t > 1$,

$$r'_t C^{-1} r_t = -2 \left[c'_t \Omega_{12} + (h_t - \Phi h_{t-1})' \Psi^{-\frac{1}{2}} \Omega_{22} \right] \Psi^{-\frac{1}{2}} (I - \Phi) \mu + \mu' (I - \Phi) \Psi^{-\frac{1}{2}} \Omega_{22} \Psi^{-\frac{1}{2}} (I - \Phi) \mu + \cdots \Rightarrow -2z'_t \mu + \mu' Z_t \mu + \cdots ,$$

where $z'_t = \left[c'_t \Omega_{12} + (h_t - \Phi h_{t-1})' \Psi^{-\frac{1}{2}} \Omega_{22} \right] \Psi^{-\frac{1}{2}} (I - \Phi)$ and $Z_t = (I - \Phi) \Psi^{-\frac{1}{2}} \Omega_{22} \Psi^{-\frac{1}{2}} (I - \Phi)$.

For $t = 1$,

$$r'_1 C_*^{-1} r_1 = -2 \left[c'_1 \Omega_{12,*} + (h_t - \Phi h_{t-1})' \Psi^{-\frac{1}{2}} \Omega_{22,*} \right] \Psi^{-\frac{1}{2}} \mu + \mu' (I - \Phi) \Psi^{-\frac{1}{2}} \Omega_{22,*} \Psi^{-\frac{1}{2}} (I - \Phi) \mu + \cdots \Rightarrow -2z'_1 \mu + \mu' Z_t \mu + \cdots ,$$

where

$$z_t = (I - \Phi) \Psi^{-\frac{1}{2}} \left[ \Omega_{21,*} e_t + \Omega_{22,*} (h_t - \Phi h_{t-1}) \right]$$

$$Z_t = (I - \Phi) \Psi^{-\frac{1}{2}} \Omega_{22,*} (I - \Phi) \Psi^{-\frac{1}{2}}$$

Let

$$z = \sum_{t=1}^{n} z_t, \quad Z = \sum_{t=1}^{n} Z_t.$$

Then,

$$r'_1 C_*^{-1} r_1 + \sum_{t=2}^{n} r'_t C^{-1} r_t = -2z' \mu + \mu' Z \mu .$$

Hence the conditional density of $\mu$ is multivariate normal with mean $Z^{-1} z$ and covariance matrix $Z^{-1}$.

**Generating $V \mid y, h, C, \phi, \mu$**

We generate $V$ using a Metropolis-Hastings algorithm. The proposal distribution, which
is multivariate $t$ with $\nu = 6$ degrees of freedom, is centered around the mode of the log conditional distribution of $V$. Define

$$w_t = \begin{cases} h_t - \mu - \Phi(h_{t-1} - \mu), & \text{if } t > 1; \\ h_t - \mu, & \text{if } t = 1; \end{cases}$$

Then

$$\sum_{t=2}^{n} r_t' C_t^{-1} r_t + r_1' C_s^{-1} r_1 = \sum_{t=2}^{n} \left( 2c_t' \Omega_{12} \Psi^{-\frac{1}{2}} w_t + w_t' \Psi^{-\frac{1}{2}} \Omega_{22} \Psi^{-\frac{1}{2}} w_t \right) + 2c_t' \Omega_{12,*} \Psi^{-\frac{1}{2}} w_t + w_t' \Psi^{-\frac{1}{2}} \Omega_{22,*} \Psi^{-\frac{1}{2}} w_1 + \cdots$$

$$= 2\text{tr}(\Psi^{-\frac{1}{2}} U) + \text{tr}(\Psi^{-\frac{1}{2}} \Omega_{22} \Psi^{-\frac{1}{2}} W) + \text{tr}(\Psi^{-\frac{1}{2}} \Omega_{22,*} \Psi^{-\frac{1}{2}} W_s) + \cdots$$

$$= 2V' U_v + V' G V + \cdots$$

where

$$U = \sum_{t=2}^{n} w_t e_t' \Omega_{12} + w_1 e_1' \Omega_{12,*}, \quad W = \sum_{t=2}^{n} w_t w_t', \quad W_s = w_1 w_1',$$

$$G = \Omega_{22} \otimes W + \Omega_{22,*} \otimes W_s, \quad U_v = \text{diag}(U).$$

The log conditional density of $V$ is

$$l(V) = \log \left( p(V \mid y, h, C, \phi, \mu) p(V) \right)$$

$$= (n + 2\alpha - 1) \sum_{t=1}^{p} \log v_t - \frac{1}{2} (2V'T_v + V(G + 2\beta I)V) + \cdots$$

Let $\hat{V}$ be the mode of $l(V)$ and

$$\Delta_V = - \left( \frac{\partial^2 l(\hat{V})}{\partial V \partial V'} \right)^{-1}.$$

The proposal density is a multivariate $t$-distribution with $\nu = 6$ degrees of freedom, location parameter $\hat{V}$ and covariance matrix $\Delta_V$.

**Generating $h \mid y, \Theta$**

To generate the log variances it is again necessary to use the Metropolis-Hastings method because it is intractable to generate from the exact conditional density. Let $h_{a:b}$ be a block of volatility vectors that we wish to generate, with $1 < a \leq b < n$. Similar results are obtained for $a = 1$ and $b = n$. The conditional density of $h_{a:b}$ is

$$p(h_{a:b} \mid h_{a:b}, y, \Theta) \propto \exp \left\{ -\frac{1}{2} \sum_{t=a}^{b} 1'h_t \right\} \exp \left\{ -\frac{1}{2} \sum_{t=a}^{b+1} r_t' C_t^{-1} r_t \right\}.$$
The log conditional density, \( l(h_{a:b}) = \log p(h_{a:b} \mid h_{a:b}, y, \Theta) \) is
\[
l(h_{a:b}) = -\frac{1}{2} \left[ \sum_{t=a}^{b} 1' h_t + \sum_{t=a}^{b+1} r_t' C^{-1} r_t \right].
\]

To generate \( h_{a:b} \) we use a multivariate Gaussian distribution based on a quadratic approximation to the log-likelihood that is centered around the mode of \( l(h_{a:b}) \). We now obtain expressions for the first and second derivatives of \( l \).

\[
\frac{\partial r_t}{\partial h_t'} = \begin{pmatrix} \frac{\partial r_t}{\partial h_t} \\ \frac{\partial r_t'}{\partial h_t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \text{diag}(S_t^{-\frac{1}{2}} y_t) \\ \Psi^{-\frac{1}{2}} \end{pmatrix}
\]

\[
\frac{\partial r_{t+1}}{\partial h_t} = \begin{pmatrix} \frac{\partial r_{t+1}}{\partial h_t} \\ \frac{\partial r_{t+1}'}{\partial h_t} \end{pmatrix} = \begin{pmatrix} 0 \\ \Psi^{-\frac{1}{2}} \Phi \end{pmatrix}.
\]

Hence for \( a \leq t \leq b \),
\[
\frac{\partial l(h_{a:b})}{\partial h_t} = -\frac{1}{2} \frac{\partial}{\partial h_t} \left( 1' h_t + r_t' C^{-1} r_t + r_{t+1}' C^{-1} r_{t+1} \right)
\]
\[
= -\frac{1}{2} \left( 1 + \left( \frac{2}{\partial h_t} r_t' \right) C^{-1} r_t + 2 \left( \frac{\partial r_{t+1}'}{\partial h_t} \right) C^{-1} r_{t+1} \right)
\]

To obtain the second derivative of \( l \), let \( \eta_t = (\Omega_{11}, \Omega_{12}) r_t \). Then we can show that
\[
\frac{\partial^2 (r_t' \Omega r_t)}{\partial h_t \partial h_t'} = 2 \frac{\partial r_t'}{\partial h_t} \Omega \frac{\partial r_t}{\partial h_t'} + \frac{1}{2} \text{diag}(S_t^{-\frac{1}{2}} y_t) \odot \eta_t,
\]
and
\[
\frac{\partial^2 l(h_{a:b})}{\partial h_t \partial h_t'} = -\left( \frac{\partial r_t'}{\partial h_t} \Omega \frac{\partial r_t}{\partial h_t'} + \frac{1}{4} \text{diag}(S_t^{-\frac{1}{2}} y_t) \odot \eta_t + \frac{\partial r_{t+1}'}{\partial h_t} C^{-1} \frac{\partial r_{t+1}}{\partial h_t'} \right),
\]
\[
\frac{\partial^2 l(h_{a:b})}{\partial h_t \partial h_{t+1}'} = -\frac{\partial r_{t+1}'}{\partial h_t} C^{-1} \frac{\partial r_{t+1}}{\partial h_{t+1}'}
\]
\[
\frac{\partial^2 l(h_{a:b})}{\partial h_t \partial h_{t+j}'} = 0 \quad \text{for } j > 1,
\]
and \( \partial^2 l(h_{a:b}) / \partial h_{a:b} \partial h_{a:b}' \) is a block tri-diagonal matrix.

The proposal density for \( h_{a:b} \) is a multivariate Gaussian obtained from a quadratic approximation to \( l(h_{a:b}) \) that is centered at the mode of \( l(h_{a:b}) \).

**Generating \( C \mid y, h, \Theta \)**

Conditional on \( \Theta \) and \( h \) the errors \( r_t, t = 1, \ldots, n \) are \( N(0, C) \) and independent and the problem of generating \( C \) from its conditional distribution reduces to generating \( C \) from the conditional distribution of the errors. To do so we use the method of Pitt et al. (2005) who adapt the covariance selection method of Wong et al. (2003) to parsimoniously estimate a correlation matrix.
Let $r = (r_1, \ldots, r_n)$ and let the matrix $D$ be defined as in 2.2. The elements $\{D_{ij}, i = 1, \ldots, 2p, j < i\}$ in the lower triangle of $D$ are generated one at a time from an approximation to the conditional density

$$p(dD_{ij}|r, D_{\{\langle ij\}}) \propto p(r|C)p(dD_{ij}|r, D_{\{\langle ij\}})I(D \in C_{2p}),$$

where $C_{2p}$ is the class of $2p \times 2p$ positive definite matrices. Given $D_{\{\langle ij\}}$, there exist numbers $a_{ij}, b_{ij}$ such that $D$ is positive definite if and only if $|D_{ij} - a_{ij}| < b_{ij}$. From Wong et al. (2003), the conditional prior density for $D_{ij}$ is

$$p(D_{ij}|D_{\{\langle ij\}}) = I(|D_{ij} - a_{ij}| < b_{ij})I(D_{ij} = 0) + dD_{ij}h(S(J_{\{\langle ij\}}))I(|a_{ij}| < b_{ij}) + 2b_{ij}h(S(J_{\{\langle ij\}}))$$

(A1)

where

$$h(S(J_{\{\langle ij\}})) = \frac{S(J_{\{\langle ij\}}) + 1}{m - S(J_{\{\langle ij\}})} \times \frac{V(S(J_{\{\langle ij\}}))}{V(S(J_{\{\langle ij\}}) + 1)},$$

and $J_{\{\langle ij\}}$ and the average volumes $V(\cdot)$ are defined in the main article. We simplified the sampling of $D_{ij}$ by approximating $h(S(J_{\{\langle ij\}}))$ by

$$\frac{S(J_{\{\langle ij\}}) + 1}{m - S(J_{\{\langle ij\}})},$$

i.e., by taking the ratio of average volumes $V(S(J_{\{\langle ij\}}))/V(S(J_{\{\langle ij\}}) + 1)$ as 1. This approximation does not alter the results appreciably for the sizes of matrices considered in our article. We now show how to generate $D_{12}$ from its conditional density using the Metropolis-Hastings method. The other o-diagonal elements of $D$ are generated similarly. Write $D$ as

$$D = \begin{pmatrix} U & V' \\ V & W \end{pmatrix}, \quad \text{with} \quad U = \begin{pmatrix} 1 & D_{12} \\ D_{12} & 1 \end{pmatrix}.$$

The matrices $V$ and $W$ are functionally independent of $D_{12}$. Suppose that the matrix $D$ is positive definite for a given value of $D_{12}$. This means that the bottom right $(2p - 2) \times (2p - 2)$ submatrix of $D$ is positive definite and that the necessary and sufficient condition for $D$ to be positive definite is that $\det(D) > 0$. Now

$$\det(D) = \det(W)\det(U - V'W^{-1}V) = \det(W)\det(U - K)$$

where $K = V'W^{-1}V$. Hence, $D$ is positive definite matrix, as a function of $D_{12}$, if and only if

$$\det(U - K) = (1 - K_{11})(1 - K_{22}) - (D_{12} - K_{12})^2 > 0,$$

that is,

$$|D_{12} - a_{12}| < b_{12},$$

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where \( a_{12} = K_{12} \) and \( b_{12} = \sqrt{(1 - K_{11})(1 - K_{22})} \).

To sample the element \( D_{12} \), it is necessary to have a fast way of computing the matrix \( T \) as a function of \( D_{12} \), with the other elements of \( D \) held constant. This can be done efficiently as follows. From standard results on partitioned matrices,

\[
D^{-1} = \begin{pmatrix}
  D_u^{-1} & -D_u^{-1}VW^{-1} \\
  -W^{-1}VD_u^{-1} & W^{-1}(I + V'D_u^{-1}VW^{-1})
\end{pmatrix}
\]

where \( D_u = U - V'W^{-1}V = U - K \) and \( K \) is a constant with respect to \( D_{12} \). By using this result, the matrix \( D^{-1} \) can be updated quickly with respect to changes in \( D_{12} \), even for \( p \) large.

We now show how to sample \( D_{12} \) from the conditional distribution \( p(D_{12} \mid r, D_{\setminus 12}) \). The likelihood of \( r \) is

\[
p(r \mid D) \propto \det(T)^n \det(D)^{n/2} \exp\left\{ -\frac{1}{2} \trace(TDT_S) \right\} = g(D_{12}) ,
\]

where \( S_r = \sum_{i=1}^n r_i'r'_i \) and the notation \( g(D_{12}) \) shows that we consider the likelihood as a function of \( D_{12} \), with the other elements of \( D \) considered fixed. Then,

\[
p(D_{12} \mid r, D_{\setminus 12}) \propto g(D_{12}) p(D_{12} \mid D_{\setminus 12}) ,
\]

where the expression for \( p(D_{12} \mid D_{\setminus 12}) \) is given by (A1). It follows that \( p(D_{12} \mid r, D_{\setminus 12}) \) is a mixture of a discrete and a continuous component. We generate \( D_{12} \) from this mixed density, by first generating \( J_{12} \) and then \( D_{12} \), conditionally on the generated value of \( J_{12} \). For conciseness, we write \( a_{12}, b_{12} \) and \( h(S(J_{\setminus 12})) \) as \( a, b, \) and \( h \). The joint distribution of \( J_{12} \) and \( D_{12} \) is

\[
p(J_{12}, D_{12} \mid r, D_{\setminus 12}) = p(J_{12} \mid r, D_{\setminus 12}) p(D_{12} \mid r, J_{12}, D_{\setminus 12})
\]

and we note that

\[
p(D_{12} = 0 \mid r, J_{12} = 0, D_{\setminus 12}) = 1 , \quad p(dD_{12} \mid r, J_{12} = 1, D_{\setminus 12}) \propto I(|D_{12} - a| < b) h g(D_{12}) .
\]

Let \( l(D_{12}) = \log(g(D_{12})) \), and obtain \( \hat{D}_{12} \), the mode of \( l(D_{12}) \) and \( \sigma^2_D = -1/l''(\hat{D}_{12}) \). We approximate the likelihood \( g(D_{12}) \) by \( t_6(D_{12}; \hat{D}_{12}, \sigma^2_D) \), a \( t \)-distribution with 6 degrees of freedom, mean \( \hat{D}_{12} \) and variance \( \sigma^2_D \). If \( J_{12} = 1 \), the proposal distribution for \( D_{12} \) is \( g_a(D_{12}) \propto I(|D_{12} - a| < b) t_6(D_{12}; \hat{D}_{12}, \sigma^2_D) \), which is a truncated \( t_6 \) distribution. We found that this proposal density dominates the conditional density in the tails.

The indicator \( J_{12} \) is generated from an approximation to the conditional density \( p(J_{12} \mid r, D_{\setminus 12}) \). To obtain the proposal density, we note that

\[
p(J_{12} = 1 \mid r, D_{\setminus 12}) \propto h \int I(|D_{12} - a| < b) g(D_{12}) dD_{12} , \approx h g(\hat{D}_{12}) / g_a(\hat{D}_{12}) .
\]
which is obtained by taking \( g(D_{12})/g_a(D_{12}) \) as approximately constant in the region \( |D_{12} - a| < b \). Based on this approximation, we take the proposal density for \( J_{12} \) as

\[
p_a(J_{12} = 1 \mid r, D_{(\setminus 12)}) = \frac{h g(\hat{D}_{12})/g_a(\hat{D}_{12})}{I(|a| < b)g(0) + h g(D_{12})/g_a(D_{12})},
\]

which we write as \( q(J_{12}) \). The proposal density for generating \( D_{12} \), when \( J_{12} = 1 \), is given above.

The Metropolis-Hastings acceptance probabilities for the transition \( J_{12}^c \rightarrow J_{12}^p, D_{12}^p \) are given similarly to Wong et al. (2003), as

\[
\alpha (J_{12}^c = 0, D_{12}^c = 0 \rightarrow J_{12}^p = 0, D_{12}^p = 0) = 1
\]
\[
\alpha (J_{12}^c = 0, D_{12}^c = 0 \rightarrow J_{12}^p = 1, D_{12}^p \neq 0) = \min \left\{ 1, \frac{g(D_{12}^p)2bh}{g(0)} \times \frac{q(J_{12}^c = 0)}{q(J_{12}^p = 1) g_a(D_{12}^p)} \right\}
\]
\[
\alpha (J_{12}^c = 1, D_{12}^c \neq 0 \rightarrow J_{12}^p = 0, D_{12}^p = 0) = \min \left\{ 1, \frac{g(0)}{g(D_{12}^c)2bh} \times \frac{q(J_{12}^c = 1) g_a(D_{12}^c)}{q(J_{12}^p = 0)} \right\}
\]
\[
\alpha (J_{12}^c = 1, D_{12}^c \neq 0 \rightarrow J_{12}^p = 1, D_{12}^p \neq 0) = \min \left\{ 1, \frac{g(D_{12}^p) g_a(D_{12}^c)}{g(D_{12}^c) g_a(D_{12}^p)} \right\}
\]

The following result

\[
\frac{p(J_{12} = 1 \mid D_{(\setminus 12)})}{p(J_{12} = 0 \mid D_{(\setminus 12)})} = \frac{2bh}{I(|a| < b)}
\]

was used to derive the acceptance probabilities. A complete iteration of the matrix \( D \) is obtained when all elements \( D_{ij}, i = 1, \ldots, 2p, j = 1, \ldots, i - 1 \) are sampled.

References


