On Measuring Sunk Capital

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Abstract. For much of its life, tangible capital goods are irreversibly committed to investment projects. The paper examines what it means for capital to be sunk. Value sunk is measured using the options available to managers of a project. External prices may or may not be relevant to an enterprise. Fixing the value of options is the only role for external prices of the assets. Moreover, marginal prices are not used in valuing sunk capital. The choice of real or nominal prices is considered in terms of measured benefits, costs, wealth and income of a project.

Key words: capital value, irreversible investment, sunk capital, options
1. INTRODUCTION

The idea of sunk capital is integrated into the static, Marshallian theory of the firm. The assets that are sunk and the value sunk are defined. In much of economic analysis, however, sunk costs are considered to be bygones and tend to be viewed as unimportant.

In recent years, the theory of investment under certainty (e.g. Dixit and Pindyck 1994) has highlighted the importance of irreversible investment and brought the implications of sunk capital back to prominence. The conditions derived tend to be instantaneous conditions for making an investment (or disinvestment) as opposed to conditions that hold for the periods between changes in the sunk capital. A firm is viewed largely as a set of options, and the conditions derived are for the exercise of an option. Cairns (forthcoming 2013) indicates that irreversibility is also the vital consideration in the accounting for, and hence valuation of, capital depreciation and hence capital itself during periods when an option is not exercised. When capital is sunk, it does not have a unique value.

The present paper revisits the question of sunk capital in a dynamic model. Sunk assets and the values sunk are defined. These determinations form a foundation for the valuation of capital in place.

2. PROJECT VALUE AND DEPRECIATION

Capital is organized into projects and projects into firms. Doms (1996: 80, n. 7) notes that much investment is lumpy, not convex. The view of a project in this section is consonant with the irreversible investment of capital of multiple types. The simplest case, of inputs at a single point of time, is considered.

The present paper departs from traditional analyses in incorporating non-marketed capital as well as marketed capital into a firm’s capital stock. Herein, capital is defined
in terms of realizations of cash flows over time. A surplus that accrues over time can be capitalized. The source is viewed as a component of the capital of the project.

**Definition 1.** Any stock that contributes to cash flow over time is a form of comprehensive capital.

Let the vector of marketed capital goods at time $t$ be represented by the $k_t$-vector $K_t$ and cost a total of $M_0$ at time 0; the vector of non-marketed capital goods by the $n_t$-vector $N_t$; and the project by the scalar $P_t$. Let the length of the project’s life be represented by $T \leq \infty$, the real interest rate in period $t$ by $r_t$ and real, variable profit in period $t$ by $m_t$. Furthermore, let the present value of the project be denoted by $V$. At $t = 0$

$$V(P_0) = \sum_{t=1}^{T} \frac{m_t}{\prod_{s=1}^{T}(1 + r_s)}.$$ (1)

Having no price and indeed no units, the non-marketed assets are confounded in the project and can be valued only as a single, residual asset. Their contribution is specific to the particular project (cf. Oliner 1996: 69). The value of the non-marketed assets is defined as a residual, $V(P_0) - M_0$.

In a traditional analysis, Hulten and Wyckoff (1996: 11) consider the price of a used machine to be both (a) the present value of the income accruing to the machine and (b) what a rational investor would pay for it. The value of the project as a whole at $t$, $V(P_t)$, satisfies these conditions, since the remaining net present value can be realized in the capital market.

The tangible capital goods invested in a project are subject to (physical) deterioration. Deterioration of non-marketed capital is not defined. Therefore, deterioration of the project is not defined. Marketable components of the project are only occasionally, and not typically, sold in a secondary market. Rather, it is the project itself (e.g. a mining property) that can be sold. The project thus becomes a primary object of economic analysis.
While there is no measure of the deterioration of a project, the analysis of the project’s depreciation has close parallels to the traditional one that involves a single capital good. It begins similarly to that of, for example, Hulten and Wyckoff (1996). As in duality theory, variable profit $m_t$ is a function of prices and wages and the irreversibly chosen (“sunk”) capital stock. The convention adopted is the usual one, that net cash flows or variable profits are received at the end of a period. If the project has a finite lifetime $T$, let $m_t = 0$ for $t > T$. For a given project, for any $t \geq 0$, the discounted-variable-profit function at $t$ is

$$V(P_t) = \sum_{s=1}^{\infty} \frac{m_{t+s}}{\Pi_{k=1}^{s} (1 + r_{t+k})}.$$  

(2)

At $t + 1$,

$$V(P_{t+1}) = \sum_{s=2}^{\infty} \frac{m_{t+s}}{\Pi_{k=2}^{s} (1 + r_{t+k})} = (1 + r_{t+1}) \sum_{s=2}^{\infty} \frac{m_{t+s+1}}{\Pi_{k=1}^{s} (1 + r_{t+k})} = (1 + r_{t+1}) V(P_t) - m_{t+1}. 

(3)

Hotelling (1925) and Samuelson (1937) provide the following definition for the depreciation of (the aggregated, comprehensive capital of) a project.

**Definition 2.** Depreciation of the project at time $t$ is the decline in its value,

$$\Delta_{t+1} = V(P_t) - V(P_{t+1}) = m_{t+1} - r_{t+1} V(P_t).$$  

(4)

**Definition 3.** The rate of depreciation is $\delta_{t+1} = \Delta_{t+1} / V(P_t)$.

Depreciation of the project is unique but it depends on the pattern of the net cash flows. These cash flows are specific to the project and depend on economic conditions.

Rearrangement of equation (3) shows that the variable profit is equal to the sum of the rate of interest and the rate of depreciation applied to value:

$$m_{t+1} = r_{t+1} V(P_t) + [V(P_t) - V(P_{t+1})] = (r_{t+1} + \delta_{t+1}) V(P_t).$$  

(5)
A rental or user cost can be identified as \((r_{t+1} + \delta_{t+1})V(P_t)\). Therefore, the net cash flow at any time is the “user cost of the project”, considered to be a composite capital good.

A further rearrangement yields a no-arbitrage condition, that the return on the original capital is equal to current profit minus depreciation, or the dividend plus the change in capital value,

\[
r_{t+1}V(P_t) = m_{t+1} + [V(P_{t+1}) - V(P_t)].
\] (6)

The following applies to the aggregate capital, the project.

**Proposition 1** The rental or user cost of a project is the sum of the rates of interest and depreciation applied to the project’s discounted net cash flow. The return on the discounted net cash flow satisfies the fundamental, no-arbitrage condition of equilibrium in asset markets. Depreciation is endogenous to economic conditions.

Since the mathematical properties of variable profit \(m_t\) or of the investment-cost function have not been used, the results are robust to whether or not the assumptions of constant returns to scale or of convexity, etc. hold. Since taxes are incorporated into variable profit, the results are also robust to the inclusion of any form of tax.\(^1\)

3. **NUMERAIRE**

Diewert (2009) observes that the user cost is a benefit in terms of cash flow, what can be called a *user benefit*, and \(V(P_t)\) is the capitalized total benefit anticipated by the owner. Viewed retrospectively (with the understanding that “sunk costs are bygones”), the full cash flow is indeed a benefit to the project’s owner in the current period.

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\(^1\)The capitalized values of taxes are *social* assets but not private ones.
For example, the Hotelling rent of a canonical exhaustible resource rises at the rate of interest. Many resource economists consider Hotelling rents to be costs; they are indeed opportunity costs and are often called user costs. They are costs, however, only because in equilibrium the maximum benefit is being obtained, as compared to the benefit available in other time periods. Having the use of the resource at the optimal time (or any time) is, consistently with Diewert’s reasoning, a benefit.

On the other hand, depreciation is the repayment of a cost. The anticipation of that repayment is necessary if the project is to be initiated. Indeed, both recovery and interest, \( \delta_{t+1} V(P_t) \) and \( r_{t+1} V(P_t) \), are components of the required repayments in the context of an anticipated investment. Viewed prospectively, the rental comprises a cost and a benefit: One may consider the depreciation, \( \delta_{t+1} V(P_t) \), to be recovery of capital cost and the interest, \( r_{t+1} V(P_t) \), to be an income or a benefit.

The discussion identifies a source of income to be an interest rate multiplied by a capital value. Income and value (wealth) are linked dynamically: the capitalized value of any form of income is the value of an asset, interest on which is the income. It depreciates according to equation (4).

Firms invest in inflationary or deflationary times as well as in times of stable prices. Suppose, then, that \( \bar{m}_t \) denotes variable profit in nominal terms and \( \bar{r}_t \) the nominal rate of interest. Let the rate of inflation be denoted by \( \pi_t \). The project’s value in money terms is

\[
\bar{V}(P_t) = \sum_{s=1}^{\infty} \frac{\bar{m}_{t+s}}{\prod_{k=1}^{s} (1 + \bar{r}_{t+k})} = \sum_{s=1}^{\infty} \frac{\bar{m}_{t+s}}{\prod_{k=1}^{s} (1 + \bar{r}_{t+k})} = \sum_{s=1}^{\infty} \frac{m_{t+s}}{\prod_{k=1}^{s} (1 + r_{t+k})} = V(P_t)
\]

Results are, then, equivalent whether nominal or real values are used. A case can be made for using the numeraire that is observed (for a preference for using nominal values) rather than real values calculated using an estimated price index.

**Proposition 2** The method of calculation of capital value and depreciation is robust
to (a) changes in the price level, (b) the mathematical properties of current variable cost, (c) taxation, (d) the retirement, deterioration, aging, obsolescence or replacement cost of tangible capital, etc.

4. SUNK ASSETS AND SUNK VALUE

At any time $t < T$, a particular marketed, tangible capital good $i$ has deteriorated through use in the project. It remains with the project at time $t$ so long as its productive capacity in the project, represented by $u^i_t K^i_0, u^i_t \leq 1$, contributes to the project more than its current market value net of transactions cost, denoted by $p^i_t (K^i_t)$, with $K^i_t$ not necessarily equal to $u^i_t K^i_0$.

$$V(P_t) > V\left(P_t \setminus u^i_t K^i_0\right) + p^i_t \left(K^i_t\right).$$

(7)

Sometimes, used capital goods are sold. Far more often they are not sold. Projects remain in operation because their assets, tangible and intangible, marketed and non-marketed, are better deployed in the project than elsewhere, as in equation (7). So long as a capital good remains with its project, in many cases until it is scrapped, its contribution comes not solely on its own account but as a result of complementarity with other assets. The value of a sunk asset satisfies condition (7).

**Definition 4.** Capital is irreversibly invested or sunk so long as its incremental contribution to its current project exceeds its market value net of transactions costs.

In traditional capital theory, the relative prices of assets of different vintages are considered to be proportional to their relative productivities in the project (e.g., Jorgenson 1996: 29). In contrast to equation (7), the following conditions are implied:

$$K^i_t = u^i_t K^i_0$$

and

$$V(P_t) = V\left(P_t \setminus u^i_t K^i_0\right) + p^i_t \left(K^i_t\right).$$

(8)

(9)
The identification of depreciation with physical deterioration of a tangible, marketable capital good is possible only if conditions (8) and (9) hold. These conditions imply that the capital good is not sunk in that it can be economically disengaged from the project at any time and its full value to the project realized in the capital market. In terms of the production function $F(K, L)$, a non-sunk asset, which satisfies conditions (8) and (9), has $\frac{\partial F}{\partial K^i} = r + \delta^i$.

The definition of the remaining capital as $u^i_tK^i_t$, however, reminds one that the stock of capital good $i$ has links to the other periods through the factors $u^i_t$. If condition (7) holds, the value of the good in the second-hand market, $p^i_t(K^i_t)$, is not realized. The asset is not sold, is not removed from the project. The productivity of a sunk asset is linked to the other periods of its life.

An asset is either sunk or not, sold or not. Once a particular unit of an asset is sunk it acquires some properties of a non-marketed asset, even if it can be traded in a second-hand market. Usually, the value sunk in a project is not specified.

**Definition 5.** Value sunk in a project is the difference between the value of the project and what can be realized by selling some or all of its marketable constituents.

For a single type of capital, the value sunk is

$$V(P_t) - \left[ V\left(P_t \setminus u^i_0K^i_0\right) + p^i_t(K^i_t) \right].$$

(10)

The values of similar assets in a project are not necessarily equal to that of the marginal unit of that type of asset, nor, if sold, are necessarily sold at once. A different value is implied if not all of the units of a given type of capital are being considered for possible sale. Some molecules of water evaporate from a lake. Others do not. There are a multitude of such values, then, depending on what comparison is made. Also, through complementarities, non-marketed capital contributes to value sunk. If the project is imagined as being possibly completely wound up, the value sunk is $V(P_t) - \left[ V(P_t \setminus (u^1_tK^1_t, ..., u^k_tK^k_t)) + \sum_{i=1}^{k^u} p^i_t(K^i_t) \right]$. 

9
Dixit and Pindyck (1994: 211) view a firm’s value as being largely the value of a set of options. The external (second-hand-market) price, \( p_t^i (K_t^i) \), is the price of an option to sell, as is implied by equations (7) through (10). If a transaction is the best option, as in equation (9), it is acted upon.

**Remark 1** *The value of the assets sunk in a project is the value of an option to continue with the project rather than to sell some or all of the marketable capital.*

If the best option is to keep the asset in the project (to continue), the external price, the price of an inefficient option, is not an appropriate criterion for action. For purposes of valuation, it is not necessary to take the outside opportunity into account. As with a non-binding constraint, whose shadow value is zero, the value of the outside opportunity may as well be zero. The best option is chosen and that option provides the realized cash flow.

One of the options may be to rent out the capital for a period of time less than its remaining life. If the rent flow at any time \( t \) is less than \( m_t \), one might suggest that the rent puts a floor on the user cost of capital at \( t \). But the level of and the opportunity of obtaining that rent does not affect the realized cash flow from the project at any time (including \( t \)) and does not affect the choice of the capital stock. Again, the value of the opportunity to rent the asset out is the value of a sub-optimal option and should not affect what is deemed to be the range of possible values of the asset’s current rental, and hence not affect its implied depreciation.

**Proposition 3** *The value of an outside option is not used in valuing sunk capital.*

In traditional capital theory, capital is valued at marginal values. Since for a function \( f \), \( df(x) = \nabla f(x) \cdot dx \), marginal values are additive. However, only if Euler’s (very strong) condition, that \( f(x) = \nabla f(x) \cdot x \), holds is value \( f(x) \) additive or given by marginal prices. Furthermore, a marginal analysis does not give a complete analysis
of capital: Non-marketed capital does not have physical units and does not have a marginal value.²

When an irreversible investment is being contemplated, even under certainty, there is an optimal strike or stopping time for which the net present value of the investment option or opportunity is maximized (Cairns and Davis 2007). A marginal condition is also a form of stopping condition, expressed in terms of a quantity rather than of time. It signals how much to invest (when “to stop”). Only the marginal unit bought or sold can be valued at the marginal price. The market price is possibly a lower bound on the value of the item to the owner—but not if the owner is considering selling the asset and is waiting for the optimal time. Then the value to the owner includes an option premium.

**Proposition 4** Marginal values or marginal shadow prices are inputs to stopping conditions in quantity space. They are relevant to tangible capital only. They are not inputs to the measurement of capital value.

Three properties that militate against the use of marginal or shadow prices to value capital are (1) non-constant returns to scale and non-convexity, (2) non-optimality and (3) the importance of non-marketed capital. Even though marginal values are defined for the marketed capital, they cannot be used to measure the value of capital. The primary capital entity is the project. Its value is well defined. Measures of its components require economic assessments of their contributions to discounted net cash flow.

In reality, firms also invest and divest capital, refit, merge and so on. They are made up of a portfolio of projects and bundled together using yet more non-marketed

²The incremental value of asset $i$ (at time 0) is $V(P_0 | K_i) - \sum_{j \neq i} M_j$. However, this value is equal to the sum of (a) the $M_j$ and (b) the non-marketed residual. Since each incremental value includes non-marketed capital value, they do not sum to total value, $V(P_0)$. 
capital. The value of a firm at any date is its net present value plus the values of the net investments in comprehensive capital made at that date. Values are updated recursively using a perpetual inventory similarly to values defined in the national accounts. When new capital (including non-marketed capital as a residual) is added, it is coalesced into the project or firm, so that condition (7) holds. The analyst simply adds its value to that of the project. When capital is retired or sold its market value is subtracted from the project value at that time. Under certainty, that value is foreseen in the rental schedule. Some forms of non-marketed capital may be common to more than one project.

5. CONCLUSION

In much of capital theory, the value of capital is measured through the equality of the marginal value to the firm and the external, market value. The paper has argued (a) that sunk costs are important for the continuing valuation of the firm and (b) that there are important forms of non-marketed capital in any project or firm. Marginal values are not appropriate for valuation. Moreover, the opportunity to sell a marketable asset is viewed as a real option. Even if external market prices exist for some forms of such capital, they have a specific role in determining whether an asset is sold or not but not in valuation of sunk capital. If the asset is not sold (remains sunk) then, like non-marketed capital, it has only a derived price, which does not affect optimal decisions in the project.

The paper also argues for caution in the choice of numeraire. Use of observed, nominal prices may be preferable to estimated, real prices for valuation.
REFERENCES


