An Analysis Of Large Multi-Unit Auctions With Bundling

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Abstract  
Should multi-unit auctions be used to sell multiple identical and indivisible units of a good? We show that in large markets, the multi-unit format tends to be more efficient than its bundled counterpart in a broad range of situations. However, the revenue performance depends on the fraction of demand that is met asymptotically. In course of deriving our results we apply some new techniques that is likely to be useful for similar analysis.

Key Words: Bundle Auction, Multi-Unit, Competitive Prices, Efficiency.

JEL Classification: D44

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1. Introduction
Much of the research in auction theory has concentrated on the study of single-object auctions. In reality, however, a large number of situations involve either multiple goods that are sold separately or in lots, or (bundles of) goods that can be split into multiple parts. Some recent work has correctly pointed out that conclusions drawn about single-object auctions do not generally extend to their multi-object counterparts. This makes it necessary to develop a separate theory of multi-object auctions. But should we have a multi-object auction at the first place?

Conceivably, there are several reasons (some even regulatory) due to which multi-object auctions may need to be held. In this paper, we consider two commonly taken perspectives from which auction markets are evaluated, viz. social efficiency and seller’s revenue, and identify those situations where there are performance based reasons to conduct a multi-object auction. One way to evaluate a multi-object auction is to compare it with the obvious alternative – the single-object auction for the bundle(s). In a large number of markets, for all practical purposes, an auctioneer has little or no control on the institutional details of the market. The only choice she has is on her decision to aggregate the objects on sale or not making this a simple search for the optimal auction within that restricted framework.

The problem of comparing a multi-object auction to its bundled counterpart arises in two different contexts. First, an object (or a bundle of objects) may be split into multiple dissimilar objects, (e.g., an apple and an orange). Second, the bundle may be split into multiple identical objects/units, (e.g., two apples). Each scenario gives
rise to a different theoretical framework that must be studied separately. Palfrey (1983) considered the first framework where the seller sells multiple dissimilar objects for which the bidders have independent private values. Clearly, under a second-price rule without reserves (or other rules that allocate the object to the bidder with the highest value) selling the objects separately is allocatively efficient while bundling is not. Palfrey (1983) and Chakraborty (1999) showed that whether bundling the objects before the auction generates a higher or lower expected revenue than selling the objects separately, depends, respectively, on whether the number of bidders is “small” or “large.” Thus, objects should be sold separately for efficiency reasons regardless of the size of the auction, and for revenue reasons when there are sufficiently many bidders.

We consider the second scenario where the objects are identical units of an object. The results for dissimilar objects are not useful for drawing any conclusion in the multi-unit framework. In fact, the multi-unit framework gives rise to a much more complex problem because the equilibrium bidding strategies for multi-unit auctions cannot be described in a useful manner except in special cases. For instance, Engelbrecht-Wiggans and Kahn (1998) showed that multi-unit uniform-price auctions give rise to demand reduction in which bidders tend to shade greater and greater amounts on their equilibrium bids for the successive units (relative to their values). The amount by which bidders shade their bids cannot be expressed in a closed form except in some special cases, thus making the comparison difficult. Moreover, differential bidding on successive units in multi-unit auctions gives rise to inefficiency. Thus, unlike the case of Palfrey (1983), whether multi-unit auctions are more or less efficient than the bundle auction is a difficult question to answer.

Our analysis of multi-unit auction relative to the bundle auction is also interesting from a market decentralization point of view. Economists have long been interested in identifying conditions under and the extent to which decentralized finite markets come close being efficient (see Satterthwaite and Williams, 1989, for a good discussion on this). We identify conditions under which the objective of a revenue maximizing auctioneer is and is not perfectly aligned with the social objective of efficiency when it
comes to a choice between the bundle and multi-unit auction. In course of developing our main result we also describe the rate (bounds on the rate when the actual rate cannot be calculated) at which the auctions tend to their limit outcomes of competitive price and efficiency.

This is certainly not the first attempt to evaluate the multi-unit auction relative to the single-object auction for the bundle. Wilson (1979) considered some tractable examples of pure common value auctions without reserves for a perfectly divisible object. He demonstrated that when bidders are allowed to submit continuous bid-price schedules for the different shares of the object under the uniform-price rule the problem of demand reduction may give rise to low revenues in the share auction relative to a single-object auction for the whole object. Moreover, he showed that demand reduction can increase in severity (in the sense that each bidder demands a smaller fraction of the item for a positive price) as the number of bidders increases. This prevents the seller from receiving any advantage from increased competition.

In many situations, however, objects cannot be divided infinitely for physical or other practical reasons. In those situations there is a limit beyond which demands cannot reduce in equilibrium. If identical but indivisible units are on sale, a bidder’s demand cannot reduce below a unit of the object and still remain positive. In this paper, we show that in such auctions with many bidders a multi-unit auction generally performs better than the bundle auction when a small fraction of demand is met asymptotically. A multi-unit auction offers a greater scope for efficiency and, thus, higher revenue by allowing more allocative flexibility. However, in small auctions demand reduction can leave both possibilities unfulfilled. When there are a large number of bidders but a relatively small supply the likelihood of any bidder receiving more than one unit, and the price being set by a bidder’s second bid can be expected to become very small. In such situations the problem of demand reduction, even in its severest form, tends to become sufficiently irrelevant when there are many bidders, making it perform better than the bundle auction. That does not happen when the supply is not small relative

\footnote{Obviously, efficiency is not an issue in the pure common value model.}
to the demand.

2. The Auction Model
Consider a sequence of auctions \( \{A_n\}_{n \geq 1} \). In auction \( A_n \) the seller wants to sell \( 2M_n \) identical units of an object in a single auction.\(^6\) There are \( n \) risk-neutral bidders with independent private values for the objects. To avoid some trivial cases we assume \( 1 \leq M_n < n \). Bidder \( i \) has diminishing marginal values \( V_{1i} \) for the first unit and \( V_{2i} \) for the second, \( i.e. \) a value \( V_{1i} + V_{2i} \) for the bundle.\(^7\) The values \((V_{1i}, V_{2i})\)-s are random vectors, each having a joint distribution \( F(v_1, v_2) \) with a density \( f(v_1, v_2) \) on \( S = [(v_1, v_2) : 0 \leq v_2 \leq v_1 \leq 1] \), that are independent across bidders.\(^8,9\) The marginal distribution functions of \( V_1 \) and \( V_2 \) are denoted by \( F_1 \) and \( F_2 \), and their density functions denoted by \( f_1 \) and \( f_2 \). We denote the distribution of \( V_1 + V_2 \) by \( F_{V_1+V_2}(\cdot) \), and its density function by \( f_{V_1+V_2}(\cdot) \).

**Definition.** By the upper end of the support for a distribution with support \([a, b]\) we mean \( b \), the lower end of the support is \( a \) in that case.

Each bidder privately observes his values and then participates in the auction. The auction is held under a sealed-bid uniform-price rule with the price set equal to the highest-losing bid.\(^{10}\) The remaining details of the rule depend on whether the auction is carried out under a bundle format or a multi-unit format.

In a bundle auction the seller sells \( M_n \) bundles each consisting of 2 units, and a bidder submits a single sealed bid for the bundle. The price for each bundle is set equal

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\(^6\) Restriction to an even number of units is for expositional reasons only.

\(^7\) Most results and conclusions in the paper extend in their appropriate forms to the more general case where each bidder has demand for \( m \) \(( \leq M_n \) bundles, \( i.e. \), \( 2m \) units.

\(^8\) It is not difficult to see that in our set-up when bidders have increasing marginal values, instead, there is an equilibrium where bidders submit the same bid on both units in the multi-unit auction, thus effectively reducing it to a bundle auction in equilibrium.

\(^9\) Throughout the paper we make a series of stronger than necessary assumptions to make the reading relatively easier.

\(^{10}\) A large part of our results continue to hold under the Vickrey and pay-as-bid auction rules, as well.
to the $M_n+1$-st highest bid in the auction.\footnote{In the bundle auction the price refers to the price for the entire bundle. We refer to the unit price by the “price per unit.”} A bundle is awarded to each bidder whose bid is among the highest $M_n$ bids.

In the \textit{multi-unit auction} each bidder submits two sealed bids $b_1$ and $b_2$. A bidder receives one or two units depending on whether one or both his bids are among the $2M_n$ highest bids in the auction. The price paid for every unit won is equal to the $2M_n+1$-st highest bid. We assume, \textit{without loss of generality}, that $b_1 \geq b_2$.\footnote{In this version of the paper we assume a zero reserve in the auctions. In auctions with reserves the expected price depends on the level at which the reserve is set. In fact, results can be obtained in any direction by setting the reserves appropriately. Nonetheless, it is possible to write down conditions on the reserve that ensure that the reserves play a sufficiently passive role in large auctions, so that our results continue to hold in auctions with reserves. These details are avoided here to keep the presentation simple.}

The number of bidders, the auction format, and the value distributions are all exogenously given and common knowledge before the auction begins.

\textit{Strategies and Equilibrium}

In each auction we consider (symmetric Bayes-Nash) equilibria in weakly undominated strategies. A strategy in the bundle auction is a function $b : S \rightarrow \mathbb{R}_+$. The bundle auction has an equilibrium in weakly dominant strategy of truthful bidding (\textit{i.e.}, bidding the true value of the bundle), thus in equilibrium $b(v_1, v_2) = v_1 + v_2$.

A strategy in the multi-unit auction is a pair of functions $(b_1, b_2) : S \rightarrow \{(b_1, b_2) \in \mathbb{R}_+^2 | b_1 \geq b_2\}$. Thus $b_1(v_1, v_2)$ is the bid for the first unit and $b_2(v_1, v_2)$ is the bid for the second unit with $b_1(v_1, v_2) \geq b_2(v_1, v_2)$.\footnote{In equilibrium the bidding strategy may depend on $n$ but we suppress this dependence in our notation.} Unlike the bundle auction, sincere bidding is a weakly dominant strategy in the multi-unit auction only for the first unit, so that in equilibrium $b_1(v_1, v_2) = v_1$. The equilibrium bidding strategy for the second unit cannot be described as a closed form mathematical expression except in special cases (see Engelbrecht-Wiggans and Kahn, 1998).

\section*{3. Rates of Convergence}
The impossibility of describing the equilibrium bidding strategy of the multi-unit auction as a closed form solution in general, makes the revenue comparison between the multi-unit and bundle auctions much more difficult than its dissimilar object counterpart. Moreover, while the equilibrium property of bidding strategies (i.e. differential bidding on successive units) in a multi-unit auction gives rise to allocative inefficiency in all finite auctions, the equilibrium in a bundle auction is also allocatively inefficient.\footnote{The traditional view is that all “standard rules” (without reserves) for single-object auctions give rise to allocative efficiency. Note that whenever an object is divisible with diminishing marginal values, even the single-object auction (for the whole) is allocatively inefficient.} However, an equilibrium bidding strategy in a multi-unit auction can be bounded above and below by the strategies \(b_1(v_1, v_2) = v_1, b_2(v_1, v_2) = v_2\) and \(b_1(v_1, v_2) = v_1, b_2(v_1, v_2) = 0\), respectively. This, in turn, makes it possible to describe explicit upper and lower bounds for the relevant quantities to make revenue and efficiency comparisons.

Let \(X_{j:n}\) denote the \(j\)-th highest order statistic from independent random variables \(X_1, ..., X_n\). Denote, also, by \(W_{j:2n}\) the \(j\)-th highest random variables from the ranked (marginal) values from \(n\) bidders \(V_{11}, V_{21}, V_{12}, V_{22}, ..., V_{1n}, V_{2n}\).\footnote{Note that these random variables are not all independent. Therefore, the standard results on order statistics cannot be used in this situation directly.} Using the above bound on \(b_2(v_1, v_2)\) the expected price in the multi-unit auction can be bounded above and below by \(E[W_{2M_n+1:2n}]\) and \(E[(V_1)_{2M_n+1:n}]\), respectively. The expected price per unit in the bundle auction is given by \(E[(V_1 + V_2)_{2n}/2]\). Hence, we can infer that the expected revenue in the multi-unit auction is greater than or less than that in the bundle auction based on whether

\[
E[(V_1)_{2M_n+1:n}] > E[(V_1 + V_2)_{M_n+1:n}/2] \tag{1}
\]

or,

\[
E[W_{2M_n+1:2n}] < E[(V_1 + V_2)_{M_n+1:n}/2]. \tag{2}
\]

It is straightforward to compare the auctions for large \(n\) if the two sides of one of the above inequalities tend to distinct limits and the limits can be ordered similarly. If, however, these quantities converge to the same limit (which happens when the prices
converge to 0 or 1), one way to obtain (1) and (2) for all large $n$ is to compare the rates of convergence. Specifically, we show that when the prices converge to 0 or 1, in a broad range of situations (as described by the regularity conditions below), the expected price in the multi-unit auction converges at a faster rate than that in the bundle auction.

We start by describing the rates at which the expected price and surplus (their relevant bounds whenever the quantities cannot be calculated exactly) in each auction converge. While the rate of convergence is of interest in its own right we focus on obtaining the above inequalities for large $n$.

**Notation.** In what follows, we keep the exposition simple by assuming that the fraction of demand that is met converges to, say, the limit $\alpha$, *i.e.*

$$\alpha \equiv \lim_n \frac{M_n}{n}$$

exists.\(^{16}\)

**Notation.** For an absolutely continuous function $H(x)$ with $H'(x) = h(x)$ on $[0, a]$ define

$$U(H) \equiv \min \{ l : h^{(l-1)}(a) \neq 0 \}$$

$$L(H) \equiv \min \{ l : h^{(l-1)}(0) \neq 0 \}$$

where $U(H)$ and $L(H)$, and all the lower order derivatives are assumed to exist.

The three Propositions below describe the rates of convergence in each of the cases where (i) a vanishing fraction of demand is met asymptotically, *i.e.* $\alpha = 0$ (ii) the fraction of demand met asymptotically is strictly between 0 and 1, *i.e.* $\alpha \in (0, 1)$, and (iii) a vanishing fraction of demand is left unfilled asymptotically, *i.e.* $\alpha = 1$.

**Proposition 1 ($\alpha = 0$).** The price in the multi-unit auction converges to 1 at a

\(^{16}\)The generalization to the case where the limit does not exist is straightforward.
rate not slower than \( \left( \frac{n}{M_n} \right)^{-\frac{1}{\tau(\frac{1}{2})}} \) and not faster than \( \left( \frac{n}{M_n} \right)^{-\frac{1}{\tau(\frac{1}{2})}} \). The price in the bundle auction converges at the rate \( n^{-\frac{1}{\tau(\frac{1}{2})}} \). The surplus per unit of supply in the multi-unit auction converges to 1 at a rate not slower than \( \left( \frac{n}{M_n} \right)^{-\frac{1}{\tau(\frac{1}{2})}} \). The surplus per unit of supply in the bundle auction converges to the limit of 1 at a rate of \( n^{-\frac{1}{\tau(\frac{1}{2})}} \).

**Proposition 2** \((\alpha \in (0, 1))\). The price in the multi-unit auction converges to the competitive prices in the limit at a rate no faster than \( n^{-\frac{1}{\alpha}} \) when \( \sqrt{n} \left( \frac{M_n}{n} - \alpha \right) \to c \in R \), and no faster than \( \frac{n}{M_n-\alpha} \) when \( \sqrt{n} \left( \frac{M_n}{n} - \alpha \right) \to \pm \infty \). The convergence of price in the bundle auction takes place exactly at these rates.

**Proposition 3** \((\alpha = 1)\). The expected price in the multi-unit auction converges to 0 at a rate not slower than \( n^{-\frac{1}{\tau(\frac{1}{2})}} \) if \( n - M_n \) is bounded, and \( \left( \frac{n}{2M_n-n+1} \right)^{-\frac{1}{\tau(\frac{1}{2})}} \) if \( n - M_n \to \infty \). The price in the bundle auction converges to 0 at a rate \( \left( \frac{n}{n-M_n} \right)^{-\frac{1}{\tau(\frac{1}{2})}} \).

The expected surplus per unit of supply in the multi-unit auction converges to \( E[V_1 + V_2]/2 \) at a rate not slower than \( n^{-\frac{1}{\tau(\frac{1}{2})}} \) if \( n - M_n \) is bounded above and converges to \( \frac{n}{2M_n} E[V_1 + V_2] \) at a rate not slower than \( \left( \frac{n}{2M_n-n+1} \right)^{-\frac{1}{\tau(\frac{1}{2})}} \) if \( n - M_n \to \infty \). The expected surplus per unit of supply in the bundle auction converges to 0 when centered at \( \frac{1}{2} E[V_1 + V_2] \) at a rate \( n^{-\frac{1}{\tau(\frac{1}{2})}} \) when \( n - M_n \) is bounded above, and to 0 when centered at \( \frac{n}{2M_n} E[V_1 + V_2] \) at a rate \( \left( \frac{n}{n-M_n} \right)^{-\frac{1}{\tau(\frac{1}{2})}} \) when \( n - M_n \to \infty \).\(^\text{19}\)

\(^{17}\) We will not use these upper bounds on the rates of convergence in the multi-unit auction when \( \alpha = 0 \) to derive our main result. This is just to illustrate that both the upper and lower bounds can be calculated. In the rest of this Proposition and the remaining Propositions we will avoid repetition by presenting only the rates that are relevant for obtaining the main result of the paper. The other bounds on the rates of convergence can be calculated similarly in all those cases.

\(^{18}\) \( M_n \) in these rates can be dropped whenever \( M_n \) bounded. Similar simplifications can be made in the rates in Proposition 3.

\(^{19}\) When \( n - M_n \to \infty \) the surplus per unit of supply in fact converges to \( E[V_1 + V_2]/2 \) at the rate \( n^{-\frac{1}{\alpha M_n}} \) in both the multi-unit and the bundle auction, and is thus independent of the behavior of the relevant distribution at 0. This, as will be understood later, is not useful for comparing the two auction formats. The dependence of the convergence on the behavior of the distribution at 0 can be identified by calculating the rate of convergence relative to the sequence \( \frac{n}{M_n} E[V_1 + V_2] \), instead. Hence, we
4. Multi-Unit vs. Bundle Auctions

When the values follow one of many standard distributions, the expected revenue in the multi-unit auction is larger (respectively, smaller) than that in the bundle auction when the fraction of supply met asymptotically is in a small neighborhood of 0 (respectively, 1). We therefore, look for general conditions on value distributions that guarantee similar behavior. As discussed earlier, when \( \alpha \in (0, 1) \) the limiting prices are distinct and can be compared directly to draw inferences about large auctions. However, when \( \alpha = 0 \) or \( \alpha = 1 \) the only way that the prices and surpluses can be compared for large auctions is by comparing the rates at which these quantities converge to their common limits. The following regularity conditions guarantee that the convergence of price, for instance, is faster under the multi-unit format when \( \alpha = 0 \) and \( \alpha = 1 \), thus making price per unit of supply in the multi-unit auction higher (respectively, lower) than that in the bundle auction in all large auctions whenever \( \alpha = 0 \) (respectively, \( \alpha = 1 \)).

(R1) Regularity condition at the upper end of the support

There is a \( k \geq 1 \) such that (i) \( F_1(\cdot) \) and \( F_{V_1+V_2}(\cdot) \) are \( k \)-times continuously differentiable in some left neighborhoods of 1 and 2, respectively, (ii) \( f_1^{(l)}(1) = 0 \) for \( l = 0, 1, \ldots, k - 2 \), and \( f_1^{(k-1)}(1) \neq 0 \), and (iii) \( f_{V_1+V_2}^{(l)}(2) = 0 \) for \( l = 0, 1, \ldots, k - 1 \), where \( f_1^{(0)} \equiv f_1 \) and \( f_{V_1+V_2}^{(0)} \equiv f_{V_1+V_2} \).

Remark 1. It is straightforward to check that the condition is automatically satisfied if \( f_1(1) \neq 0 \) (this case corresponds to \( k = 1 \)), and \( f_1 \) and \( f_{V_1+V_2} \) are differentiable in some left neighborhoods of the upper ends of their supports. In that case, \( f_{V_1+V_2}(2) = 0 \) and \( f_{V_1+V_2}^{(1)}(2) = 0 \). In fact, the condition \( f_{V_1+V_2}(2) = 0 \) always holds.

Sometimes the analysis of multi-unit auction is made more tractable by assuming that the values for the successive units are the higher and lower order statistics for two independent draws from a distribution with density, say, \( h(\cdot) \). The regularity condition is satisfied in that special framework regardless of the particular distribution \( h(\cdot) \) whenever \( h(\cdot) \) is continuous in a left neighborhood of 1.

\[ \frac{n}{2M_n} E[V_1 + V_2] \] present the rate of convergence relative to this case to extract the dependence of the rate of convergence on the distribution.
Remark 2. If for some $k > 1$ (i) $f_{V_1}(\cdot)$ and $f_{V_1 + V_2}(\cdot)$ are $k$-times differentiable in a left neighborhood of 1 and 2, respectively, and $f^{(k)}_{V_1}(1) = 0, l = 0, \ldots, k - 1$, and (ii) $f_{V_2|V_1}(y|x)$ is uniformly bounded in $y$ for $x > 1 - \delta^*$ for some $\delta^* > 0$, then $f^{(l)}_{V_1 + V_2}(2) = 0$, $l = 0, \ldots, k$ and the regularity condition (R1) is satisfied whenever $f^{(k-1)}_{V_1}(1) \neq 0$. This sufficient condition (for (R1) to hold) essentially means that the values $V_1$ and $V_2$ do not become increasingly and highly correlated conditional on $V_1$ tending to its highest possible value of 1. This also gives an idea on how the condition can be violated. Examples of probability distributions can in fact be constructed to violate (R1). For instance, define $\tilde{S}$ to be the set bounded by the lines that join $(0, \frac{1}{2}), (\frac{1}{2}, 0), (1, 1)$. Let $F$ be the joint distribution of the higher and lower order statistics of two independent draws from a uniform distribution on $\tilde{S}$. Then $F$ violates (R1).

(R2) Regularity condition at the lower end of the support

There is a $k \geq 1$ such that (i) $F_2(\cdot)$ and $F_{V_1 + V_2}(\cdot)$ are $k$-times continuously differentiable in some left neighborhood of 0, respectively, (ii) $f^{(l)}_2(0) = 0$ for $l = 0, 1, \ldots, k - 2$, and $f^{(k-1)}_2(1) \neq 0$, and (iii) $f^{(l)}_{V_1 + V_2}(0) = 0$ for $l = 0, 1, \ldots, k - 1$.\(^{20}\)

(R1) guarantees that when $\alpha = 0$, $E[(V_1)_{2M_n+1:n}]$, a lower bound on the price in the multi-unit auction, converges to 1 at a rate faster than $E[(V_1 + V_2)_{M_n+1:n}]/2$, the price per unit of supply in the bundle auction. Similarly, (R2) guarantees that when $\alpha = 1$, an upper bound on the price in the multi-unit auction converges to 0 at a rate faster than $E[(V_1 + V_2)_{M_n+1:n}]/2$. When a fraction $\alpha \in (0, 1)$ of the demand is met asymptotically the price in the bundle auction converges to $F_{V_1 + V_2}^{-1}(1 - \alpha)$ - the probability limit of the order statistic $(V_1 + V_2)_{M_n+1:n}$. Now consider the multi-unit auction. It follows upon applying the results of Swinkels (2001) and that of Chakraborty and Engelbrecht-Wiggans (2005) that the price in the multi-unit auction converges to $\left(\frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2}\right)^{-1}(1 - \alpha)$. A “convex ordering” based argument then makes a direct comparison of these limits possible to give us the following Theorem:

**Theorem.** Suppose that (R1) and (R2) hold.

\(^{20}\)(R2) is a mirror image of (R1) and the counterparts of Remarks 1 and 2 hold in this case.
There is a $\alpha_* \leq \alpha^*$ in the interval $(0, 1)$ such that if $\alpha \in (0, \alpha_*)$ the multi-unit auction generates a strictly lower revenue than the bundle auction in all sufficiently large auctions. If $\alpha \in (\alpha^*, 1)$ the multi-unit auction generates a strictly larger revenue than the bundle auction in all sufficiently large auctions and that the number of units supplied is bounded above.

The multi-unit auction is more efficient than the bundle auction for all large number of bidders if $\alpha \notin E = \{x \in (0, 1) : F_1(x) + F_2(x) = 2F_{V_1+V_2}(2x)\}$.

**Remark 3.** Now, let us examine the effect of violating (R1) on our Theorem. From Remark 2 one would guess that one way to violate the regularity condition is to consider values that become highly correlated towards the upper end of the support which would imply that the behavior of $f_1(\cdot)$ does not differ very much from the behavior of $f_{X+Y}(\cdot)$ in the limit. Let us consider the extreme case where both values are equal with probability 1, and has a distribution $H(\cdot)$ on $\{(v,v) : v \in [0,1]\}$. In that case, the equilibrium in the multi-unit auction involves bidding zero on the second unit if the “bounded hazard rate” condition of Engelbrecht-Wiggans and Kahn (1998) is satisfied.

Standard arguments show that regardless of the number of bidders the multi-unit auction in this case generates a lower revenue than a bundle-auction. Intuitively, if the marginal value for the second unit is very likely to be as large as that for the first unit (and increasingly so for the top few bidders who matter as the number of bidders increases), demand reduction (from the high bidders) can be very costly to the seller; so costly that even a large competitive effect with many bidders cannot compensate the revenue loss in the multi-unit auction.

**Remark 4.** The multi-unit auction under the “Vickrey auction rule” generates a higher expected revenue than under the uniform-price rule with complete demand reduction. Therefore, the revenue performance of multi-unit auction is better than the bundle auction under the Vickrey rule, as well, for all large $n$ when a vanishing fraction of the the demand is met asymptotically.\(^{21}\) Moreover, it is easy to see that the multi-unit

\(^{21}\)We thank Ron Harstad for pointing out this fact.
Vickrey auction is more efficient than the bundle auction regardless of the size of the auction and supply.

**Remark 5.** It is easy to see that the set $E$ is nonempty whenever (R1) and (R2) hold. Moreover, that it is at most countable for several standard probability distributions. In fact, the only way that it can be larger than countable is if $F_V(x) + F_2(x) = 2F_{V_1+V_2}(2x)$ over some interval $[x_*, x^*] \subset [0, 1]$. In particular, we are not aware of an example of $f(\cdot, \cdot)$ that satisfies this condition as well as $f(v_1, v_2) > 0$ in the interior of $S$.

While the relative efficiency of large multi-unit auctions holds unambiguously (except, perhaps, on at most countably many points), whether the relative revenue-performance of large multi-unit auctions is better or worse than its bundled counterpart depends on the fraction of demand that is met asymptotically. This switch in revenue performances takes place monotonically when the following regularity condition holds.

(R3) A “single-crossing” condition for probability distributions

There is a unique $x^* \in (0, 1)$ such that

$$
\frac{1}{2}F_{V_1}(x) + \frac{1}{2}F_{V_2}(x) > \frac{1}{2}F_{V_1+V_2}(x) \quad \text{if} \quad x \in (0, x^*)
$$

$$
< \frac{1}{2}F_{V_1+V_2}(x) \quad \text{if} \quad x \in (x^*, 1).
$$

**Corollary.** Suppose a fraction $\alpha \in [0, 1]$ of the demand is met asymptotically. The multi-unit auction generates a higher or lower revenue than the bundle auction for all large number of bidders depending on whether the fraction of demand met asymptotically is smaller or larger than $\alpha^*$ for some $\alpha^* \in (0, 1)$ if and only if condition (R3) holds.$^{22}$

**Remark 6.** The regularity condition (R3) implies that the convolution of random variables $V_1$ and $V_2$ brings the probability mass towards its “center.” In other words, the

$^{22}$It is easy to see that when $\alpha = \alpha^*$ the revenue performance of multi-unit auction relative to the bundle auction is ambiguous even in large auctions.
mixture distribution $\frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2}$ is larger than $F_{\frac{V_1+V_2}{2}}$ in the sense of a dispersion-based ranking. In fact, part of the proof of the Theorem involves showing that in general $\frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2} \succ F_{\frac{V_1+V_2}{2}}$ in the sense of convex ordering. 

(R3) requires an ordering in a stronger sense that involves a type of “single-crossing property” for distributions. Nonetheless, (R3) is satisfied by several distributions. Suppose that $V_1$ and $V_2$ are the higher and lower order statistics of two independent draws $X$ and $Y$ from a distribution $H(\cdot)$. Then (R3) is satisfied whenever $X$ and $Y$ satisfy the regularity condition of Chakraborty (1999) (which is itself known to be satisfied by several classes of distributions). Incidentally, (R3) also means that $F_{\frac{V_1+V_2}{2}}$ has a greater kurtosis than the distribution $\frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2}$ (see van Zwet, 1967).

At this point it may seem that our main result is qualitatively the same as that of Chakraborty (1999). We present an example below to show that the monotonicity as stated in the Corollary does not hold as generally here.

**Example: Increasing $\alpha$ may not have a monotonic effect on revenue.**

Let $X \sim U[0, 1]$ and $Y \sim U[0.4, 0.6]$, with the distribution functions being denoted by $G_X$ and $G_Y$. Suppose that $V_1$ and $V_2$ are the higher and lower order statistics for independent draws from $G_X$ and $G_Y$. Then we have $\frac{1}{2}F_{V_1}(x) + \frac{1}{2}F_{V_2}(x) = \frac{1}{2}G_X(x) + \frac{1}{2}G_Y(x)$ and $F_{\frac{V_1+V_2}{2}}(x) = G_{\frac{X+Y}{2}}(x)$ where $G_{\frac{X+Y}{2}}(\cdot)$ is the distribution of $\frac{X+Y}{2}$. It follows that there are $x_1, x_2, x_3$ with $0 < x_1 < x_2 < x_3 < 1$ such that

$$
\frac{1}{2}F_{V_1}(x) + \frac{1}{2}F_{V_2}(x) - F_{\frac{V_1+V_2}{2}}(x) > 0 \text{ for } x \in (0, x_1) \text{ and } x \in (x_2, x_3)
$$

$$
< 0 \text{ for } x \in (x_1, x_2) \text{ and } x \in (x_3, 1).
$$

Thus, (R3) is violated by $F$. This has the implication that there exist $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$ such that if $\alpha \in (0, \alpha_1)$ or $\alpha \in (\alpha_2, \alpha_3)$ then multi-unit auction generates a higher expected revenue than bundle auction for all large number of bidders. However, if $\alpha \in (\alpha_1, \alpha_2)$ or $\alpha \in (\alpha_3, 1)$ then the bundle auction generates a strictly higher expected revenue than the multi-unit auction for all large number of bidders. In short, as the fraction of demand met asymptotically increases, it is no more the case that
the revenue-performance of large multi-unit auctions changes in the monotonic manner that simple intuition on the competitive effect of a decreasing supply would make it appear.

It is also interesting to note that a direct application of the Theorem 1 of Watson and Gordon (1986) implies that the convolution of $G_X$ and $G_Y$ have the spread-reducing effect in the sense of the regularity condition of Chakraborty (1999). In short, monotonicity of the competitive effect as the fraction of demand met asymptotically decreases continues to hold with these same distributions in the context of the dissimilar object (as discussed above). This further highlights the difference between the multi-unit set-up and auctions with dissimilar objects.

7. Concluding Discussion
We showed that in a broad class of situations large markets tend to favor unbundled sales from an efficiency point of view. The impact of bundling on the seller’s revenue in such situations, however, depends on the fraction of the demand that is met. A profit-maximizing seller tends to prefer the socially efficient multi-unit format only if the supply is small relative to the demand. Thus, from a market decentralization point of view, the market does not necessarily become efficient by virtue of having a sufficiently large number of buyers.

Similar revenue effect of changing demand relative to the supply is also observed in Chakraborty (1999). The difference is that there the variation in the demand-supply ratio is obtained as the number of bidders increases while the supply of objects remains fixed. The expected price in that case traverses across the support of the value distribution (although in discrete jumps since the number of bidders can only increase discretely) as does the price in our model (although continuously) when the fraction of demand met asymptotically changes (continuously). However, the switch in the relative revenue performances of the bundled and unbundled auctions as the fraction of demand that is met, changes monotonically under far less general conditions relative to Chakraborty (1999). This difference highlights the dissimilarity between auctions for multiple units and those for multiple (dissimilar) objects.
Our results show that for each level of $\alpha$ – the fraction of demand met asymptotically – there is a $n^*$ such that a decisive revenue ranking between bundle and multi-unit auctions holds for all number of bidders larger than $n^*$. However, in more than a subtle way throughout the paper we emphasized the “tendency of large market” aspect of the results rather than the particular value of the cut-off $n^*$. The reason is that when the switch in the revenue performance does not happen at a unique $n^*$, identifying a critical number at which a cross-over takes place is not sufficient for the identification of the $n^*$ at which the crossover takes place permanently as the number of bidders increases. The rate of convergence approach can show the existence of $n^*$ for each $\alpha$ and turns out to be very useful for a general analysis of large markets. However, a useful empirical or econometric application of this theory along the lines of, say, Brendstrup and Paarsch (2005) is likely to require an identification of $n^*$\textsuperscript{23}. Whether a somewhat general technique can be devised for such analyses is left to future research efforts in the area.

On the technical side, we have developed some useful tools to analyze the asymptotics of an interesting class of problems. Ordering and ranking of actions can be an important element of market models even outside our framework (e.g. bilateral bargaining models). The typical analysis of such models make heavy use of results on order statistics. Consider the situations where each agent’s action is a multivariate random variable with a joint distribution (from the perspective of the other agents) before being pooled with the actions of the other agents and ranked. There are no existing statistical results that could be used to analyze such order statistics. In course of proving our results on multi-unit auctions we have developed results on the behavior of such order statistics that is likely to make an asymptotic analysis of these other models possible.

Consider, for instance, the multi-unit version of the double-bid auction model of Rustichini, Satterthwaite and Williams (1994). Suppose that there are $M^B_n$ buyers and $M^S_n$ sellers and $\alpha = \lim_{n \to \infty} M^S_n / M^B_n$. Each buyer has diminishing (or increasing) marginal

values for $m^B$ units and each seller has $m^S$ units to sell with diminishing (or increasing) opportunity costs for the successive units. Let $F^B_r$ and $F^S_r$ be the marginal distribution corresponding to the $r$-th unit for the buyer and the seller, respectively. Whenever such a double-bid market converges to efficiency in the limit, similar techniques can be applied to show that the demand (respectively, supply) in the limit competitive market on a per buyer (respectively, per seller) basis is given by $m^B - \sum_{r=1}^{m^B} F^B_r(p)$ (respectively, $\sum_{r=1}^{m^S} F^S_r(p)$). The limit competitive price\textsuperscript{24} can then be used to analyze the large double-bid market in question. In fact, a number of numerical examples indicate that as $\alpha$ increases from 0 to $\infty$ the ranking of the price in such large markets under bundle and multi-unit formats behave in a similar manner as do the prices in the Corollary when $\alpha$ increases from 0 to 1. The details of a more general observation will, of course, require an elaborate investigation.

8. Appendix

**Notations.** We shall denote weak convergence of distributions/random variables by $\rightarrow$ and convergence in probability by $\rightarrow_p$. In order to keep the presentation uniform we refrain from considering convergence in the almost sure sense. All limits are, unless specified otherwise, as $n$ approaches infinity. By $X \doteq Y$ we mean that the random variables $X$ and $Y$ have the same distribution.

Gamma$(\alpha, \beta)$ will be used, depending on the context, to denote both a gamma variate and a gamma distribution with mean $\alpha \beta$ and variance $\alpha \beta^2$. Similarly, $N(\mu, \sigma^2)$ will denote the normal distribution with mean $\mu$ and variance $\sigma^2$, and $U(0, 1)$ the uniform distribution on $(0, 1)$. The standard normal distribution function will be denoted by $\Phi(\cdot)$. The norm $\| \cdot \|$ will denote the total variation norm on the space of probability measures.

In the following $U_{j,n}$ will stand for the $j$-th highest order statistic from a random sample of size $n$ from the uniform distribution on $(0, 1)$. All results that we use for

\textsuperscript{24}As characterized by $m^B - \sum_{r=1}^{m^B} F^B_r(p) = \alpha \sum_{r=1}^{m^S} F^S_r(p)$, i.e. equating the demand and the supply in the limit competitive market.
order statistics from $U(0, 1)$ can be found in Reiss (1989). We summarize these in the following two lemmas - the first deals with weak convergence results and the second with the asymptotic behavior of moments.

**Lemma A1.** (i) For any positive integer $j$,

$$n(1 - U_{j:n}) \xrightarrow{d} \text{Gamma}(j, 1).$$

(ii) For any sequence of positive integers $\{j_n\}_{n \geq 1}$ satisfying

$$\frac{j_n}{n} \rightarrow 0 \text{ and } j_n \rightarrow \infty,$$

we have

$$\left( \frac{n}{j_n} \right) \left( 1 - U_{j_n:n} \right) \xrightarrow{P} 1.$$

(iii) For any sequence of positive integers $\{j_n\}_{n \geq 1}$ satisfying

$$j_n \rightarrow \infty \text{ and } (n - j_n) \rightarrow \infty,$$

we have

$$\left( \frac{n}{j_n} \left( 1 - j_n/n \right)\right)^{\frac{1}{2}} \left( \left( 1 - \frac{j_n}{n} \right) - U_{j_n:n} \right) \xrightarrow{d} N(0, 1)$$

**Proof.** (i) The proof follows simply as a weaker version of Lemma 5.1.5 of Reiss (1989).

(ii) To prove this part we use the fact that

$$\left( \frac{n}{\sqrt{j_n}} \right) \left( U_{j_n:n} - 1 + \frac{j_n}{n} \right) \xrightarrow{d} N(0, 1),$$

which can be deduced from theorem 5.1.7 of Reiss (1989). Hence, using Slutsky’s theorem we have

$$\left( \frac{n}{j_n} \right) (1 - U_{j_n:n}) = \frac{1}{\sqrt{j_n}} \left[ \left( \frac{n}{\sqrt{j_n}} \left( 1 - \frac{j_n}{n} - U_{j_n:n} \right) \right) \right] + 1 \xrightarrow{d} 1.$$

(This result follows much more easily from the second part of the following Lemma.)

(iii) This is a standard asymptotic normality property of central sequences. See, for instance, Reiss (1989).
Lemma A2. (i) For any positive integer $j$,
\[
n^\zeta E[(1 - U_{j:n})^\zeta] \longrightarrow \frac{\Gamma(j + \zeta)}{\Gamma(j)}, \forall \zeta \geq 0.\]

(ii) For any sequence of positive integers $\{j_n\}_{n \geq 1}$ satisfying
\[
\frac{j_n}{n} \longrightarrow 0 \text{ and } j_n \longrightarrow \infty,
\]
we have
\[
\left(\frac{n}{j_n}\right)^\zeta E[(1 - U_{j_n:n})^\zeta] \longrightarrow 1, \forall \zeta \geq 0.
\]

(iii) For every positive integer $k$ and $j \in \{1, 2, ..., n\}$,
\[
E[|U_{j:n} - \mu|^k] \leq 2k!5^k\sigma^k n^{-\frac{k}{2}},
\]
with
\[
\mu = 1 - \frac{j}{n+1} \text{ and } \sigma^2 = \mu(1 - \mu).
\]

Proof. Parts (i) and (ii) follow from expressing the concerned moments in terms of the beta function and then using the Stirling’s approximation for the gamma function.

Part (iii) is Lemma 3.1.3 of Reiss (1989).

We use the Stein-Chen method for Poisson approximation and in particular the following result from Lindvall (1992).

Lemma A3. Let $\{Z_i\}_{i \leq n}$ be a finite sequence of Bernoulli variables with expectations $\{p_i\}_{i \leq n}$, respectively. Let $S \equiv \sum_{i=1}^n Z_i$ and $\lambda \equiv E[S]$. Moreover, let $\{R_i\}_{i \leq n}$ and $\{T_i\}_{i \leq n}$ be such that
\[
R_i \overset{d}{=} S \text{ and } 1 + T_i \overset{d}{=} P[S \in \cdot | Z_i = 1].
\]

Then,
\[
\|P[S \in \cdot] - \text{Poisson}(\lambda)\| \leq 2(1 + \lambda^{-1}) \sum_{i=1}^n p_i E[|R_i - T_i|].
\]
In the following two lemmas, $G(\cdot)$ will denote a univariate distribution function with
density $g(\cdot)$ and support $[0, a]$. Moreover, we assume that $G(\cdot)$ is $k$-times
differentiable in a left neighborhood of $a$ with $g^{(l)}(a) = 0$ for $l = 0, 1, \ldots, k - 2$ and $g^{(k-1)}(a) \neq 0$. For
such a $G(\cdot)$, let
\[ C(G, k) \equiv \left( \frac{k!(-1)^{k-1} g^{(k-1)}(a)^{\frac{1}{k}}}{g^{(k-1)}(a)} \right). \]
$V_{j:n}$, for a non-negative $j$ between $0$ and $n$, will denote the $j$-th highest order statistic
from a random sample of size $n$ from $G(\cdot)$.

**Lemma A4.**
\[ n^\frac{1}{k} E[(a - V_{j:n})] \longrightarrow C(G, k) \frac{\Gamma(j + \frac{1}{k})}{\Gamma(j)}. \]

**Proof.** Using the transformation $G^{-1}(\cdot)$, we see that
\[ (a - V_{j:n}) \overset{d}{=} (G^{-1}(1) - G^{-1}(U_{j:n})) , \]
which converts the problem to one in order statistics from $U(0, 1)$. Therefore, it suffices to show that
\[ n^\frac{1}{k} E[G^{-1}(1) - G^{-1}(U_{j:n})] \longrightarrow C(G, k) \frac{\Gamma(j + \frac{1}{k})}{\Gamma(j)}. \]
Towards showing the weak convergence of $n^\frac{1}{k} (G^{-1}(1) - G^{-1}(U_{j:n}))$ and finding its weak
limit, note that writing
\[ X_n \equiv \frac{G^{-1}(1) - G^{-1}(U_{j:n})}{(1 - U_{j:n})^{\frac{1}{k}}} \quad \text{and} \quad Y_n \equiv (n(1 - U_{j:n}))^{\frac{1}{k}}. \]
we have
\[ n^\frac{1}{k} \left( G^{-1}(1) - G^{-1}(U_{j:n}) \right) = X_n \cdot Y_n \]
where by Lemma A1(i), $Y_n \overset{d}{\longrightarrow} Y$ with $Y \overset{d}{=} \text{Gamma}(j, 1)$ and $X_n \overset{p}{\longrightarrow} C(G, k)$. The
latter holds as $U_{j:n} \overset{p}{\longrightarrow} 1$ and $X_n = \psi(U_{j:n})$ where $\psi(\cdot)$, defined by
\[ \psi(x) \equiv \left( \frac{G^{-1}(1) - G^{-1}(x)}{(1 - x)^{\frac{1}{k}}} \right), \forall x \in [0, 1] \]
is a continuous function with $C(G,k)$ as the limit at 1. The limit is derived using the Young’s form of the Taylor’s theorem and our assumptions on the behavior of $G(\cdot)$ at $a$. Hence, using Slutsky’s theorem, we have

$$n^\frac{1}{k} (G^{-1}(1) - G^{-1}(U_{j:n})) \xrightarrow{d} C(G,k) \cdot Y.$$  

Now all that remains is to show the $L^1$ convergence which we prove by showing that the sequence is bounded in the $L^2$ sense. Now $\{n^\frac{1}{k}(G^{-1}(1) - G^{-1}(U_{j:n}))\}_{n \geq 1}$ will be $L^2$-bounded if so is $\{Y_n\}_{n \geq 1}$ as $\psi(\cdot)$ is non-negative and bounded (due to continuity on a compact interval). But $\{Y_n\}_{n \geq 1}$ is $L^2$-bounded by Lemma A2(i). The proof then becomes complete upon observing that $E[Y] = \frac{r(j+\frac{1}{2})}{r(j)}$.

**Lemma A5.** For any sequence of positive integers $\{j_n\}_{n \geq 1}$ satisfying

$$\frac{j_n}{n} \rightarrow 0 \text{ and } j_n \rightarrow \infty$$

we have

$$\left(\frac{n}{j_n}\right)^\frac{1}{k} E[(a - V_{j_n:n})] \rightarrow C(G,k).$$

**Proof.** The proof follows along similar lines as Lemma A4 - the changes being using part of (ii) Lemma A1 and Lemma A2 instead of the part (i) used above and the rate of convergence now is in terms of $j_n^{-1}n$ instead of $n$.

**Lemma A6.** Suppose that $\frac{M_n}{n} \rightarrow 0$ and $M_n \rightarrow \infty$. Then

$$\frac{1}{M_n} \sum_{j=1}^{M_n} (a - E[V_{j:n}])$$

converges to 0 at the rate $(\frac{M_n}{n})^{\frac{1}{k}}$ where $k$ is the first non-zero derivative of $G(\cdot)$.

**Proof.** Let

$$\psi_n \equiv \frac{1}{M_n} \sum_{j=1}^{M_n} (a - V_{j:n}).$$

Then

$$\psi_n \leq a - V_{M_n:n}, \quad (3)$$

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and if $k_n \equiv \lfloor M_n(1 - \varepsilon) \rfloor$ for some $0 < \varepsilon < 1$, then

$$\psi_n \geq \varepsilon (a - V_{k_n:n}). \quad (4)$$

From (3) and Lemma 5 we have

$$\limsup_{n \to \infty} \left( \frac{n}{M_n} \right)^{\frac{1}{k}} E\psi_n \leq \limsup_{n \to \infty} \left( \frac{n}{M_n} \right)^{\frac{1}{k}} (a - E[V_{M_n:n}])$$

$$= C(G, k).$$

Using (4) and Lemma 5 we have

$$\liminf_{n \to \infty} \left( \frac{n}{M_n} \right)^{\frac{1}{k}} E\psi_n = (1 - \varepsilon)^{\frac{1}{k}} \liminf_{n \to \infty} \left( \frac{n}{k_n} \right)^{\frac{1}{k}} E\psi_n$$

$$= (1 - \varepsilon)^{\frac{1}{k}} \varepsilon C(G, k) > 0$$

Hence the proof. \qed

In the following corollaries, instead of the earlier assumptions on the behavior of $G(\cdot)$ at the upper end of its support, we will assume that it is $k$-times differentiable in a right neighborhood of 0 with $g^{(l)}(0) = 0$ for $l = 0, 1, \ldots, k - 2$ and $g^{(k-1)}(0) \neq 0$. For such a $G(\cdot)$, define

$$C^*(G, k) \equiv \left( \frac{k!( -1)^{k-1}}{g^{(k-1)}(0)} \right)^{\frac{1}{k}}.$$

**Corollary A7.** For a positive integer $j$, we have

$$n^\frac{1}{k} E[V_{n+1-j:n}] \to C^*(G, k) \frac{\Gamma(j + \frac{1}{k})}{\Gamma(j)}.$$

**Proof.** If $F(\cdot)$ is defined by,

$$F(x) = 1 - G(a - x), \quad \forall x \in [0, a]$$

then

$$G^{-1}(y) = a - F^{-1}(y) = F^{-1}(1) - F^{-1}(1-y), \quad \forall y \in [0, 1],$$

\footnote{Note that $\frac{k_n}{n} \to 0$ as $k_n \to \infty$.}
$F$ satisfies all the conditions of Lemma A4, and $C^*(G, k) = C(F, k)$. Moreover, as $U_{j:n} \xrightarrow{d} 1 - U_{n+1-j:n}$, we have

$$V_{n+1-j:n} \xrightarrow{d} G^{-1}(U_{n+1-j:n}) = F^{-1}(1) - F^{-1}(1-U_{n+1-j:n}) \xrightarrow{d} F^{-1}(1) - F^{-1}(U_{j:n}) = F_{j:n}^*,$$

where $V_{j:n}^*$ is the $j$-th highest order statistic from a random sample of size $n$ from $F(\cdot)$. Hence the result follows from Lemma A4.

**Corollary A8.** For any sequence of positive integers $\{j_n\}_{n \geq 1}$ satisfying

$$\frac{j_n}{n} \to 0 \quad \text{and} \quad j_n \to \infty,$$

we have

$$\left( \frac{n}{j_n} \right)^{\frac{1}{2}} E[V_{j_n:n}^*] \to C^*(G, k).$$

**Proof.** By an argument similar to that of Corollary A7, the result follows from Lemma A5.

**Corollary A9.** Suppose that $\frac{M_n}{n} \to 1$ and $n - M_n \to \infty$. Then

$$\frac{1}{2M_n} \sum_{r=1}^{M_n} E(V_1 + V_2)_{r:n}$$

and

$$\frac{1}{2M_n} \left[ \sum_{r=1}^{n} E(V_1)_{r:n} + \sum_{r=1}^{2M_n-n} E(V_2)_{r:n} \right]$$

centered at $\frac{n}{2M_n} E[V_1 + V_2]$ converges to 0 at rates $(1 - \frac{M_n}{n})^{1 + \frac{1}{\pi(\sqrt{2})}}$ and $(1 - \frac{M_n}{n})^{1 + \frac{1}{\pi(\sqrt{2})}}$, respectively.

When $n - M_n$ is finite,

$$\frac{1}{2M_n} \sum_{r=1}^{M_n} E[(V_1 + V_2)_{r:n}]$$

and

$$\frac{1}{2M_n} \left[ \sum_{r=1}^{n} E[(V_1)_{r:n}] + \sum_{r=1}^{2M_n-n} E[(V_2)_{r:n}] \right]$$

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centered at $E[V_1 + V_2]/2$ at the rate $n^{-\left(1 + \frac{1}{\kappa_0(V_1 + V_2)}\right)}$ and not slower than $n^{-\left(1 + \frac{1}{\kappa_0(V_1 + V_2)}\right)}$, respectively.

**Proof.** Define $\mu \equiv E[V_1 + V_2]/2$. We have

$$
\frac{1}{2M_n} \left[ \sum_{r=1}^{M_n} E[(V_1 + V_2)_{r:n}] \right] - \frac{n}{M_n} \mu
= - \frac{\sum_{r=M_n+1}^{n} E[(V_1 + V_2)_{r:n}]}{2M_n}
= - \left( \frac{n - M_n}{M_n} \right) \sum_{r=1}^{k_n} E[(V_1 + V_2)_{n-r+1:n}] \frac{1}{2k_n}
= - \left( \frac{n}{M_n} \right) \left( \frac{n - M_n}{n} \right) O \left( \frac{k_n}{n} \right)^{\frac{1}{\kappa_1(V_1 + V_2)}}
$$

where $k_n \equiv n - M_n$. Hence the LHS is $O \left( \left( \frac{n-M_n}{n} \right)^{1+\frac{1}{\kappa_0(V_1 + V_2)}} \right)$. Similarly,

$$
\frac{1}{2M_n} \left[ \sum_{r=1}^{n} E[(V_1)_{r:n}] + \sum_{r=1}^{2M_n-n} E[(V_2)_{r:n}] \right] - \frac{n}{M_n} \mu
= - \frac{\sum_{r=2M_n-n+1}^{n} E[(V_2)_{r:n}]}{2M_n}
$$

is $O \left( \left( \frac{n-M_n}{n} \right)^{1+\frac{1}{\kappa_0(V_2)}} \right)$.

If $n - M_n$ is bounded, then we have

$$
\frac{1}{2M_n} \left[ \sum_{r=1}^{M_n} E[(V_1 + V_2)_{r:n}] \right] - \frac{n}{M_n} \mu
= - \left[ \frac{n - M_n}{M_n} \right] \sum_{r=1}^{k_n} E[(V_1 + V_2)_{r:n}] \frac{1}{k_n}
= - \left( \frac{n}{M_n} \right) \left[ \frac{n - M_n}{M_n} \right] O \left( n \right)^{-\frac{1}{\kappa_0(V_1 + V_2)}}
$$

since each term in the summation is $n^{-\frac{1}{\kappa_0(V_1 + V_2)}}$. Therefore, $\frac{1}{2M_n} \sum_{r=1}^{M_n} E[(V_1 + V_2)_{r:n}]$ centered at $E[V_1 + V_2]/2$ converges to 0 at the rate $n^{-\left(1 + \frac{1}{\kappa_0(V_1 + V_2)}\right)}$. 

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Similarly, \( \frac{1}{2M_n} \left[ \sum_{r=1}^{n} E[(V_1)_r:n] + \sum_{r=1}^{2M_n-n} E[(V_2)_r:n] \right] \) centered at \( E[V_1+V_2]/2 \) converges to 0 at the rate \( n^{-(1+1/k_0(F_2))} \).

**Lemma A10.** Let \( G \) be a distribution function with density \( g \) and \( V_{j:n} \), for a nonnegative integer \( j \) between 0 and \( n \) be the \( j \)-th highest order statistic from a random sample of size \( n \) from \( G \). Let \( \alpha \in (0, 1) \) be such that \( g(G^{-1}(\alpha)) > 0 \). Then the following hold:

(i) For \( \{j_n\}_{n \geq 1} \) satisfying,

\[
\sqrt{n} \left( \frac{j_n}{n} - (1 - \alpha) \right) \longrightarrow c \in R,
\]

we have

\[
\sqrt{n} E[G^{-1}(\alpha) - V_{j_n:n}] \longrightarrow \frac{c}{g(G^{-1}(\alpha))}.
\]

(ii) For \( \{j_n\}_{n \geq 1} \) satisfying,

\[
\sqrt{n} \left( \frac{j_n}{n} - (1 - \alpha) \right) \longrightarrow \pm \infty \quad \text{and} \quad \frac{j_n}{n} \longrightarrow 1 - \alpha,
\]

we have

\[
\left( \frac{n}{j_n - n(1 - \alpha)} \right) E[G^{-1}(\alpha) - V_{j_n:n}] \longrightarrow \frac{1}{g(G^{-1}(\alpha))}.
\]

**Proof.** (i) We start by observing that \( \sqrt{n}(\alpha - U_{j_n:n}) \overset{d}{\longrightarrow} N(c, \alpha(1 - \alpha)) \) since

\[
\sqrt{n}(\alpha - U_{j_n:n}) = \sqrt{n} \left( \left( 1 - \frac{j_n}{n + 1} \right) - U_{j_n:n} \right) + \sqrt{n} \left( \frac{j_n}{n + 1} - (1 - \alpha) \right),
\]

and

\[
\sqrt{n} \left( \frac{1}{n + 1} - U_{j_n:n} \right) \overset{d}{\longrightarrow} N(0, \alpha(1 - \alpha)) \quad \text{and} \quad \sqrt{n} \left( \frac{j_n}{n + 1} - (1 - \alpha) \right) \longrightarrow c,
\]

where the weak convergence of the first term follows from Lemma 1(iii). Now by the device of transformation, use of Young’s form of the Taylor’s theorem at \( G^{-1}(\alpha) \) and the above result, similar to the proof of Lemma 4, we have

\[
\sqrt{n} E[G^{-1}(\alpha) - V_{j_n:n}] \overset{d}{\longrightarrow} N \left( \frac{c}{g(G^{-1}(\alpha))}, \frac{\alpha(1 - \alpha)}{(g(G^{-1}(\alpha)))^2} \right)
\]

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For the convergence of the first moment we observe that sequence \( \{ \sqrt{n}(\alpha - U_{jn:n}) \}_{n \geq 1} \) is \( L^2 \)-bounded, using Minkowski’s inequality we have
\[
\left( E \left[ n \left( \left( \frac{1 - j_n}{n} \right) - U_{jn:n} \right)^2 \right] \right)^{\frac{1}{2}} - \left( E \left[ n (\alpha - U_{jn:n})^2 \right] \right)^{\frac{1}{2}} \leq n \left( \frac{j_n}{n} - (1 - \alpha) \right)^2 \rightarrow c^2 < \infty
\]
and the sequence \( \{ \sqrt{n} \left( (1 - \frac{j_n}{n}) - U_{jn:n} \right) \}_{n \geq 1} \) is \( L^2 \)-bounded by Lemma 2 (iii).

(ii) By arguments similar to the first part above, it can be shown that
\[
\left( \frac{n}{j_n - n(1 - \alpha)} \right) (\alpha - U_{jn:n}) \overset{d}{\rightarrow} 1,
\]
and hence
\[
\left( \frac{n}{j_n - n(1 - \alpha)} \right) (G^{-1}(\alpha) - V_{jn:n}) \overset{d}{\rightarrow} \frac{1}{g(G^{-1}(\alpha))}.
\]

Use of Minkowski’s inequality coupled with an application of Lemma 2(iii) proves, as above, the convergence of the first moment.

Let \( F \) denote a distribution with support \([0, a]^2\), for some positive \( a \), and \( F_1 \) and \( F_2 \) be its marginals.\(^{26}\) For \( \{ X_i = (X_i^1, X_i^2) \}_{1 \leq i \leq n} \) a random sample of size \( n \) from \( F \), we define \( W_{j;2n} \) be the \( j \)-th highest value among the \( 2n \) values \( \{ X_i^k \}_{1 \leq i \leq n, k=1,2} \), for \( j = 1, 2, ..., 2n \). The following two lemmas and a corollary study the asymptotics of \( W_{jn;2n} \) under different assumptions on the behavior of \( \{ j_n \}_{n \geq 1} \).

As the sequence \( W_{jn;2n} \) is invariant with respect to permutations of the coordinates of \( X_i \), we could equivalently work with \( \{ Y_i = (Y_i^1, Y_i^2) = (X_i^{\eta_1}, X_i^{\eta_2}) \}_{i \geq 1} \) is an i.i.d. sequence of random uniform permutations of \( (1, 2) \). Note that \( Y_i \) is symmetric in its coordinates, \( i.e. \), the joint distribution function is permutation invariant, which in particular implies that the marginals are identically equal to \( G = \frac{1}{2} F_1 + \frac{1}{2} F_2 \). By \( g \) we shall denote the first derivative of \( G \).

Let \( \sigma_{\alpha}^2 \), for \( \alpha \in (0, 1) \), denote the variance of \( \frac{1}{2} I_{\{ Y_i^1 > G^{-1}(\alpha) \}} + \frac{1}{2} I_{\{ Y_i^2 > G^{-1}(\alpha) \}} \). It can be shown that
\[
\sigma_{\alpha}^2 = \alpha(1 - \alpha) - \frac{1}{2} E[I_{\{ Y_i^1 \leq G^{-1}(\alpha) < Y_i^2 \}}].
\]

\(^{26}\)In our case, we will be using support \( S \). The results are presented in this case to recognize their usefulness beyond our specific framework.
which reduces to \( \frac{1}{2} \alpha (1 - \alpha) \) in the case of independent coordinates.

We will also find use for the order statistics from a random sample of each of the coordinates - hence we denote by \( W_{j:n}^1 \) and \( W_{j:n}^2 \) the \( j \)-th highest 1-st and 2-nd coordinate of \( \{Y_i\}_{1 \leq i \leq n} \), respectively. Note that \( W_{j:n}^1 \) and \( W_{j:n}^2 \) are identically distributed though not necessarily independent.

**Lemma A11.** Let \( \alpha \in (0, 1) \) be such that \( g(G^{-1}(\alpha)) > 0 \). Then the following hold:

(i) For \( \{j_n\}_{n \geq 1} \) satisfying,
\[
\sqrt{n} \left( \frac{j_n}{n} - (1 - \alpha) \right) \rightarrow c \in \mathbb{R},
\]
we have
\[
\sqrt{n} E[G^{-1}(\alpha) - W_{2j_n:2n}] \rightarrow \frac{c}{g(G^{-1}(\alpha))}.
\]

(ii) For \( \{j_n\}_{n \geq 1} \) satisfying,
\[
\sqrt{n} \left( \frac{j_n}{n} - (1 - \alpha) \right) \rightarrow \pm \infty \quad \text{and} \quad \frac{j_n}{n} \rightarrow 1 - \alpha,
\]
we have
\[
\left( \frac{n}{j_n - n(1 - \alpha)} \right) E[G^{-1}(\alpha) - W_{2j_n:2n}] \rightarrow \frac{1}{g(G^{-1}(\alpha))}.
\]

**Proof.** (i) First, we show that
\[
\sqrt{n}(G^{-1}(\alpha) - W_{2j_n:2n}) \overset{d}{\rightarrow} N \left( \frac{c}{g(G^{-1}(\alpha))}, \left[ \frac{\sigma \alpha}{g(G^{-1}(\alpha))} \right]^2 \right).
\]
Towards this end we employ another standard device of expressing the event
\[
\{ \sqrt{n}(G^{-1}(\alpha) - W_{2j_n:2n}) \leq x \}
\]
by
\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \sum_{k=1}^{2} \left[ I_{Y_i > G^{-1}(\alpha) - \frac{x}{\sqrt{n}}} \right] \geq j_n \right\}
\]
and working with the latter event to establish the desired weak convergence. Rewriting
the latter as

\[
\frac{1}{2\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{2} \left[ I\{Y_i^k \geq G^{-1}(\alpha)\} - (1 - \alpha) \right] + \frac{1}{2\sqrt{n}} \sum_{i=1}^{n} \sum_{k=1}^{2} \left[ I\{Y_i^k \in [G^{-1}(\alpha) - \frac{x}{\sqrt{n}}, G^{-1}(\alpha)]\} \right]
\]

\[
\xrightarrow{d, N(0, \sigma^2)} \quad \xrightarrow{P, xg(G^{-1}(\alpha))}
\]

\[
\geq \sqrt{n} \left( \frac{j_n}{n} - (1 - \alpha) \right),
\]

where the first weak convergence follows using the central limit theorem on the zero
mean, i.i.d. random variables \( \frac{1}{2} \sum_{k=1}^{2} \left[ I\{Y_i^k \geq G^{-1}(\alpha)\} - (1 - \alpha) \right] \) with variance \( \sigma^2_\alpha \). The
convergence of the second term follows by combining the convergence of its expectation,
given by

\[
G(G^{-1}(\alpha)) - G \left( G^{-1}(\alpha) - \frac{x}{\sqrt{n}} \right).
\]

to \( xg(G^{-1}(\alpha)) \) and that its variance, given by

\[
\frac{1}{4} \left[ \left( G(G^{-1}(\alpha)) - G \left( G^{-1}(\alpha) - \frac{x}{\sqrt{n}} \right) \right) \right] \left[ 1 - G(G^{-1}(\alpha)) + G \left( G^{-1}(\alpha) - \frac{x}{\sqrt{n}} \right) \right] = O \left( n^{-\frac{1}{2}} \right).
\]

Having established the required convergences, by Slutsky’s theorem and Pólya’s theorem
the probability of the inequality being satisfied converges to

\[
1 - \Phi \left( \frac{c - xg(G^{-1}(\alpha))}{\sigma_\alpha} \right) = \Phi \left( \frac{x - \frac{c}{g(G^{-1}(\alpha))}}{\frac{\sigma_\alpha}{g(G^{-1}(\alpha))}} \right),
\]

which establishes the asymptotic behavior of \( W_{2j_n:2n} \). Now, for the convergence of the
first moment observe that

\[
|W_{2j_n:2n} - G^{-1}(\alpha)| \leq \max \{ |W_{j_n:n}^1 - G^{-1}(\alpha)|, |W_{j_n:n}^2 - G^{-1}(\alpha)| \},
\]

which in turn implies that

\[
nE \left[ (W_{2j_n:2n} - G^{-1}(\alpha))^2 \right] \leq 2nE \left[ (W_{j_n:n}^1 - G^{-1}(\alpha))^2 \right].
\]

Moreover, since \( nE \left[ (W_{j_n:n}^1 - G^{-1}(\alpha))^2 \right] \) is bounded, as was \( nE \left[ (V_{j_n:n} - G^{-1}(\alpha))^2 \right] \) in
Lemma A10, the convergence of the first moment is established.
(ii) By arguments similar to part (ii) of Lemma A10 and part (i) above, the convergence of the first moment can be established.

In the following we study the behavior of $W_{jn:2n}$ when $\frac{j_n}{n} \rightarrow 0$ which will depend on the behavior of the previously defined $G$ at the upper end of the support, i.e., at $a$. We assume that $G(\cdot)$ is $k$-times differentiable in a left neighborhood of $a$ with $g^{(l)}(a) = 0$ for $l = 0, 1, \ldots, k - 2$ and $g^{(k-1)}(a) \neq 0$. We define $C(G, k)$ by,

$$C(G, k) = \left[ \frac{(-1)^{k-1}k!}{g^{(k-1)}(a)} \right]^\frac{1}{k}$$

First, we study the case when $j_n = j$, for some positive integer $j$, for which we will need, for which we will need, for any $x \geq 0$, the asymptotic behavior of

$$S_n \equiv \sum_{i=1}^{n} \sum_{m=1}^{2} I\{Y^m_i \geq a - xn^{-\frac{1}{k}}\} = \sum_{i=1}^{2n} Z^n_i,$$

where, for convenience, we define the sequence $\{Z^n_i\}_{1 \leq i \leq 2n}$ to be

$$Z^n_{2(i-1)+m} = I\{Y^m_i \geq a - xn^{-\frac{1}{k}}\}, \quad i = 1, 2, \ldots, n \text{ and } m = 1, 2.$$

Observe that the summands are identically distributed Bernoulli variables with probability of taking the value 1, say $p_n$, satisfying

$$np_n = n \left(1 - G\left(a - xn^{-\frac{1}{k}}\right)\right) \rightarrow \left[\frac{x}{C(G, k)}\right]^k.$$

The fact that $p_n = O(n^{-1})$ immediately points in the direction of a Poisson limiting distribution for $S_n$; but since the summands are dependent, we need some condition based on a measure of dependence which will make the limiting distribution Poisson, the same that holds under independence. Note that the dependence in $Z_i$'s is only between $Z^n_{2i-1}$ and $Z^n_{2i}$.

Our choice for the measure of dependence is $\epsilon_n$, defined as,

$$\epsilon_n = F\left(a - xn^{-\frac{1}{k}}, a - xn^{-\frac{1}{k}}\right) - \left[G\left(a - xn^{-\frac{1}{k}}\right)\right]^2, \forall n \geq 1.$$
It is a measure of dependence as it is the deviation of the probability mass function from that under independence as,

\[ P[Z_1^n = i, Z_2^n = j] = (p_n)^{i+j} (1 - p_n)^{2-(i+j)} + (-1)^{i+j} \epsilon_n, \quad \forall i, j = 0, 1 \quad \text{and} \quad \forall n \geq 1. \]

More specifically, the total variation distance between the distribution of \((Z_1^n, Z_2^n)\) and that of two independent Bernoulli variables with parameter \(p_n\) is equal to \(4\epsilon_n\). Also, note that \(\text{Cov}(Z_1^n, Z_2^n) = \epsilon_n\) and the correlation coefficient between \(Z_1^n\) and \(Z_2^n\), denoted by \(\rho_{Z_1^n, Z_2^n}\), satisfies

\[ \rho_{Z_1^n, Z_2^n} = \rho_{Z_{2i-1}^n, Z_{2i}^n} = \frac{\epsilon_n}{p_n(1 - p_n)}, \quad \forall n \geq 1. \]

Interestingly, even though \(\epsilon_n\) can be negative, we have

\[ 0 \leq \lim_{n \to \infty} \inf n\epsilon_n \leq \lim_{n \to \infty} \sup n\epsilon_n \leq \left[ \frac{x}{C(G, k)} \right]^k. \]

The following lemma proves the Poisson convergence under the condition that \(\lim_{n \to \infty} n\epsilon_n = 0\), which can be seen to be equivalent to both \(\lim_{n \to \infty} n\epsilon_n\) and, more importantly, \(\lim_{n \to \infty} \rho_{Z_1^n, Z_2^n} = 0\). An interesting sufficient condition is \(\epsilon_n \leq 0, \forall n \geq 0, \text{i.e., negative dependence.}\)

Although there are other ways of proving the following Poisson convergence, we choose the Stein-Chen method as it gives a better appreciation of the condition, \(\lim_{n \to \infty} n\epsilon_n = 0\).

**Lemma A12.** If

\[ \lim_{n \to \infty} n\epsilon_n = 0, \]

then

\[ S_n \xrightarrow{d} \text{Poisson} \left( 2 \left[ \frac{x}{C(G, k)} \right]^k \right). \]

**Proof.** First, using the Stein-Chen method for Poisson approximation, we will show that

\[ \| P[S_n \in \cdot] - \text{Poisson}(2np_n) \| \leq 4 \left( 1 + \frac{1}{2np_n} \right) np_n \left( p_n + \frac{\epsilon_n}{p_n} \right). \]
As \( \{(Z_{2i-1}^n, Z_{2i}^n)\}_{i \leq n} \) are i.i.d. random vectors which are symmetric in their coordinates, it is sufficient to show that we can achieve a coupling of \( R_1 \) and \( T_1 \) of Lemma 3 in our case such that

\[
E[|R_1 - T_1|] = \left( p_n + \frac{|\epsilon_n|}{p_n} \right).
\]

One such coupling is the following:

\[
R_1 \equiv \sum_{i=1}^{2n} Z_i^n \quad \text{and} \quad T_1 \equiv Z_2^* + \sum_{i=3}^{2n} Z_i^n \quad \text{where} \quad Z_2^* \overset{d}{=} P[Z_2^n \in \cdot |Z_1^n = 1]
\]

where the coupling of \((Z_1^n, Z_2^n, Z_2^*)\) is defined by the joint distribution of \((Z_1, Z_2)\) and

\[
P[Z_1^n = 1; Z_2^n = 0; Z_2^* = 0] = p_n(1 - p_n) - \epsilon_n \quad \text{and} \quad P[Z_1^n = 1; Z_2^n = 1; Z_2^* = 1] = p_n^2 + \epsilon_n
\]

\[
P[Z_1^n = 0; Z_2^n = 0; Z_2^* = 1] = \frac{\epsilon_n}{p_n} \quad \text{and} \quad P[Z_1^n = 0; Z_2^n = 1; Z_2^* = 0] = \left( \frac{-\epsilon_n}{p_n} \right) \quad \text{and}
\]

Now, since for discrete distributions we have equivalence of weak convergence and convergence in total variation, we have

\[
\| \text{Poisson}(2np_n) - \text{Poisson} \left( 2 \left[ \frac{x}{C(G,k)} \right]^k \right) \| \rightarrow 0, \quad \text{as} \quad np_n \rightarrow \left[ \frac{x}{C(G,k)} \right]^k.
\]

Combining the above, we have the convergence to Poisson of \( S_n \).

\[\Box\]

**Lemma A13.**

\[
n^{-\frac{1}{k}} E(a - W_{j:2n}) \rightarrow 2^{-\frac{1}{k}} C(G,k) \Gamma \left( j + \frac{1}{k} \right) \frac{\Gamma \left( j + \frac{1}{k} \right)}{\Gamma \left( j \right)} .
\]

**Proof.** First, we show that

\[
n^{-\frac{1}{k}} (a - W_{j:2n}) \overset{d}{\rightarrow} \text{Gamma} \left( j, \frac{1}{2} C(G,k)^k \right)^{\frac{1}{k}} .
\]

Using the equivalence,

\[
\left\{ n^{-\frac{1}{k}} (a - W_{j:2n}) \leq x \right\} = \left\{ \sum_{i=1}^{2n} Z_i^n \geq j \right\}
\]
and Lemma A12, we have
\[
P \left[ n^\frac{1}{k} (a - W_{j;2n}) \leq x \right]
= P \left[ \sum_{i=1}^{2n} Z_i^n \geq j \right] \longrightarrow 1 - \exp \left\{ -2 \left[ \frac{x}{C(G,k)} \right]^k \right\} \sum_{i=0}^{j-1} \frac{1}{i!} \left( 2 \left[ \frac{x}{C(G,k)} \right]^k \right)^i,
\]
which is a restatement of the weak convergence above. The proof of the convergence of the first moment follows along the same lines as in Lemma A11.

Now we deal with the case when, \( \frac{j_n}{n} \longrightarrow 0 \) and \( j_n \longrightarrow \infty \). Here we will need, for any \( x \geq 0 \), the asymptotic behavior of
\[
S_n^* \equiv \sum_{i=1}^{n} \sum_{m=1}^{2} I \left\{ Y_{i,m}^{n} \geq a - \left( \frac{j_n}{n} \right)^{\frac{k}{x}} \right\} ;
\]
Observe that the summands are identically distributed Bernoulli variables with probability of taking the value 1, say \( p_n^* \), satisfying
\[
\left( \frac{n}{j_n} \right) p_n^* = \left( \frac{n}{j_n} \right) \left( 1 - G \left( a - \left( \frac{j_n}{n} \right)^{\frac{1}{x}} \right) \right) \longrightarrow \left[ \frac{x}{C(G,k)} \right]^k
\]
Also, the first two central moments of \( S_n^* \) are
\[
E[S_n^*] = np_n^* \text{ and } \text{Var}(S_n^*) = 2n(p_n^*(1 - p_n^*) + \epsilon_n^*),
\]
where \( \epsilon_n^* \) is defined as,
\[
\epsilon_n^* = F \left( a - \left( \frac{j_n}{n} \right)^{\frac{1}{x}}, a - \left( \frac{j_n}{n} \right)^{\frac{1}{x}} \right) - \left[ G \left( a - \left( \frac{j_n}{n} \right)^{\frac{1}{x}} \right) \right]^2 \forall n \geq 1.
\]
Similar to \( \epsilon_n \), \( \epsilon_n^* \) satisfies
\[
0 \leq \lim_{n \to \infty} \inf \left( \frac{n}{j_n} \right) \epsilon_n^* \leq \left[ \frac{x}{C(G,k)} \right]^k.
\]
The above, in particular imply that
\[
E \left[ \frac{S_n^*}{j_n} \right] = \left( \frac{2n}{j_n} \right) p_n^* \longrightarrow 2 \left[ \frac{x}{C(G,k)} \right]^k
\]
and 
\[ \lim_{n \to \infty} \sup \text{Var} \left( \frac{S_n}{j_n} \right) = \lim_{n \to \infty} \sup \frac{2}{j_n} \left[ \left( \frac{n}{j_n} \right) p_n^* (1 - p_n^*) + \left( \frac{n}{j_n} \right) \epsilon_n^* \right] = 0, \]
which together with Tchebycheff’s inequality imply
\[ \frac{S_n}{j_n} \xrightarrow{p} 2 \left[ \frac{x}{C(G, k)} \right]^k. \]
The above lead to the following lemma.

**Lemma A14.** For any sequence of positive integers \( \{j_n\}_{n \geq 1} \) satisfying
\[ \frac{j_n}{n} \to 0 \quad \text{and} \quad j_n \to \infty, \]
we have
\[ \left( \frac{n}{j_n} \right)^{\frac{1}{k}} E[a - W_{j_n,2n}] \to 2^{-\frac{1}{k}} C(G, k). \]

**Proof.** First, we show that
\[ \left( \frac{n}{j_n} \right)^{\frac{1}{k}} (a - W_{j_n,2n}) \xrightarrow{p} 2^{-\frac{1}{k}} C(G, k). \]
Using the equivalence,
\[ \left\{ \left( \frac{n}{j_n} \right)^{\frac{1}{k}} (a - W_{j_n,2n}) \leq x \right\} = \{ S_n^* \geq j_n \}, \]
and the discussion above, we have
\[ \lim_{n \to \infty} P \left[ n^{\frac{1}{k}} (a - W_{j_n,2n}) \leq x \right] = \begin{cases} 0 & \text{if } x < 2^{-\frac{1}{k}} C(G, k) \\ 1 & \text{if } x > 2^{-\frac{1}{k}} C(G, k) \end{cases} \]
which is a restatement of the convergence in probability above. The proof of the convergence of the first moment then follows along the same lines as in Lemma A11.

In the following corollaries, instead of the earlier assumptions on the behavior of \( G(\cdot) \) at the upper end of its support, we will assume that it is \( k \)-times continuously
differentiable in a right neighborhood of 0 with \( g^{(l)}(0) = 0 \) for \( l = 0, 1, \ldots, k - 2 \) and \( g^{(k-1)}(0) \neq 0 \). For such a \( G(\cdot) \), Let us define
\[
C^*(G, k) \equiv \left( \frac{k!(-1)^{k-1}}{g^{(k-1)}(0)} \right).
\]

**Corollary A15.** For a positive integer \( j \), we have
\[
n^\frac{1}{k} E[W_{2n+1-j_n:2n}] \longrightarrow 2^{-\frac{1}{k}} C^*(G, k) \frac{\Gamma(j + \frac{1}{k})}{\Gamma(j)}.
\]

**Proof.** The proof is similar to Corollary A7, using Lemma A13 instead of Lemma A4. \( \blacksquare \)

**Corollary A16.** For any sequence of positive integers \( \{j_n\}_{n \geq 1} \) satisfying
\[
\frac{j_n}{n} \longrightarrow 0 \quad \text{and} \quad j_n \longrightarrow \infty,
\]
we have
\[
\left( \frac{n}{j_n} \right)^{\frac{1}{k}} E[W_{2n+1-j_n:2n}] \longrightarrow 2^{-\frac{1}{k}} C^*(G, k).
\]

**Proof.** By an argument similar to that of Corollary A7, using Lemma A14 instead of Lemma A13. \( \blacksquare \)

**Proof of Proposition 1.** Suppose that \( \frac{M_n}{n} \rightarrow 0 \). It is enough to consider the two cases where (i) \( M_n \) is bounded and (ii) \( M_n \rightarrow \infty \). First consider the case where \( M_n \) is bounded. Without loss of generality, suppose that \( M_n = M \). The expected price in the multi-unit auction is bounded below and above by \( E[(V_1)_{2M+1:2n}] \) and \( E[W_{2M+1:2n}] \), respectively. The lower bound converges to 1 at the rate \( n^{-\frac{1}{k+1}} \), by Lemma A4, whereas the upper bound converges to 1 at the rate \( n^{-\frac{1}{(k+1)+\frac{1}{k}}} \) by Lemma A13. The expected price in the bundle auction is \( E[(V_1 + V_2)_{M+1:n}] \) which goes to 2 at the rate \( n^{-\frac{1}{k(V_1+V_2)}} \) by Lemma A4.

The expected social surplus in the multi-unit auction is bounded below by \( \sum_{r=1}^{2M} E[(V_1)_{r:n}] \).
By Lemma A4 each term in the lower bound converges at the rate \( n^{-\frac{1}{k(V_1+V_2)}} \). Hence the
rate of convergence of the sum. The expected social surplus generated in the bundle auction is 
\[ P \left( \sum_{r=1}^{M} E \left[ (V_1 + V_2)_{r:n} \right] \right) \] which converges at the rate \( n^{-\frac{1}{2M(V_1 + V_2)^{\mu}}} \) since each term converges at that rate.

Next, suppose that \( M_n \to \infty \). The expected price in the multi-unit auction is bounded below and above by \((V_1)_{2M_n+1:n}\) and \(W_{2M_n+1:2n}\), respectively. The rates of convergence are then obtained upon applying Lemma A5 and Lemma A14 on the expectations. Similarly, the expected price in the bundle auction is \( E \left[ (V_1 + V_2)_{M_n+1:n} \right] \) which converges to 2 at the rate \( O \left( \frac{1}{M_n^{\mu}} \right) \) by Lemma A5.

The expected surplus per unit of supply in the multi-unit auction is bounded below by \( \frac{1}{2M_n} \sum_{r=1}^{2M_n} E[ (V_1)_{r:n} ] \) and that in the bundle auction is given by \( E \left[ (V_1 + V_2)_{M_n+1:n} \right] \). Applying Lemma A6 these quantities converge to 1 and 2 at the rates \( \left( \frac{M_n}{n} \right)^{\frac{1}{2M(V_1 + V_2)^{\mu}}} \) and \( \left( \frac{M_n}{n} \right)^{\frac{1}{2M(V_1 + V_2)^{\mu}}} \), respectively.

**Proof of Proposition 2.** The first part follows from Lemma A11 while the second part follows from Lemma A10.

**Proof of Proposition 3.** Suppose that \( \frac{M_n}{n} \to 1 \). It is enough to consider the cases where (i) \( n - M_n \) is bounded and (ii) \( n - M_n \to \infty \). Suppose that \( n - M_n \) is bounded. The expected price in the multi-unit auction is bounded above by \( E \left[ (V_2)_{2M_n-n+1:n} \right] \) and that in the bundle auction is given by \( E \left[ (V_1 + V_2)_{M_n+1:n} \right] \). Therefore, the price in the multi-unit auction converges to zero at a rate not slower than \( n^{-\frac{1}{2M(V_2)^{\mu}}} \) if \( n - M_n \) is bounded, by Corollary A7, and \( \left( \frac{n}{2M_n-n+1} \right)^{-\frac{1}{2M(V_2)^{\mu}}} \) if \( n - M_n \to \infty \), by Corollary A8.

The price in the bundle auction converges to zero at the rate \( n^{-\frac{1}{2M(V_1 + V_2)^{\mu}}} \).

The expected surplus per unit of supply in the multi-unit auction is no less than
\[
\frac{1}{2M_n} \left[ \sum_{r=1}^{n} E[ (V_1)_{r:n} ] + \sum_{r=1}^{2M_n-n} E[ (V_2)_{r:n} ] \right]
\]
and that in the bundle auction is given by
\[
\frac{1}{2M_n} \sum_{r=1}^{M_n} E[ (V_1 + V_2)_{r:n} ].
\]
Using Corollary A9 the expected surplus per unit of supply in the multi-unit auction converges to $E[(V_1 + V_2)/2]$ at a rate not slower than $n^{-\left(1+\frac{1}{\pi F_{V_1 + V_2}}\right)}$ when $n - M_n$ is bounded and $(\frac{n-M_n}{n})^{1+\frac{1}{\pi F_{V_1 + V_2}}}$ when $n - M_n \to \infty$. The expected surplus per unit of (bundled) supply in the bundle auction converges to $E[(V_1 + V_2)/2]$ at a rate $\frac{\ln(n)}{n^{\left(1+\frac{1}{\pi F_{V_1 + V_2}}\right)}}$ when $n - M_n$ is bounded and $(\frac{n-M_n}{n})^{1+\frac{1}{\pi F_{V_1 + V_2}}}$ when $n - M_n \to \infty$. 

The following lemma demonstrates that the average of the marginal distributions of two random variables is greater in the sense of Lorenz ordering (denoted $\geq_{\text{Lorenz}}$) than the distribution of their average.

**Lemma A17.** Let $(X,Y) \sim F$, where $F(\cdot,\cdot)$ is a distribution function on $S \subseteq \mathbb{R}^2$. Let $F_{\bar{X},\bar{Y}}$ be the distribution function of the average of $X$ and $Y$, $F_X$ the marginal distribution of $X$, and $F_Y$ the marginal distribution of $Y$. Defining $F_M$ as the distribution function of the symmetric mixture of $X$ and $Y$, i.e.,

$$F_M \equiv \frac{F_X + F_Y}{2},$$

we have

$$\int_0^t F^{-1}_M(u) du \geq \int_0^t F^{-1}_{\bar{X},\bar{Y}}(u) du, \quad \forall t \in [0,1]$$

i.e., $F_M \geq_{\text{Lorenz}} F_{\bar{X},\bar{Y}}$.

**Proof.** Let $I$ be a Bernoulli variable with mean of $\frac{1}{2}$ and independent of $(X,Y)$. Then

$$Z \equiv IX + (1-I)Y \overset{d}{=} \frac{F_X + F_Y}{2} \quad \text{and} \quad E[Z|X+Y] = \frac{X+Y}{2}.$$

Hence by Theorem 3.4 of Arnold (1987) we have $Z \geq_{\text{Lorenz}} \frac{X+Y}{2}$, which, in particular, implies

$$\int_0^t F^{-1}_M(u) du \leq \int_0^t F^{-1}_{\bar{X},\bar{Y}}(u) du, \quad \forall t \in [0,1].$$

Combining the above with

$$\int_0^1 F^{-1}_M(u) du = E[Z] = E\left[\frac{X+Y}{2}\right] = \int_0^1 F^{-1}_{\bar{X},\bar{Y}}(u) du$$

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completes the proof.

**Proof of Theorem.** First, we prove the result at $\alpha = 0$ when $(R1)$ holds.

Suppose that $M_n$ is bounded, or more specifically, it is equal to $M$. The expected price in the multi-unit auction *minus* that from the bundle auction is no less than

$$E[(V_1)_{2M+1:n}] - E[(V_1 + V_2)_{M:n}/2]$$

$$= E[1 - (V_1 + V_2)_{M:n}/2] - E[1 - (V_1)_{2M+1:n}]$$

Applying $(R1)$ and Proposition 1 we have that

$$n^{-\gamma_1(\gamma_1)} (E[1 - (V_1 + V_2)_{M:n}/2] - E[1 - (V_1)_{2M+1:n}]) \to \infty$$

as $n \to \infty$. Hence,

$$E[(V_1)_{2M+1:n}] > E[(V_1 + V_2)_{M:n}/2]$$

for all large enough $n$.

The expected social surplus in the multi-unit auction is no less than $\sum_{j=1}^{2M} E[(V_1)_{j:n}]$. In a bundle auction the expected social surplus is given by $\sum_{j=1}^{M} E[(V_1 + V_2)_{j:n}]$. Therefore, the expected difference is no less than

$$\sum_{j=1}^{2M} E[(V_1)_{j:n}] - \sum_{j=1}^{M} E[(V_1 + V_2)_{j:n}]$$

$$= \sum_{j=1}^{M} E[2 - (V_1 + V_2)_{j:n}] - \sum_{j=1}^{2M} E[1 - (V_1)_{j:n}]$$

Again, applying $(R1)$ and Proposition 1, the terms in the second summation are $O(n^{-\gamma_1(\gamma_1)\gamma})$ whereas the terms in the first summation are no faster than $O(n^{-\gamma_1(\gamma_1)\gamma})$. Hence,

$$n^{-\gamma_1(\gamma_1)} \left( \sum_{j=1}^{2M} E[(V_1)_{j:n}] - \sum_{j=1}^{M} E[(V_1 + V_2)_{j:n}] \right) \to \infty.$$

In other words,

$$\sum_{j=1}^{2M} E[(V_1)_{j:n}] > \sum_{j=1}^{M} E[(V_1 + V_2)_{j:n}]$$

for all large enough $n$. 
Now suppose that $M_n \to \infty$. Then following similar steps as above but using the multiplying factor $\left( \frac{n}{M_n} \right)^{\frac{1}{2}(F_{V_1})}$ instead of $n^{\frac{1}{2}(F_{V_2})}$ it follows, using (R1) and Proposition 1, that

$$E[(V_1)_{2M_n+1:n}] > E[(V_1 + V_2)_{M_n:n}/2]$$

for all large $n$. Similarly,

$$\frac{1}{2M_n} \sum_{j=1}^{2M_n} E[(V_1)_{j:n}] > \frac{1}{2M_n} \sum_{j=1}^{M_n} E[(V_1 + V_2)_{j:n}]$$

for all large $n$.

At $\alpha = 1$ the proof of the result follows steps similar to above, except that we apply (R2) and Proposition 3.

Next, combining (R1) with (5), and (R2) with (6) (so that $\frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2} \neq \frac{F_{V_1} + F_{V_2}}{2}$ in some right neighborhood of 0 and in some left neighborhood of 1) we have that there exist $x_*$ and $x^*$ ($x* \leq x^*$) such that

$$\frac{1}{2}F_{V_1}(x) + \frac{1}{2}F_{V_2}(x) > \frac{F_{V_1} + F_{V_2}}{2}(x) \forall x \in (0, x_*)$$

$$\frac{1}{2}F_{V_1}(x) + \frac{1}{2}F_{V_2}(x) < \frac{F_{V_1} + F_{V_2}}{2}(x) \text{ for all } x \in (x^*, 1).$$

These conditions along with the continuity and monotonicity of the distribution functions guarantee the existence of $t_*$ and $t^*$ such that

$$\left( \frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2} \right)^{-1}(t) < \frac{F_{V_1} + F_{V_2}}{2}(t), \forall t \in (0, t_*)$$

$$\left( \frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2} \right)^{-1}(t) > \frac{F_{V_1} + F_{V_2}}{2}(t), \forall t \in (t^*, 1).$$

Finally, observe that in the limit bidders bid truthfully whenever the marginal value for a unit is larger than the price. Hence in the limit the social surplus generated per bidder is given by

$$\int_0^1 x(f_1(x) + f_2(x))dx = 2 \int_{1-\alpha}^1 \left( \frac{1}{2}F_{V_1} + \frac{1}{2}F_{V_2} \right)^{-1}(t)dt$$
in the multi-unit auction, and by
\[
\int_{F_{V_1+V_2}^{-1}(1-\alpha)}^2 x f_{V_1+V_2}(x) dx = \int_{1-\alpha}^1 F_{V_1+V_2}^{-1}(t) dt
\]
in the bundle auction. Upon applying Lemma 11 and the fact that \( E \) is countable we have
\[
\int_{1-\alpha}^1 \left( 2 \left( \frac{1}{2} F_{V_1} + \frac{1}{2} F_{V_2} \right)^{-1} (t) - F_{V_1+V_2}^{-1}(t) \right) dt > 0
\]
for all but countably many \( \alpha \in (0,1) \). This proves that in the limit the per bidder surplus generated by the multi-unit auction is strictly larger than that from the bundle auction for all but countably many \( \alpha \in (0,1) \). Hence the same holds for all sufficiently large (but finite) number of bidders.

\textbf{Proof Corollary.} Follows easily from the Theorem upon applying Condition R(3).

9. References


